離散最適化基礎論

第3回

線形計画法の復習

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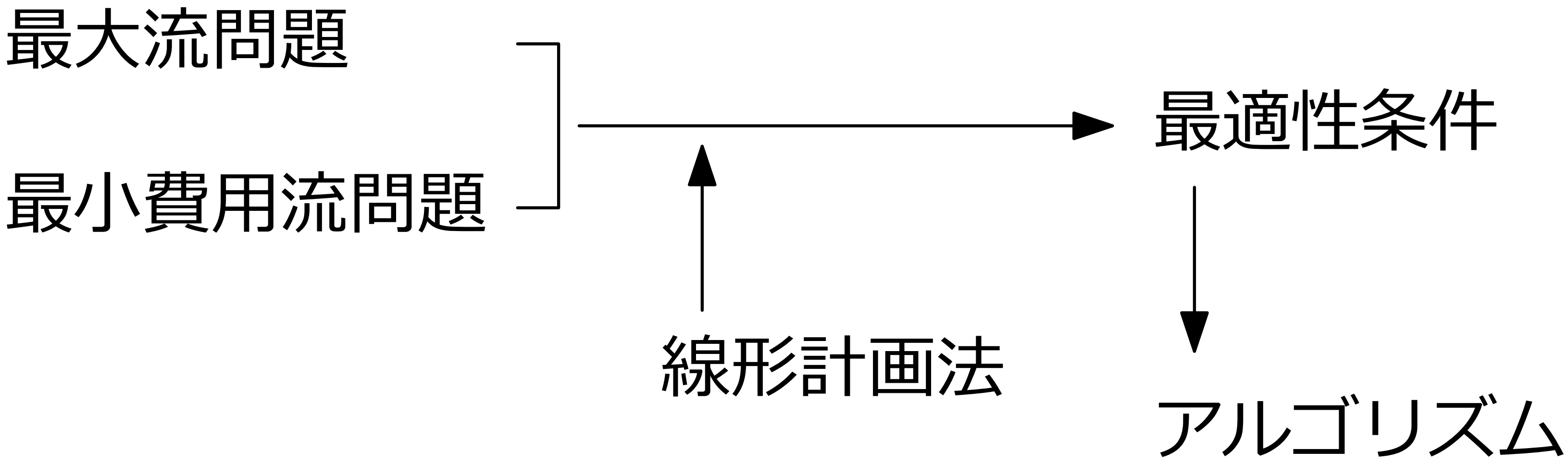


最大流問題

最小費用流問題

最適性条件

線形計画法

アルゴリズム

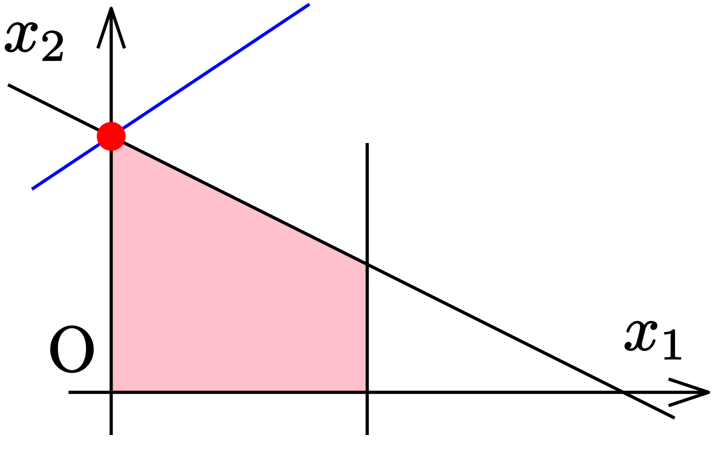
本日の目標

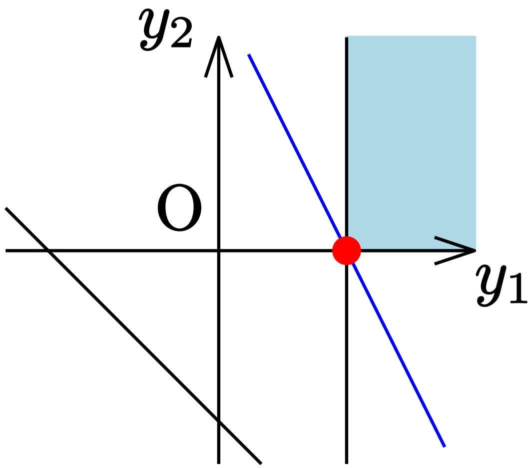
線形計画法の重要な概念を復習して，書けるようになる

- 双対性 (duality)
- 相補性 (complementarity)

本日の内容

1. **線形計画問題と線形計画法**
2. 双対定理
3. 相補性定理





線形計画問題：小さな例

$$\begin{aligned} \text{maximize} \quad & -2x_1 + 3x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \leq 2, \\ & x_1 \leq 1, \\ & x_1 \geq 0, \\ & x_2 \geq 0 \end{aligned}$$

線形計画問題：小さな例

$$\begin{aligned}\text{maximize}\quad & -2x_1 + 3x_2 \\ \text{subject to}\quad & x_1 + 2x_2 \leq 2, \\ & x_1 \leq 1, \\ & x_1 \geq 0, \\ & x_2 \geq 0\end{aligned}$$

変数は x_1 と x_2

線形計画問題：小さな例

$$\begin{aligned} \text{maximize} \quad & \boxed{-2x_1 + 3x_2} \\ \text{subject to} \quad & x_1 + 2x_2 \le 2, \\ & x_1 \le 1, \\ & x_1 \ge 0, \\ & x_2 \ge 0 \end{aligned}$$

目的関数

変数は x_1 と x_2

線形計画問題：小さな例

maximize $-2x_1 + 3x_2$ 目的関数

subject to

$$\begin{aligned} x_1 + 2x_2 &\leq 2, \\ x_1 &\leq 1, \\ x_1 &\geq 0, \\ x_2 &\geq 0 \end{aligned}$$

制約

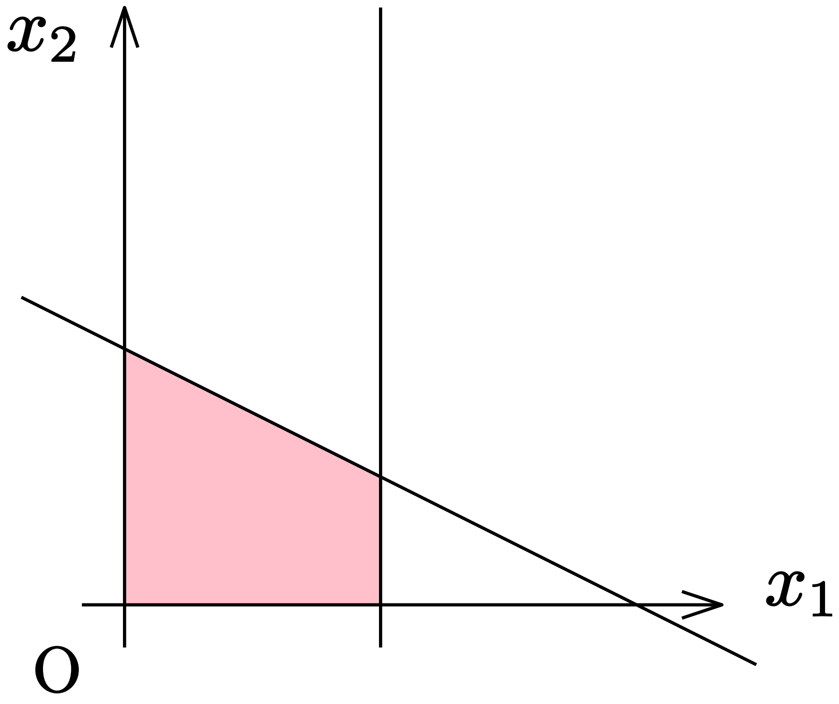
変数は x_1 と x_2

線形計画問題：小さな例

maximize $-2x_1 + 3x_2$ 目的関数

subject to
$$\begin{aligned} x_1 + 2x_2 &\leq 2, \\ x_1 &\leq 1, \\ x_1 &\geq 0, \\ x_2 &\geq 0 \end{aligned}$$
制約

変数は x_1 と x_2



線形計画問題：小さな例

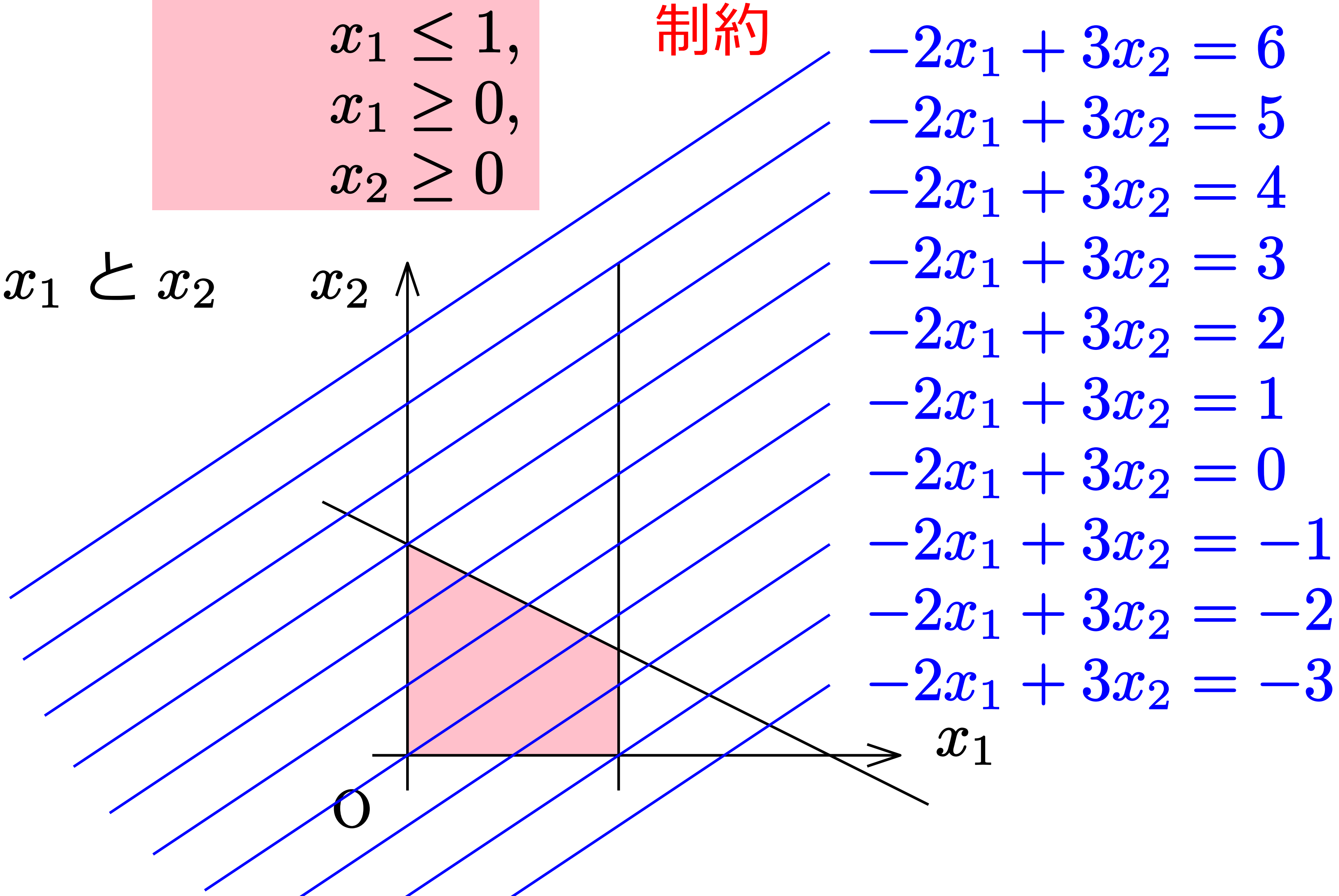
maximize $-2x_1 + 3x_2$ 目的関数

subject to

$$\begin{aligned} x_1 + 2x_2 &\le 2, \\ x_1 &\le 1, \\ x_1 &\ge 0, \\ x_2 &\ge 0 \end{aligned}$$

制約

変数は x_1 と x_2

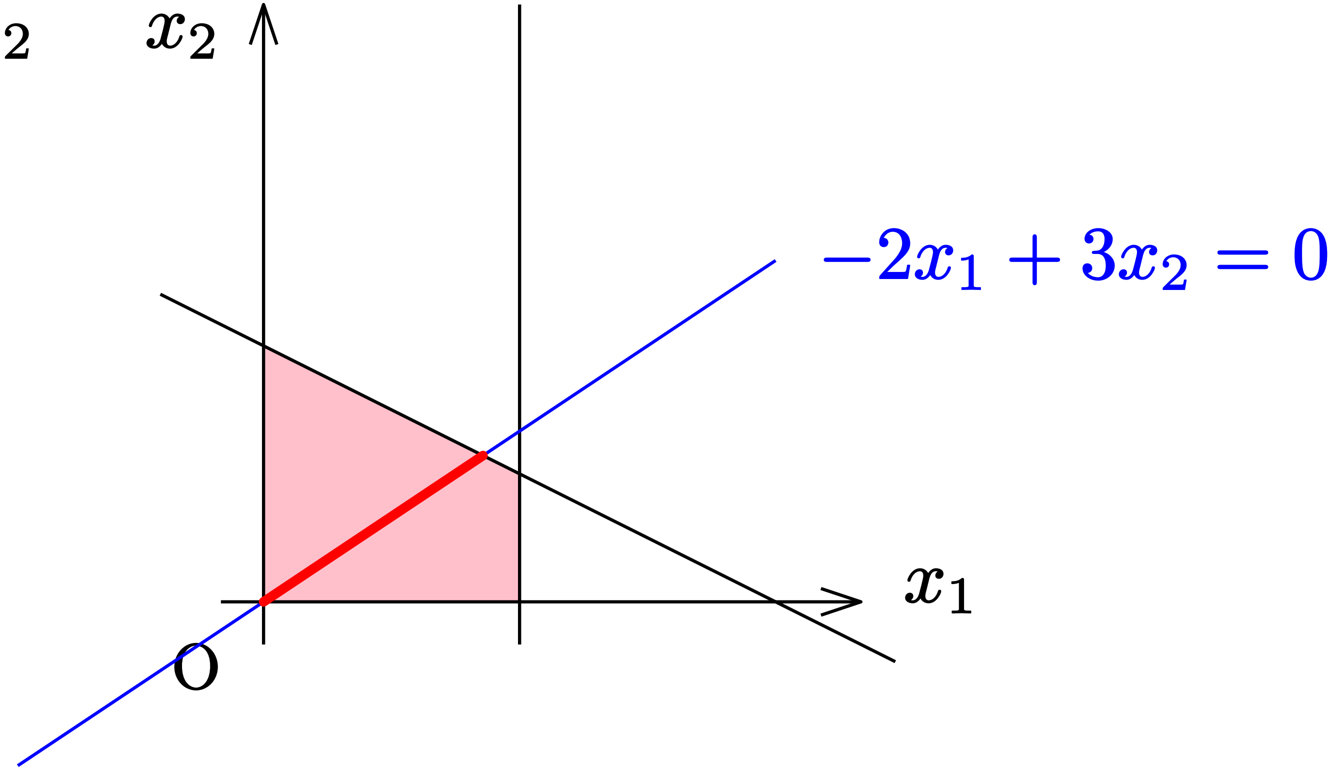


線形計画問題：小さな例

maximize $-2x_1 + 3x_2$ 目的関数

subject to
$$\begin{aligned} x_1 + 2x_2 &\leq 2,\\ x_1 &\leq 1,\\ x_1 &\geq 0,\\ x_2 &\geq 0 \end{aligned}$$
制約

変数は x_1 と x_2

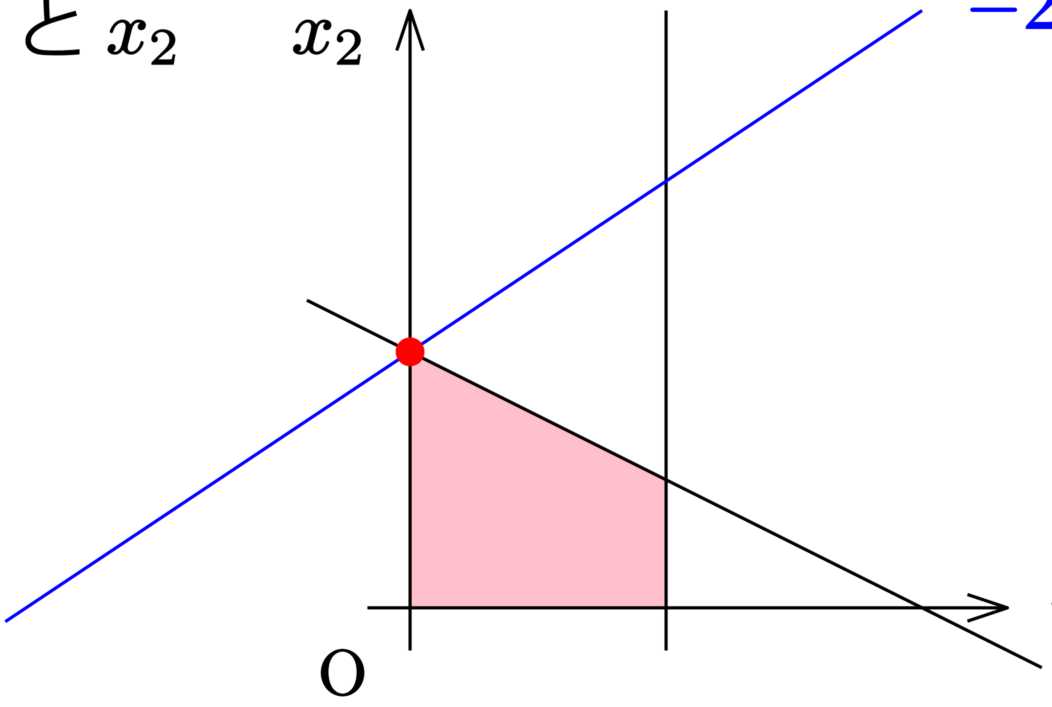


線形計画問題：小さな例

maximize $-2x_1 + 3x_2$ 目的関数

subject to
$$\begin{aligned} x_1 + 2x_2 &\le 2, \\ x_1 &\le 1, \\ x_1 &\ge 0, \\ x_2 &\ge 0 \end{aligned}$$
制約

変数は x_1 と x_2



線形計画問題：小さな例

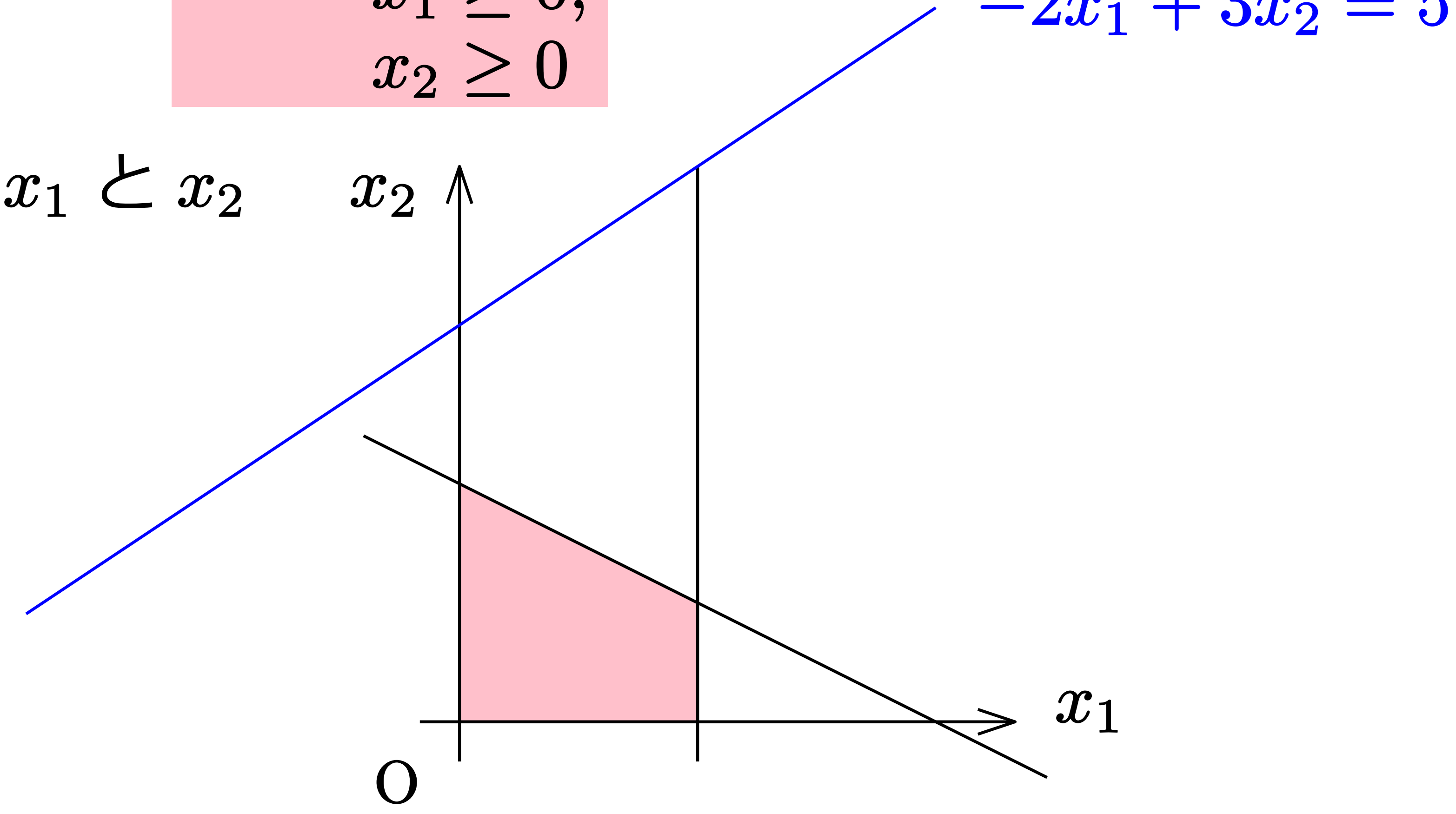
maximize $-2x_1 + 3x_2$ 目的関数

subject to

$$\begin{aligned} x_1 + 2x_2 &\le 2, \\ x_1 &\le 1, \\ x_1 &\ge 0, \\ x_2 &\ge 0 \end{aligned}$$

制約

変数は x_1 と x_2

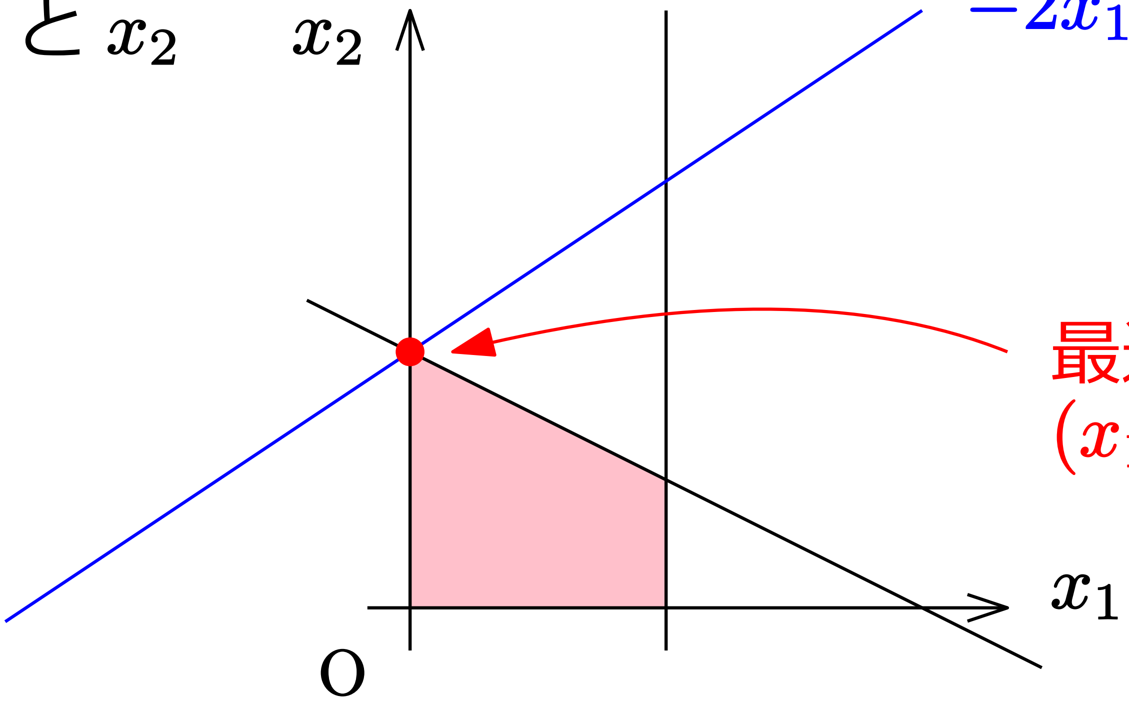


線形計画問題：小さな例 (続き)

maximize $-2x_1 + 3x_2$ 目的関数

subject to
$$\begin{aligned} x_1 + 2x_2 &\le 2, \\ x_1 &\le 1, \\ x_1 &\ge 0, \\ x_2 &\ge 0 \end{aligned}$$
制約

変数は x_1 と x_2



$-2x_1 + 3x_2 = \boxed{3}$

最適値

最適解
$(x_1, x_2) = (0, 1)$

線形計画問題：定義に向けて

線形計画問題の構成要素

変数	実数値をとる
制約	変数に関する線形の等式 または 変数に関する等号付きの線形の不等式
目的関数	変数に関する線形関数

$$
\begin{aligned}
\text{maximize} \quad & -2x_1 + 3x_2 \\
\text{subject to} \quad & x_1 + 2x_2 \leq 2, \\
& \qquad\quad\; x_1 \leq 1, \\
& \qquad\quad\; x_1 \geq 0, \\
& \qquad\quad\; x_2 \geq 0
\end{aligned}
$$

線形計画問題：定義に向けて

線形計画問題の構成要素

変数	実数値をとる
制約	変数に関する線形の等式 または 変数に関する等号付きの線形の不等式
目的関数	変数に関する線形関数

ベクトルと行列で表示

$$\begin{aligned} \text{maximize} \quad & -2x_1 + 3x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \leq 2, \\ & x_1 \leq 1, \\ & x_1 \geq 0, \\ & x_2 \geq 0 \end{aligned}$$

$$\begin{bmatrix} -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

成分ごとの不等式

$$\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq \mathbf{0}$$

線形計画問題：定義 ― 最大化の場合

最大化 の **線形計画問題** (linear program) とは,

次のように書かれる問題

$$\begin{array}{ll} \text{maximize} & \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_1 + \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2 \\ \text{subject to} & A_{11}\boldsymbol{x}_1 + A_{12}\boldsymbol{x}_2 \leq \boldsymbol{b}_1, \\ & A_{21}\boldsymbol{x}_1 + A_{22}\boldsymbol{x}_2 = \boldsymbol{b}_2, \\ & \boldsymbol{x}_1 \geq \boldsymbol{0} \end{array}$$

与えられるものは $\boldsymbol{c}_1 \in \mathbb{R}^{n_1}, \boldsymbol{c}_2 \in \mathbb{R}^{n_2}, \boldsymbol{b}_1 \in \mathbb{R}^{m_1}, \boldsymbol{b}_2 \in \mathbb{R}^{m_2}$

$A_{11} \in \mathbb{R}^{m_1 \times n_1}, A_{12} \in \mathbb{R}^{m_1 \times n_2},$

$A_{21} \in \mathbb{R}^{m_2 \times n_1}, A_{22} \in \mathbb{R}^{m_2 \times n_2}$

決めるものは $\boldsymbol{x}_1 \in \mathbb{R}^{n_1}, \boldsymbol{x}_2 \in \mathbb{R}^{n_2}$

線形計画問題：定義 — 最大化の場合

最大化 の **線形計画問題** (linear program) とは,

次のように書かれる問題

$$
\begin{array}{lrl}
\text{maximize} & \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_1 + \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2 & \longleftarrow \text{目的関数} \\
\text{subject to} & A_{11}\boldsymbol{x}_1 + A_{12}\boldsymbol{x}_2 \le \boldsymbol{b}_1, & \longleftarrow \text{不等式制約} \\
 & A_{21}\boldsymbol{x}_1 + A_{22}\boldsymbol{x}_2 = \boldsymbol{b}_2, & \longleftarrow \text{等式制約} \\
 & \boldsymbol{x}_1 \ge \boldsymbol{0} & \longleftarrow \text{非負制約}
\end{array}
$$

与えられるものは (定数)

$$\boldsymbol{c}_1 \in \mathbb{R}^{n_1}, \boldsymbol{c}_2 \in \mathbb{R}^{n_2}, \boldsymbol{b}_1 \in \mathbb{R}^{m_1}, \boldsymbol{b}_2 \in \mathbb{R}^{m_2}$$

$$A_{11} \in \mathbb{R}^{m_1 \times n_1}, A_{12} \in \mathbb{R}^{m_1 \times n_2},$$

$$A_{21} \in \mathbb{R}^{m_2 \times n_1}, A_{22} \in \mathbb{R}^{m_2 \times n_2}$$

決めるものは (変数)

$$\boldsymbol{x}_1 \in \mathbb{R}^{n_1}, \boldsymbol{x}_2 \in \mathbb{R}^{n_2}$$

$\boldsymbol{x}_2$ ← 自由変数と呼ぶことがある

線形計画問題：定義 — 最小化の場合

最小化 の **線形計画問題** (linear program) とは,

次のように書かれる問題

$$\begin{array}{ll} \text{minimize} & \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_1 + \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2 \qquad \longleftarrow \text{目的関数} \\ \text{subject to} & A_{11}\boldsymbol{x}_1 + A_{12}\boldsymbol{x}_2 \geq \boldsymbol{b}_1, \longleftarrow \text{不等式制約} \\ & A_{21}\boldsymbol{x}_1 + A_{22}\boldsymbol{x}_2 = \boldsymbol{b}_2, \longleftarrow \text{等式制約} \\ & \boldsymbol{x}_1 \geq \boldsymbol{0} \qquad \longleftarrow \text{非負制約} \end{array}$$

与えられるものは
(定数)

$\boldsymbol{c}_1 \in \mathbb{R}^{n_1}, \boldsymbol{c}_2 \in \mathbb{R}^{n_2}, \boldsymbol{b}_1 \in \mathbb{R}^{m_1}, \boldsymbol{b}_2 \in \mathbb{R}^{m_2}$

$A_{11} \in \mathbb{R}^{m_1 \times n_1}, A_{12} \in \mathbb{R}^{m_1 \times n_2},$

$A_{21} \in \mathbb{R}^{m_2 \times n_1}, A_{22} \in \mathbb{R}^{m_2 \times n_2}$

決めるものは
(変数)

$\boldsymbol{x}_1 \in \mathbb{R}^{n_1}, \boldsymbol{x}_2 \in \mathbb{R}^{n_2}$

$\nwarrow$ 自由変数と呼ぶことがある

許容解と許容領域

定義：許容解 (feasible solution)

制約をすべて満たす変数の値 (のベクトル)

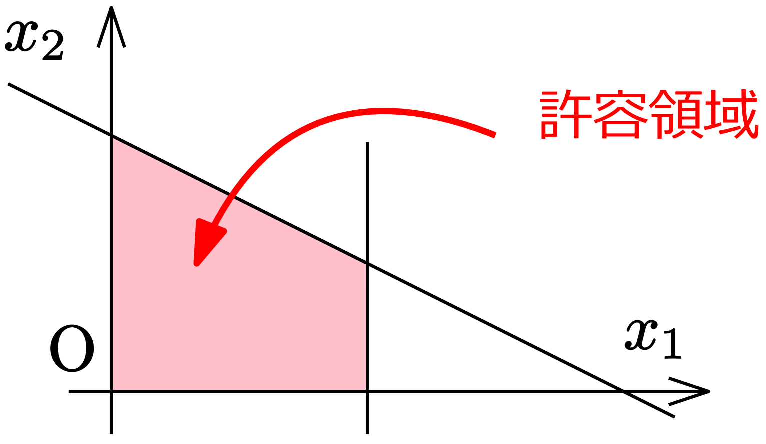
許容解の別名：実行可能解

定義：許容領域 (feasible region)

許容解全体の集合

許容領域の別名：実行可能領域

$$
\begin{array}{ll}
\text{maximize} & -2x_1 + 3x_2 \\
\text{subject to} & x_1 + 2x_2 \leq 2, \\
 & x_1 \leq 1, \\
 & x_1 \geq 0, \\
 & x_2 \geq 0
\end{array}
$$



最適解と最適値 — 最大化の場合

定義：最適解 (optimal solution)：最大化問題の場合

最適解とは，許容解 x_1^*, x_2^* で次を満たすもの

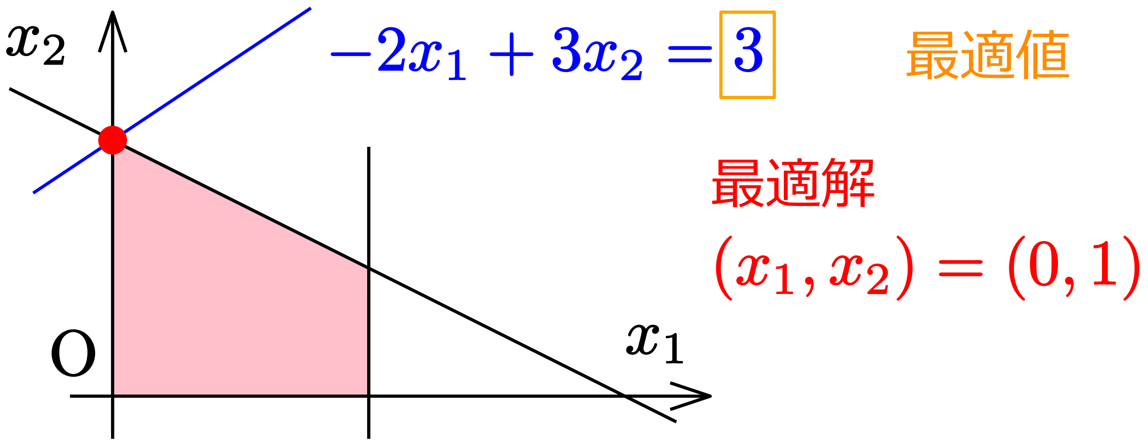
- 任意の許容解 x_1, x_2 に対して，

$$\boldsymbol{c}_1^{\mathrm{T}} \boldsymbol{x}_1^* + \boldsymbol{c}_2^{\mathrm{T}} \boldsymbol{x}_2^* \geq \boldsymbol{c}_1^{\mathrm{T}} \boldsymbol{x}_1 + \boldsymbol{c}_2^{\mathrm{T}} \boldsymbol{x}_2$$

定義：最適値 (optimal value)

最適値とは，最適解における目的関数の値

$$\begin{array}{ll} \text{maximize} & -2x_1 + 3x_2 \\ \text{subject to} & x_1 + 2x_2 \leq 2, \\ & x_1 \leq 1, \\ & x_1 \geq 0, \\ & x_2 \geq 0 \end{array}$$



最適解と最適値 — 最小化の場合

定義：最適解 (optimal solution)：最小化問題の場合

最適解とは，許容解 $\boldsymbol{x}_1^*, \boldsymbol{x}_2^*$ で次を満たすもの

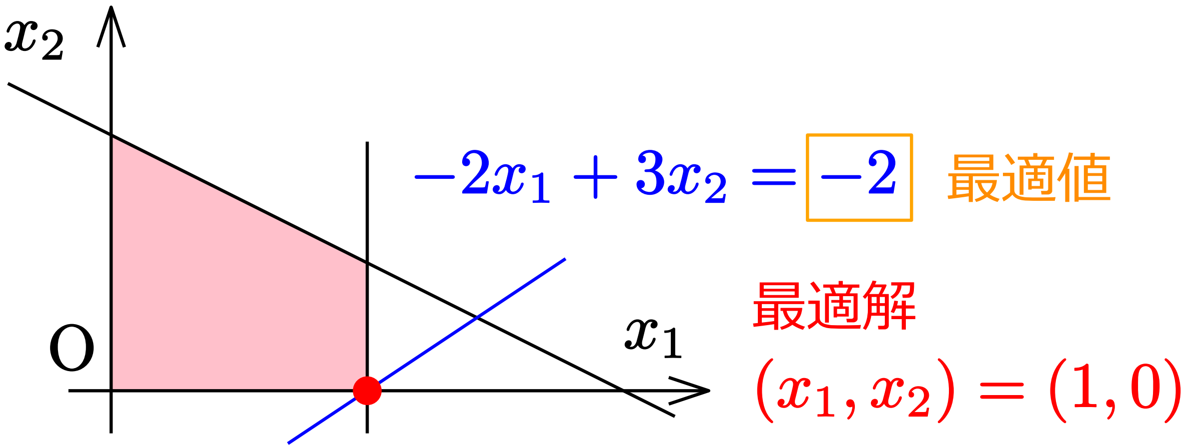
- 任意の許容解 $\boldsymbol{x}_1, \boldsymbol{x}_2$ に対して，

$$\boldsymbol{c}_1^{\mathrm{T}} \boldsymbol{x}_1^* + \boldsymbol{c}_2^{\mathrm{T}} \boldsymbol{x}_2^* \leq \boldsymbol{c}_1^{\mathrm{T}} \boldsymbol{x}_1 + \boldsymbol{c}_2^{\mathrm{T}} \boldsymbol{x}_2$$

定義：最適値 (optimal value)

最適値とは，最適解における目的関数の値

$$\begin{array}{lr} \text{minimize} & -2x_1 + 3x_2 \\ \text{subject to} & x_1 + 2x_2 \leq 2, \\ & x_1 \leq 1, \\ & x_1 \geq 0, \\ & x_2 \geq 0 \end{array}$$



線形計画問題を解く とは？

線形計画問題を解く とは？ (不正確な定義)

$A_{11}, A_{12}, A_{21}, A_{22}, b_1, b_2, c_1, c_2$ を与えて，
それらが定める線形計画問題の**最適解**を **1つ**求めること

本来は，

最適解が存在しないとき，それも判定しないといけない

$$\begin{array}{ll} \text{maximize} & c_1^{\mathrm{T}} x_1 + c_2^{\mathrm{T}} x_2 \\ \text{subject to} & A_{11} x_1 + A_{12} x_2 \le b_1, \\ & A_{21} x_1 + A_{22} x_2 = b_2, \\ & x_1 \ge 0 \end{array}$$

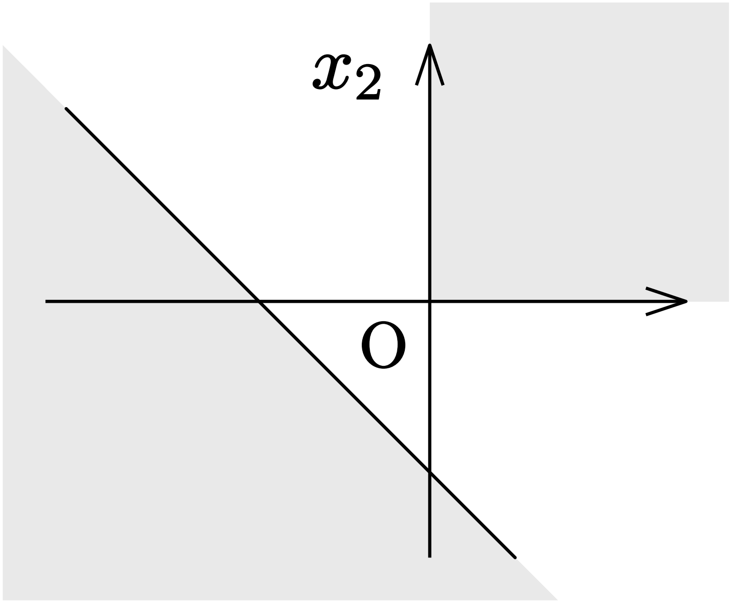
非許容な線形計画問題

非許容な線形計画問題に，最適解は存在しない

定義：非許容な線形計画問題

線形計画問題が **非許容** (infeasible) であるとは,
その許容解が存在しないこと

$$\begin{aligned} \text{maximize} \quad & x_1 - x_2 \\ \text{subject to} \quad & -x_1 - x_2 \geq 1, \\ & x_1 \geq 0, \\ & x_2 \geq 0 \end{aligned}$$



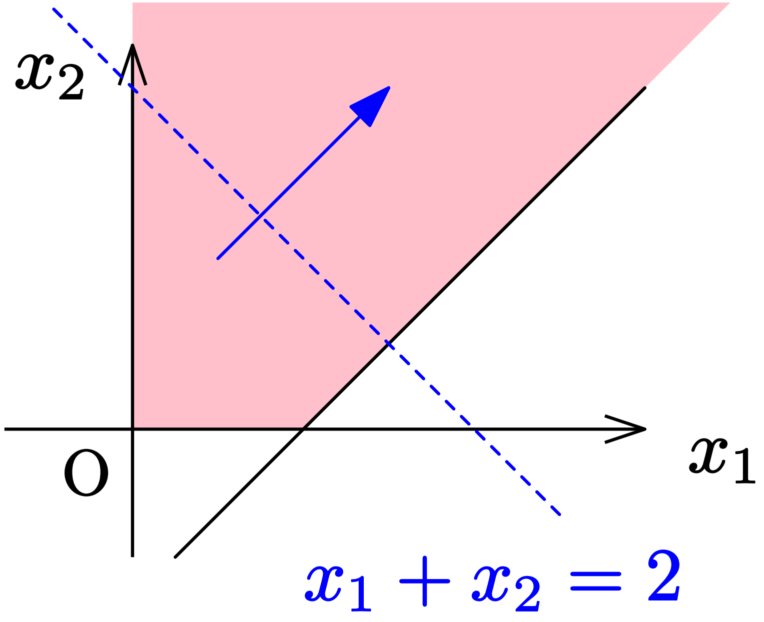
非有界な線形計画問題

非有界な線形計画問題に，最適解は存在しない

定義：非有界な線形計画問題

(最大化の) 線形計画問題が **非有界** (unbounded) とは，
目的関数値を任意に大きくする許容解の列があること

$$\begin{aligned} \text{maximize} \quad & x_1 + x_2 \\ \text{subject to} \quad & -x_1 + x_2 \geq -1, \\ & x_1 \geq 0, \\ & x_2 \geq 0 \end{aligned}$$



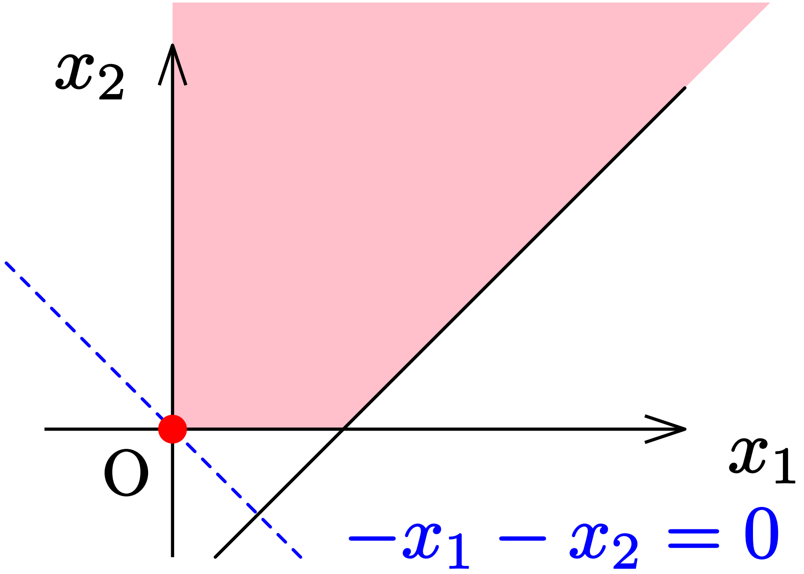
非有界な線形計画問題：注

許容領域が非有界でも，最適解が存在することはある

定義：非有界な線形計画問題

(最大化の) 線形計画問題が **非有界** (unbounded) とは，
目的関数値を任意に大きくする許容解の列があること

$$\begin{aligned} \text{maximize} \quad & -x_1 - x_2 \\ \text{subject to} \quad & -x_1 + x_2 \geq -1, \\ & x_1 \geq 0, \\ & x_2 \geq 0 \end{aligned}$$



線形計画法

線形計画法とは？　(岩波数学辞典第 4 版，’07)

線形計画問題の理論，応用，解法に関する分野の総称

この授業では，

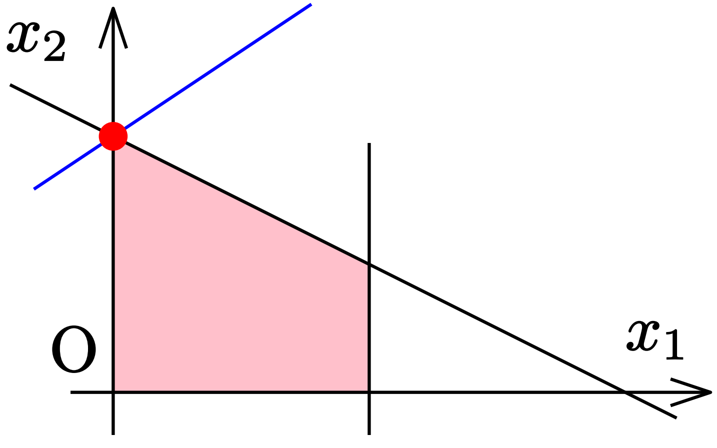
最大流問題/最小費用流問題を解くために

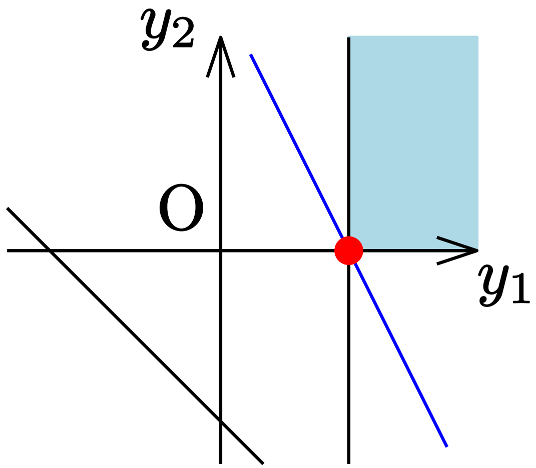
線形計画法の考え方を用いる

注：線形計画法のアルゴリズムを用いるわけではない

本日の内容

1. 線形計画問題と線形計画法
2. **双対定理**
3. 相補性定理





最適値の上界を求める (1)

$$\begin{aligned}\text{maximize}\quad & -2x_1 + 3x_2 \\ \text{subject to}\quad & x_1 + 2x_2 \le 2, \\ & x_1 \le 1, \\ & x_1 \ge 0, \\ & x_2 \ge 0\end{aligned}$$

最適値の上界を求める (1)

maximize $-2x_1 + 3x_2$

subject to

$$\begin{aligned} x_1 + 2x_2 &\leq 2, && \longleftarrow \quad y_1 \geq 0 \text{ をかける} \\ x_1 &\leq 1, && \longleftarrow \quad y_2 \geq 0 \text{ をかける} \\ x_1 &\geq 0, \\ x_2 &\geq 0 \end{aligned}$$

$$\begin{aligned} (x_1 + 2x_2)y_1 &\leq 2y_1 \\ x_1 y_2 &\leq y_2 \end{aligned}$$

最適値の上界を求める (1)

$$\begin{array}{lrll} \text{maximize} & -2x_1 + 3x_2 & & \\ \text{subject to} & x_1 + 2x_2 \leq 2, & \longleftarrow & y_1 \geq 0 \text{ をかける} \\ & x_1 \leq 1, & \longleftarrow & y_2 \geq 0 \text{ をかける} \\ & x_1 \geq 0, & & \\ & x_2 \geq 0 & & \end{array}$$

$$\begin{array}{rl} (x_1 + 2x_2)y_1 \leq 2y_1 & \\ x_1 y_2 \leq y_2 & \quad (+ \\ \hline (x_1 + 2x_2)y_1 + x_1 y_2 \leq 2y_1 + y_2 & \end{array}$$

最適値の上界を求める (1)

$$\begin{array}{lrl} \text{maximize} & -2x_1 + 3x_2 & \\ \text{subject to} & x_1 + 2x_2 \le 2, & \longleftarrow \; y_1 \ge 0 \text{ をかける} \\ & x_1 \le 1, & \longleftarrow \; y_2 \ge 0 \text{ をかける} \\ & x_1 \ge 0, & \\ & x_2 \ge 0 & \end{array}$$

$$\begin{array}{rl} (x_1 + 2x_2)y_1 \le 2y_1 & \\ x_1 y_2 \le y_2 & \qquad (+ \\ \hline (x_1 + 2x_2)y_1 + x_1 y_2 \le 2y_1 + y_2 & \end{array}$$

$$\parallel$$

$$(y_1 + y_2)x_1 + 2y_1 x_2 \qquad \therefore (y_1 + y_2)x_1 + 2y_1 x_2 \le 2y_1 + y_2$$

最適値の上界を求める (1)

$$\begin{array}{lrcll} \text{maximize} & -2x_1+3x_2 & & & \\ \text{subject to} & x_1+2x_2 & \le & 2, & \longleftarrow \quad y_1 \ge 0 \text{ をかける} \\ & x_1 & \le & 1, & \longleftarrow \quad y_2 \ge 0 \text{ をかける} \\ & x_1 & \ge & 0, & \\ & x_2 & \ge & 0 & \end{array}$$

$$\begin{array}{rl} (x_1+2x_2)y_1 \le 2y_1 & \\ x_1y_2 \le y_2 & (+ \\ \hline (x_1+2x_2)y_1+x_1y_2 \le 2y_1+y_2 & \end{array}$$

$$\|$$

$$(y_1+y_2)x_1+2y_1x_2 \qquad \therefore (y_1+y_2)x_1+2y_1x_2 \le 2y_1+y_2$$

$y_1=\frac{3}{2}, y_2=0$ とすると $\qquad \frac{3}{2}x_1+3x_2 \le 3$

最適値の上界を求める (1)

maximize $-2x_1 + 3x_2$

subject to
$$\begin{aligned} x_1 + 2x_2 &\le 2, && \longleftarrow \quad y_1 \ge 0 \text{ をかける} \\ x_1 &\le 1, && \longleftarrow \quad y_2 \ge 0 \text{ をかける} \\ x_1 &\ge 0, \\ x_2 &\ge 0 \end{aligned}$$

$$\begin{array}{rl} (x_1 + 2x_2)y_1 \le 2y_1 & \\ x_1y_2 \le y_2 & (+ \\ \hline (x_1 + 2x_2)y_1 + x_1y_2 \le 2y_1 + y_2 & \end{array}$$

$$\|$$

$$(y_1 + y_2)x_1 + 2y_1x_2 \qquad \therefore (y_1 + y_2)x_1 + 2y_1x_2 \le 2y_1 + y_2$$

$y_1 = \dfrac{3}{2}, y_2 = 0$ とすると $\qquad \dfrac{3}{2}x_1 + 3x_2 \le 3$

$$\therefore \ -2x_1 + 3x_2 \le \frac{3}{2}x_1 + 3x_2 \le 3$$

最適値の上界を求める (1)

$$\begin{array}{ll} \text{maximize} & -2x_1 + 3x_2 \\ \text{subject to} & x_1 + 2x_2 \leq 2, \\ & x_1 \leq 1, \\ & x_1 \geq 0, \\ & x_2 \geq 0 \end{array}$$

$\longleftarrow$ $y_1 \geq 0$ をかける

$\longleftarrow$ $y_2 \geq 0$ をかける

(x_1, x_2) が許容解 $\Rightarrow$

$-2x_1 + 3x_2 \leq 3$

$$\begin{array}{rl} (x_1 + 2x_2)y_1 \leq 2y_1 & \\ x_1 y_2 \leq y_2 & (+ \\ \hline (x_1 + 2x_2)y_1 + x_1 y_2 \leq 2y_1 + y_2 & \end{array}$$

$$\|$$

$$(y_1 + y_2)x_1 + 2y_1 x_2 \qquad \therefore (y_1 + y_2)x_1 + 2y_1 x_2 \leq 2y_1 + y_2$$

$y_1 = \frac{3}{2}, y_2 = 0$ とすると $\qquad \frac{3}{2}x_1 + 3x_2 \leq 3$

$$\therefore \quad -2x_1 + 3x_2 \leq \frac{3}{2}x_1 + 3x_2 \leq 3$$

最適値の上界を求める (2)

$$\begin{array}{lrl} \text{maximize} & \multicolumn{2}{l}{-2x_1 + 3x_2} & \\ \text{subject to} & x_1 + 2x_2 \le 2, & \longleftarrow \quad y_1 \ge 0 \text{ をかける} \\ & x_1 \le 1, & \longleftarrow \quad y_2 \ge 0 \text{ をかける} \\ & x_1 \ge 0, & \\ & x_2 \ge 0 & \end{array}$$

$$\begin{array}{rl} (x_1 + 2x_2)y_1 \le 2y_1 & \\ x_1y_2 \le y_2 & \qquad (+ \\ \hline (x_1 + 2x_2)y_1 + x_1y_2 \le 2y_1 + y_2 & \end{array}$$

$$\|$$

$$(y_1 + y_2)x_1 + 2y_1x_2 \qquad \therefore (y_1 + y_2)x_1 + 2y_1x_2 \le 2y_1 + y_2$$

「この方法で，できるだけ小さな上界を見つける」問題を導出したい

最適値の小さな上界を求める問題

$$\begin{aligned} \text{maximize} \quad & -2x_1 + 3x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \le 2, \\ & x_1 \le 1, \\ & x_1 \ge 0, \\ & x_2 \ge 0 \end{aligned}$$

$$\begin{aligned} \text{minimize} \quad & 2y_1 + y_2 \\ \text{subject to} \quad & y_1 + y_2 \ge -2, \\ & 2y_1 \ge 3, \\ & y_1 \ge 0, \\ & y_2 \ge 0 \end{aligned}$$

元の線形計画問題

元の線形計画問題の
最適値の小さな上界を
見つける問題
(これも線形計画問題)

主問題と双対問題：例

$$
\begin{array}{ll}
\text{maximize} & -2x_1 + 3x_2 \\
\text{subject to} & x_1 + 2x_2 \leq 2, \\
 & \qquad\quad x_1 \leq 1, \\
 & \qquad\quad x_1 \geq 0, \\
 & \qquad\quad x_2 \geq 0
\end{array}
$$

主問題 (primal problem)

$$
\begin{array}{ll}
\text{minimize} & 2y_1 + y_2 \\
\text{subject to} & y_1 + y_2 \geq -2, \\
 & \qquad\quad 2y_1 \geq 3, \\
 & \qquad\quad y_1 \geq 0, \\
 & \qquad\quad y_2 \geq 0
\end{array}
$$

双対問題 (dual problem)

主問題と双対問題：例

$$\begin{array}{ll} \text{maximize} & -2x_1 + 3x_2 \\ \text{subject to} & x_1 + 2x_2 \le 2, \\ & x_1 \le 1, \\ & x_1 \ge 0, \\ & x_2 \ge 0 \end{array}$$

$$\begin{array}{ll} \text{minimize} & 2y_1 + y_2 \\ \text{subject to} & y_1 + y_2 \ge -2, \\ & 2y_1 \ge 3, \\ & y_1 \ge 0, \\ & y_2 \ge 0 \end{array}$$

主問題 (primal problem)

双対問題 (dual problem)

許容解 (x_1, x_2)

許容解 (y_1, y_2) に対して

$$\begin{aligned} -2x_1 + 3x_2 &\le (y_1 + y_2)x_1 + 2y_1x_2 \\ &= (x_1 + 2x_2)y_1 + x_1y_2 \le 2y_1 + y_2 \end{aligned}$$

主問題の目的関数値

双対問題の目的関数値

主問題と双対問題：一般論 (1)

$$\begin{array}{ll} \text{max.} & \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_1 + \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2 \\ \text{s.t.} & A_{11}\boldsymbol{x}_1 + A_{12}\boldsymbol{x}_2 \leq \boldsymbol{b}_1, \\ & A_{21}\boldsymbol{x}_1 + A_{22}\boldsymbol{x}_2 = \boldsymbol{b}_2, \\ & \hfill \boldsymbol{x}_1 \geq \boldsymbol{0} \end{array}$$

定数：$\boldsymbol{c}_1 \in \mathbb{R}^{n_1}$, $\boldsymbol{c}_2 \in \mathbb{R}^{n_2}$, $\boldsymbol{b}_1 \in \mathbb{R}^{m_1}$, $\boldsymbol{b}_2 \in \mathbb{R}^{m_2}$

$A_{11} \in \mathbb{R}^{m_1 \times n_1}$, $A_{12} \in \mathbb{R}^{m_1 \times n_2}$, $A_{21} \in \mathbb{R}^{m_2 \times n_1}$, $A_{22} \in \mathbb{R}^{m_2 \times n_2}$

変数：$\boldsymbol{x}_1 \in \mathbb{R}^{n_1}$, $\boldsymbol{x}_2 \in \mathbb{R}^{n_2}$

主問題と双対問題：一般論 (1)

$$\begin{array}{ll} \text{max.} & \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_1 + \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2 \\ \text{s.t.} & A_{11}\boldsymbol{x}_1 + A_{12}\boldsymbol{x}_2 \leq \boldsymbol{b}_1, \\ & A_{21}\boldsymbol{x}_1 + A_{22}\boldsymbol{x}_2 = \boldsymbol{b}_2, \\ & \boldsymbol{x}_1 \geq \boldsymbol{0} \end{array}$$

$\longleftarrow$ $\boldsymbol{y}_1 \geq \boldsymbol{0}$ をかける　$\boldsymbol{y}_1 \in \mathbb{R}^{m_1}$

$\longleftarrow$ $\boldsymbol{y}_2$ をかける　$\boldsymbol{y}_2 \in \mathbb{R}^{m_2}$

定数：$\boldsymbol{c}_1 \in \mathbb{R}^{n_1}, \boldsymbol{c}_2 \in \mathbb{R}^{n_2}, \boldsymbol{b}_1 \in \mathbb{R}^{m_1}, \boldsymbol{b}_2 \in \mathbb{R}^{m_2}$

$A_{11} \in \mathbb{R}^{m_1 \times n_1}, A_{12} \in \mathbb{R}^{m_1 \times n_2}, A_{21} \in \mathbb{R}^{m_2 \times n_1}, A_{22} \in \mathbb{R}^{m_2 \times n_2}$

変数：$\boldsymbol{x}_1 \in \mathbb{R}^{n_1}, \boldsymbol{x}_2 \in \mathbb{R}^{n_2}$

主問題と双対問題：一般論 (1)

$$\begin{array}{ll} \text{max.} & \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_1 + \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2 \\ \text{s.t.} & A_{11}\boldsymbol{x}_1 + A_{12}\boldsymbol{x}_2 \leq \boldsymbol{b}_1, \\ & A_{21}\boldsymbol{x}_1 + A_{22}\boldsymbol{x}_2 = \boldsymbol{b}_2, \\ & \boldsymbol{x}_1 \geq \boldsymbol{0} \end{array}$$

← $\boldsymbol{y}_1 \geq \boldsymbol{0}$ をかける　　$\boldsymbol{y}_1 \in \mathbb{R}^{m_1}$

← $\boldsymbol{y}_2$ をかける　　$\boldsymbol{y}_2 \in \mathbb{R}^{m_2}$

定数：$\boldsymbol{c}_1 \in \mathbb{R}^{n_1}$, $\boldsymbol{c}_2 \in \mathbb{R}^{n_2}$, $\boldsymbol{b}_1 \in \mathbb{R}^{m_1}$, $\boldsymbol{b}_2 \in \mathbb{R}^{m_2}$

$A_{11} \in \mathbb{R}^{m_1 \times n_1}$, $A_{12} \in \mathbb{R}^{m_1 \times n_2}$, $A_{21} \in \mathbb{R}^{m_2 \times n_1}$, $A_{22} \in \mathbb{R}^{m_2 \times n_2}$

変数：$\boldsymbol{x}_1 \in \mathbb{R}^{n_1}$, $\boldsymbol{x}_2 \in \mathbb{R}^{n_2}$

$$\begin{array}{ll} (A_{11}\boldsymbol{x}_1 + A_{12}\boldsymbol{x}_2)^{\mathrm{T}}\boldsymbol{y}_1 \leq \boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{y}_1 & \\ (A_{21}\boldsymbol{x}_1 + A_{22}\boldsymbol{x}_2)^{\mathrm{T}}\boldsymbol{y}_2 = \boldsymbol{b}_2^{\mathrm{T}}\boldsymbol{y}_2 & (+ \\ \hline \end{array}$$

主問題と双対問題：一般論 (1)

$$\begin{array}{lrl}
\text{max.} & \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_1 + \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2 & \\
\text{s.t.} & A_{11}\boldsymbol{x}_1 + A_{12}\boldsymbol{x}_2 \le \boldsymbol{b}_1, & \longleftarrow \quad \boldsymbol{y}_1 \ge \boldsymbol{0} \text{ をかける} \qquad \boldsymbol{y}_1 \in \mathbb{R}^{m_1} \\
 & A_{21}\boldsymbol{x}_1 + A_{22}\boldsymbol{x}_2 = \boldsymbol{b}_2, & \longleftarrow \quad \boldsymbol{y}_2 \text{ をかける} \qquad \boldsymbol{y}_2 \in \mathbb{R}^{m_2} \\
 & \boldsymbol{x}_1 \ge \boldsymbol{0} &
\end{array}$$

定数：$\boldsymbol{c}_1 \in \mathbb{R}^{n_1}$, $\boldsymbol{c}_2 \in \mathbb{R}^{n_2}$, $\boldsymbol{b}_1 \in \mathbb{R}^{m_1}$, $\boldsymbol{b}_2 \in \mathbb{R}^{m_2}$

$A_{11} \in \mathbb{R}^{m_1 \times n_1}$, $A_{12} \in \mathbb{R}^{m_1 \times n_2}$, $A_{21} \in \mathbb{R}^{m_2 \times n_1}$, $A_{22} \in \mathbb{R}^{m_2 \times n_2}$

変数：$\boldsymbol{x}_1 \in \mathbb{R}^{n_1}$, $\boldsymbol{x}_2 \in \mathbb{R}^{n_2}$

$$\begin{array}{r}
(A_{11}\boldsymbol{x}_1 + A_{12}\boldsymbol{x}_2)^{\mathrm{T}}\boldsymbol{y}_1 \le \boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{y}_1 \quad \\
(A_{21}\boldsymbol{x}_1 + A_{22}\boldsymbol{x}_2)^{\mathrm{T}}\boldsymbol{y}_2 = \boldsymbol{b}_2^{\mathrm{T}}\boldsymbol{y}_2 \quad (+ \\
\hline
\end{array}$$

$$(A_{11}\boldsymbol{x}_1 + A_{12}\boldsymbol{x}_2)^{\mathrm{T}}\boldsymbol{y}_1 + (A_{21}\boldsymbol{x}_1 + A_{22}\boldsymbol{x}_2)^{\mathrm{T}}\boldsymbol{y}_2 \le \boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{y}_1 + \boldsymbol{b}_2^{\mathrm{T}}\boldsymbol{y}_2$$

主問題と双対問題：一般論 (1)

$$\begin{aligned} \text{max.}\quad & \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_1 + \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2 \\ \text{s.t.}\quad & A_{11}\boldsymbol{x}_1 + A_{12}\boldsymbol{x}_2 \le \boldsymbol{b}_1, \\ & A_{21}\boldsymbol{x}_1 + A_{22}\boldsymbol{x}_2 = \boldsymbol{b}_2, \\ & \qquad\qquad\qquad \boldsymbol{x}_1 \ge \boldsymbol{0} \end{aligned}$$

$\longleftarrow$ $\boldsymbol{y}_1 \ge \boldsymbol{0}$ をかける $\boldsymbol{y}_1 \in \mathbb{R}^{m_1}$

$\longleftarrow$ $\boldsymbol{y}_2$ をかける $\boldsymbol{y}_2 \in \mathbb{R}^{m_2}$

定数：$\boldsymbol{c}_1 \in \mathbb{R}^{n_1}, \boldsymbol{c}_2 \in \mathbb{R}^{n_2}, \boldsymbol{b}_1 \in \mathbb{R}^{m_1}, \boldsymbol{b}_2 \in \mathbb{R}^{m_2}$

$A_{11} \in \mathbb{R}^{m_1 \times n_1}, A_{12} \in \mathbb{R}^{m_1 \times n_2}, A_{21} \in \mathbb{R}^{m_2 \times n_1}, A_{22} \in \mathbb{R}^{m_2 \times n_2}$

変数：$\boldsymbol{x}_1 \in \mathbb{R}^{n_1}, \boldsymbol{x}_2 \in \mathbb{R}^{n_2}$

$$\begin{aligned} (A_{11}\boldsymbol{x}_1 + A_{12}\boldsymbol{x}_2)^{\mathrm{T}}\boldsymbol{y}_1 &\le \boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{y}_1 \\ (A_{21}\boldsymbol{x}_1 + A_{22}\boldsymbol{x}_2)^{\mathrm{T}}\boldsymbol{y}_2 &= \boldsymbol{b}_2^{\mathrm{T}}\boldsymbol{y}_2 \quad (+ \end{aligned}$$

$$\underline{(A_{11}\boldsymbol{x}_1 + A_{12}\boldsymbol{x}_2)^{\mathrm{T}}\boldsymbol{y}_1 + (A_{21}\boldsymbol{x}_1 + A_{22}\boldsymbol{x}_2)^{\mathrm{T}}\boldsymbol{y}_2} \le \boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{y}_1 + \boldsymbol{b}_2^{\mathrm{T}}\boldsymbol{y}_2$$

$$= \boldsymbol{x}_1^{\mathrm{T}}(A_{11}^{\mathrm{T}}\boldsymbol{y}_1 + A_{21}^{\mathrm{T}}\boldsymbol{y}_2) + \boldsymbol{x}_2^{\mathrm{T}}(A_{12}^{\mathrm{T}}\boldsymbol{y}_1 + A_{22}^{\mathrm{T}}\boldsymbol{y}_2)$$

主問題と双対問題：一般論 (1)

$$\begin{array}{llcll} \text{max.} & \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_1 + \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2 & & & \\ \text{s.t.} & A_{11}\boldsymbol{x}_1 + A_{12}\boldsymbol{x}_2 \le \boldsymbol{b}_1, & \longleftarrow & \boldsymbol{y}_1 \ge \boldsymbol{0} \text{ をかける} & \boldsymbol{y}_1 \in \mathbb{R}^{m_1} \\ & A_{21}\boldsymbol{x}_1 + A_{22}\boldsymbol{x}_2 = \boldsymbol{b}_2, & \longleftarrow & \boldsymbol{y}_2 \text{ をかける} & \boldsymbol{y}_2 \in \mathbb{R}^{m_2} \\ & \qquad\qquad \boldsymbol{x}_1 \ge \boldsymbol{0} & & & \end{array}$$

定数：$\boldsymbol{c}_1 \in \mathbb{R}^{n_1}, \boldsymbol{c}_2 \in \mathbb{R}^{n_2}, \boldsymbol{b}_1 \in \mathbb{R}^{m_1}, \boldsymbol{b}_2 \in \mathbb{R}^{m_2}$

$A_{11} \in \mathbb{R}^{m_1 \times n_1}, A_{12} \in \mathbb{R}^{m_1 \times n_2}, A_{21} \in \mathbb{R}^{m_2 \times n_1}, A_{22} \in \mathbb{R}^{m_2 \times n_2}$

変数：$\boldsymbol{x}_1 \in \mathbb{R}^{n_1}, \boldsymbol{x}_2 \in \mathbb{R}^{n_2}$

$$\begin{array}{rl} (A_{11}\boldsymbol{x}_1 + A_{12}\boldsymbol{x}_2)^{\mathrm{T}}\boldsymbol{y}_1 \le \boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{y}_1 & \\ (A_{21}\boldsymbol{x}_1 + A_{22}\boldsymbol{x}_2)^{\mathrm{T}}\boldsymbol{y}_2 = \boldsymbol{b}_2^{\mathrm{T}}\boldsymbol{y}_2 & (+ \\ \hline \end{array}$$

$$\underline{(A_{11}\boldsymbol{x}_1 + A_{12}\boldsymbol{x}_2)^{\mathrm{T}}\boldsymbol{y}_1 + (A_{21}\boldsymbol{x}_1 + A_{22}\boldsymbol{x}_2)^{\mathrm{T}}\boldsymbol{y}_2} \le \boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{y}_1 + \boldsymbol{b}_2^{\mathrm{T}}\boldsymbol{y}_2$$

$$= \boldsymbol{x}_1^{\mathrm{T}}(A_{11}^{\mathrm{T}}\boldsymbol{y}_1 + A_{21}^{\mathrm{T}}\boldsymbol{y}_2) + \boldsymbol{x}_2^{\mathrm{T}}(A_{12}^{\mathrm{T}}\boldsymbol{y}_1 + A_{22}^{\mathrm{T}}\boldsymbol{y}_2)$$

$$\ge \boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{c}_1 + \boldsymbol{x}_2^{\mathrm{T}}\boldsymbol{c}_2$$

$A_{11}^{\mathrm{T}}\boldsymbol{y}_1 + A_{21}^{\mathrm{T}}\boldsymbol{y}_2 \ge \boldsymbol{c}_1, A_{12}^{\mathrm{T}}\boldsymbol{y}_1 + A_{22}^{\mathrm{T}}\boldsymbol{y}_2 = \boldsymbol{c}_2$ とすると

主問題と双対問題：一般論 (2)

$$\begin{array}{ll} \text{max.} & \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_1 + \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2 \\ \text{s.t.} & \boldsymbol{A}_{11}\boldsymbol{x}_1 + \boldsymbol{A}_{12}\boldsymbol{x}_2 \le \boldsymbol{b}_1, \\ & \boldsymbol{A}_{21}\boldsymbol{x}_1 + \boldsymbol{A}_{22}\boldsymbol{x}_2 = \boldsymbol{b}_2, \\ & \qquad\qquad\qquad \boldsymbol{x}_1 \ge \boldsymbol{0} \end{array}$$

$$\begin{array}{ll} \text{min.} & \boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{y}_1 + \boldsymbol{b}_2^{\mathrm{T}}\boldsymbol{y}_2 \\ \text{s.t.} & \boldsymbol{A}_{11}^{\mathrm{T}}\boldsymbol{y}_1 + \boldsymbol{A}_{21}^{\mathrm{T}}\boldsymbol{y}_2 \ge \boldsymbol{c}_1, \\ & \boldsymbol{A}_{12}^{\mathrm{T}}\boldsymbol{y}_1 + \boldsymbol{A}_{22}^{\mathrm{T}}\boldsymbol{y}_2 = \boldsymbol{c}_2, \\ & \qquad\qquad\qquad \boldsymbol{y}_1 \ge \boldsymbol{0} \end{array}$$

主問題 $\longrightarrow$ 双対問題

主問題と双対問題：一般論 (2)

$$\begin{array}{ll} \text{max.} & \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_1 + \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2 \\ \text{s.t.} & A_{11}\boldsymbol{x}_1 + A_{12}\boldsymbol{x}_2 \leq \boldsymbol{b}_1, \\ & A_{21}\boldsymbol{x}_1 + A_{22}\boldsymbol{x}_2 = \boldsymbol{b}_2, \\ & \boldsymbol{x}_1 \geq \boldsymbol{0} \end{array}$$

$$\begin{array}{ll} \text{min.} & \boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{y}_1 + \boldsymbol{b}_2^{\mathrm{T}}\boldsymbol{y}_2 \\ \text{s.t.} & A_{11}^{\mathrm{T}}\boldsymbol{y}_1 + A_{21}^{\mathrm{T}}\boldsymbol{y}_2 \geq \boldsymbol{c}_1, \\ & A_{12}^{\mathrm{T}}\boldsymbol{y}_1 + A_{22}^{\mathrm{T}}\boldsymbol{y}_2 = \boldsymbol{c}_2, \\ & \boldsymbol{y}_1 \geq \boldsymbol{0} \end{array}$$

主問題 → 双対問題

双対問題の導出法から，次の性質の成立が分かる

性質：線形計画法の弱双対定理 (weak duality theorem)

主問題　の任意の許容解 $\boldsymbol{x}_1, \boldsymbol{x}_2$ と
双対問題の任意の許容解 $\boldsymbol{y}_1, \boldsymbol{y}_2$ に対して

$$\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_1 + \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2 \quad \leq \quad \boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{y}_1 + \boldsymbol{b}_2^{\mathrm{T}}\boldsymbol{y}_2$$

主問題と双対問題：一般論 (3)

$$\begin{array}{ll} \text{max.} & \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_1 + \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2 \\ \text{s.t.} & A_{11}\boldsymbol{x}_1 + A_{12}\boldsymbol{x}_2 \leq \boldsymbol{b}_1, \\ & A_{21}\boldsymbol{x}_1 + A_{22}\boldsymbol{x}_2 = \boldsymbol{b}_2, \\ & \boldsymbol{x}_1 \geq \boldsymbol{0} \end{array}$$

$$\begin{array}{ll} \text{min.} & \boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{y}_1 + \boldsymbol{b}_2^{\mathrm{T}}\boldsymbol{y}_2 \\ \text{s.t.} & A_{11}^{\mathrm{T}}\boldsymbol{y}_1 + A_{21}^{\mathrm{T}}\boldsymbol{y}_2 \geq \boldsymbol{c}_1, \\ & A_{12}^{\mathrm{T}}\boldsymbol{y}_1 + A_{22}^{\mathrm{T}}\boldsymbol{y}_2 = \boldsymbol{c}_2, \\ & \boldsymbol{y}_1 \geq \boldsymbol{0} \end{array}$$

主問題 ⟶ 双対問題

双対問題 ⟵ 主問題

右の問題の最適値の

大きな下界を見つける問題

双対定理

性質：線形計画法の弱双対定理 (weak duality theorem)

主問題　の任意の許容解 $\boldsymbol{x}_1, \boldsymbol{x}_2$ と
双対問題の任意の許容解 $\boldsymbol{y}_1, \boldsymbol{y}_2$ に対して

$$\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_1 + \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2 \quad \leq \quad \boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{y}_1 + \boldsymbol{b}_2^{\mathrm{T}}\boldsymbol{y}_2$$

双対定理

性質：線形計画法の弱双対定理 (weak duality theorem)

主問題　の任意の許容解 $\boldsymbol{x}_1, \boldsymbol{x}_2$ と
双対問題の任意の許容解 $\boldsymbol{y}_1, \boldsymbol{y}_2$ に対して

$$\boldsymbol{c}_1^{\mathrm{T}} \boldsymbol{x}_1 + \boldsymbol{c}_2^{\mathrm{T}} \boldsymbol{x}_2 \quad \leq \quad \boldsymbol{b}_1^{\mathrm{T}} \boldsymbol{y}_1 + \boldsymbol{b}_2^{\mathrm{T}} \boldsymbol{y}_2$$

性質：線形計画法の強双対定理 (strong duality theorem)

主問題と双対問題に許容解が存在する $\Rightarrow$
主問題　のある許容解 $\boldsymbol{x}_1^*, \boldsymbol{x}_2^*$ と
双対問題のある許容解 $\boldsymbol{y}_1^*, \boldsymbol{y}_2^*$ が存在して，次が成立

$$\boldsymbol{c}_1^{\mathrm{T}} \boldsymbol{x}_1^* + \boldsymbol{c}_2^{\mathrm{T}} \boldsymbol{x}_2^* \quad = \quad \boldsymbol{b}_1^{\mathrm{T}} \boldsymbol{y}_1^* + \boldsymbol{b}_2^{\mathrm{T}} \boldsymbol{y}_2^*$$

この講義で，強双対定理の証明は行わない

双対定理：帰結

弱双対定理

$\forall\, \boldsymbol{x}_1, \boldsymbol{x}_2$：主問題の許容解，$\forall\, \boldsymbol{y}_1, \boldsymbol{y}_2$：双対問題の許容解

$\boldsymbol{c}_1^{\mathrm{T}} \boldsymbol{x}_1 + \boldsymbol{c}_2^{\mathrm{T}} \boldsymbol{x}_2 \leq \boldsymbol{b}_1^{\mathrm{T}} \boldsymbol{y}_1 + \boldsymbol{b}_2^{\mathrm{T}} \boldsymbol{y}_2$

強双対定理 主問題，双対問題に許容解がある $\Rightarrow$

$\exists\, \boldsymbol{x}_1^*, \boldsymbol{x}_2^*$：主問題の許容解，$\exists\, \boldsymbol{y}_1^*, \boldsymbol{y}_2^*$：双対問題の許容解

$\boldsymbol{c}_1^{\mathrm{T}} \boldsymbol{x}_1^* + \boldsymbol{c}_2^{\mathrm{T}} \boldsymbol{x}_2^* = \boldsymbol{b}_1^{\mathrm{T}} \boldsymbol{y}_1^* + \boldsymbol{b}_2^{\mathrm{T}} \boldsymbol{y}_2^*$

双対定理：帰結

弱双対定理

$\forall\, \boldsymbol{x}_1, \boldsymbol{x}_2$：主問題の許容解，$\forall\, \boldsymbol{y}_1, \boldsymbol{y}_2$：双対問題の許容解

$\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_1 + \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2 \leq \boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{y}_1 + \boldsymbol{b}_2^{\mathrm{T}}\boldsymbol{y}_2$

強双対定理 主問題，双対問題に許容解がある $\Rightarrow$

$\exists\, \boldsymbol{x}_1^*, \boldsymbol{x}_2^*$：主問題の許容解，$\exists\, \boldsymbol{y}_1^*, \boldsymbol{y}_2^*$：双対問題の許容解

$\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_1^* + \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2^* = \boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{y}_1^* + \boldsymbol{b}_2^{\mathrm{T}}\boldsymbol{y}_2^*$

$\boldsymbol{x}_1^*, \boldsymbol{x}_2^*$ は主問題の最適解，$\boldsymbol{y}_1^*, \boldsymbol{y}_2^*$ は双対問題の最適解

$\because$ 主問題の任意の許容解 $\boldsymbol{x}_1, \boldsymbol{x}_2$ に対して

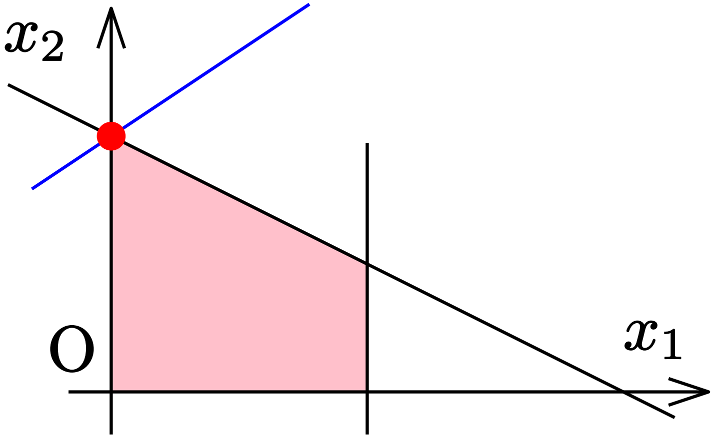
$$\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_1^* + \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2^* = \boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{y}_1^* + \boldsymbol{b}_2^{\mathrm{T}}\boldsymbol{y}_2^* \geq \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_1 + \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2$$

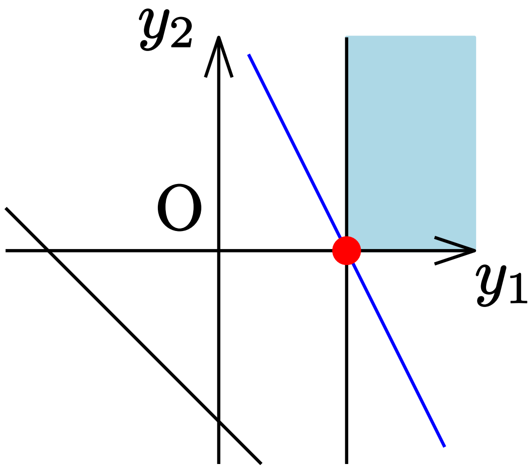
双対問題の任意の許容解 $\boldsymbol{y}_1, \boldsymbol{y}_2$ に対して

$$\boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{y}_1^* + \boldsymbol{b}_2^{\mathrm{T}}\boldsymbol{y}_2^* = \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_1^* + \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2^* \leq \boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{y}_1 + \boldsymbol{b}_2^{\mathrm{T}}\boldsymbol{y}_2$$

本日の内容

1. 線形計画問題と線形計画法
2. 双対定理
3. **相補性定理**

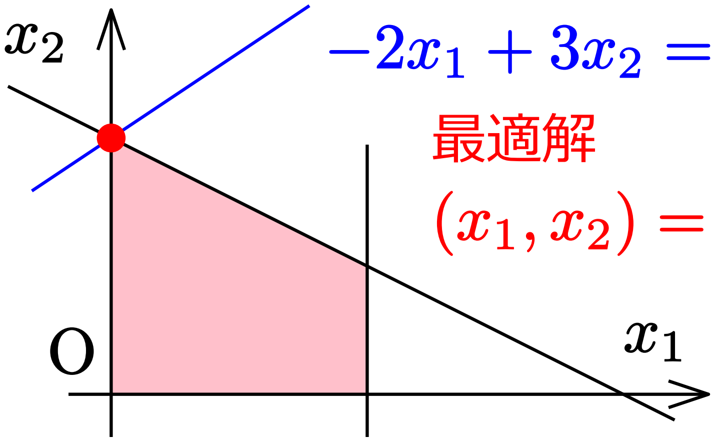




相補性定理：例

$$\begin{array}{ll} \text{maximize} & -2x_1 + 3x_2 \\ \text{subject to} & \begin{aligned} x_1 + 2x_2 &\leq 2, \\ x_1 &\leq 1, \\ x_1 &\geq 0, \\ x_2 &\geq 0 \end{aligned} \end{array}$$

$$\begin{array}{ll} \text{minimize} & 2y_1 + y_2 \\ \text{subject to} & \begin{aligned} y_1 &\geq 0, \\ y_2 &\geq 0, \\ y_1 + y_2 &\geq -2, \\ 2y_1 &\geq 3 \end{aligned} \end{array}$$

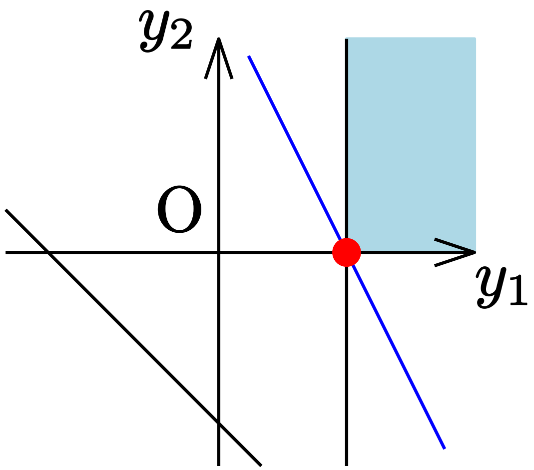


最適値

$-2x_1 + 3x_2 = \boxed{3}$

最適解

$(x_1, x_2) = (0, 1)$



最適値

$2y_1 + y_2 = \boxed{3}$

最適解

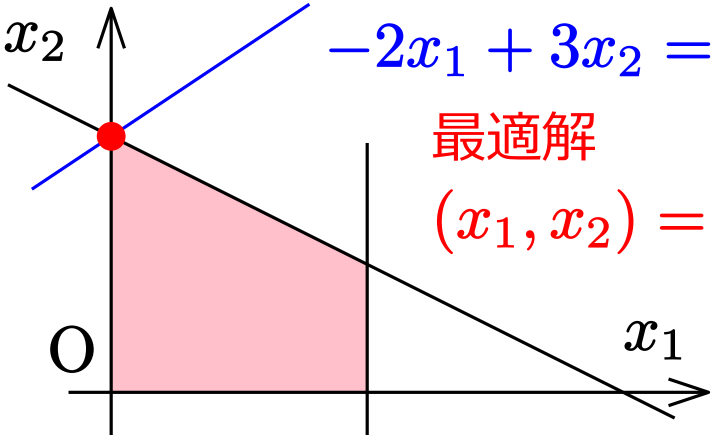
$(y_1, y_2) = (3/2, 0)$

相補性定理：例

$$\begin{aligned} \text{maximize} \quad & -2x_1 + 3x_2 \\ \text{subject to} \quad & \boxed{x_1 + 2x_2 \le 2,} \\ & x_1 \le 1, \\ & \boxed{x_1 \ge 0,} \\ & x_2 \ge 0 \end{aligned}$$

$$\begin{aligned} \text{minimize} \quad & 2y_1 + y_2 \\ \text{subject to} \quad & y_1 \ge 0, \\ & \boxed{y_2 \ge 0,} \\ & y_1 + y_2 \ge -2, \\ & \boxed{2y_1 \ge 3} \end{aligned}$$

この最適解において，等号成立

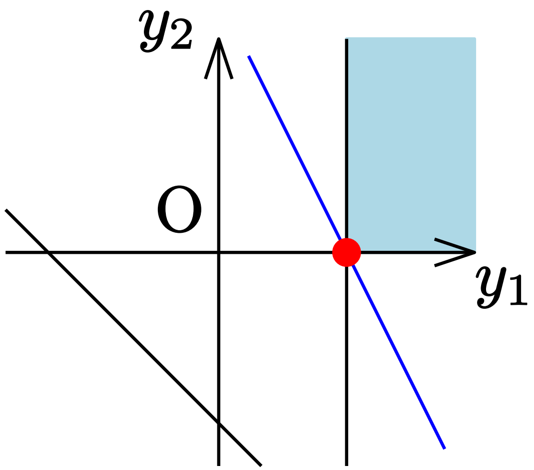


最適値

$-2x_1 + 3x_2 = \boxed{3}$

最適解

$(x_1, x_2) = (0, 1)$



最適値

$2y_1 + y_2 = \boxed{3}$

最適解

$(y_1, y_2) = (3/2, 0)$

相補性定理：一般論

主問題.

$$\begin{array}{ll} \text{max.} & c_1^{\mathrm{T}} x_1 + c_2^{\mathrm{T}} x_2 \\ \text{s.t.} & A_{11} x_1 + A_{12} x_2 \le b_1, \\ & A_{21} x_1 + A_{22} x_2 = b_2, \\ & \hfill x_1 \ge 0 \end{array}$$

双対問題

$$\begin{array}{ll} \text{min.} & b_1^{\mathrm{T}} y_1 + b_2^{\mathrm{T}} y_2 \\ \text{s.t.} & A_{11}^{\mathrm{T}} y_1 + A_{21}^{\mathrm{T}} y_2 \ge c_1, \\ & A_{12}^{\mathrm{T}} y_1 + A_{22}^{\mathrm{T}} y_2 = c_2, \\ & \hfill y_1 \ge 0 \end{array}$$

性質：相補性定理 (complementary slackness theorem)

x_1, x_2 が主問題の最適解，y_1, y_2 が双対問題の最適解 $\Leftrightarrow$

- x_1, x_2 は主問題の許容解 (主許容性)
- y_1, y_2 は双対問題の許容解 (双対許容性)
- $x_1^{\mathrm{T}}(A_{11}^{\mathrm{T}} y_1 + A_{21}^{\mathrm{T}} y_2 - c_1) = 0$ (相補性)
- $y_1^{\mathrm{T}}(b_1 - A_{11} x_1 - A_{12} x_2) = 0$ (相補性)

相補性定理：一般論 − 補足

主問題.

$$\begin{array}{ll} \text{max.} & c_1^{\mathrm{T}} x_1 + c_2^{\mathrm{T}} x_2 \\ \text{s.t.} & A_{11} x_1 + A_{12} x_2 \le b_1, \\ & A_{21} x_1 + A_{22} x_2 = b_2, \\ & x_1 \ge 0 \end{array}$$

双対問題

$$\begin{array}{ll} \text{min.} & b_1^{\mathrm{T}} y_1 + b_2^{\mathrm{T}} y_2 \\ \text{s.t.} & A_{11}^{\mathrm{T}} y_1 + A_{21}^{\mathrm{T}} y_2 \ge c_1, \\ & A_{12}^{\mathrm{T}} y_1 + A_{22}^{\mathrm{T}} y_2 = c_2, \\ & y_1 \ge 0 \end{array}$$

$$x_1^{\mathrm{T}}(A_{11}^{\mathrm{T}} y_1 + A_{21}^{\mathrm{T}} y_2 - c_1) = \sum_{i=1}^{n_1} x_{1i}(A_{11}^{\mathrm{T}} y_1 + A_{21}^{\mathrm{T}} y_2 - c_1)_i$$

ゆえに,

$$x_1^{\mathrm{T}}(A_{11}^{\mathrm{T}} y_1 + A_{21}^{\mathrm{T}} y_2 - c_1) = 0 \quad \Leftrightarrow$$

任意の $i \in \{1, 2, \ldots, n_1\}$ に対して,

$x_{1i} = 0$ または $(A_{11}^{\mathrm{T}} y_1 + A_{21}^{\mathrm{T}} y_2 - c_1)_i = 0$

相補性定理：一般論 ― 証明 (1)

主問題.

$$\begin{aligned} \text{max.} \quad & c_1^{\mathrm{T}} x_1 + c_2^{\mathrm{T}} x_2 \\ \text{s.t.} \quad & A_{11} x_1 + A_{12} x_2 \leq b_1, \\ & A_{21} x_1 + A_{22} x_2 = b_2, \\ & x_1 \geq 0 \end{aligned}$$

双対問題

$$\begin{aligned} \text{min.} \quad & b_1^{\mathrm{T}} y_1 + b_2^{\mathrm{T}} y_2 \\ \text{s.t.} \quad & A_{11}^{\mathrm{T}} y_1 + A_{21}^{\mathrm{T}} y_2 \geq c_1, \\ & A_{12}^{\mathrm{T}} y_1 + A_{22}^{\mathrm{T}} y_2 = c_2, \\ & y_1 \geq 0 \end{aligned}$$

x_1, x_2：主最適解
y_1, y_2：双対最適解

$\Leftrightarrow$

x_1, x_2：主許容解，y_1, y_2：双対許容解
$x_1^{\mathrm{T}}(A_{11}^{\mathrm{T}} y_1 + A_{21}^{\mathrm{T}} y_2 - c_1) = 0$
$y_1^{\mathrm{T}}(b_1 - A_{11} x_1 - A_{12} x_2) = 0$

相補性定理：一般論 — 証明 (1)

主問題.

$$\begin{aligned} \text{max.} \quad & \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_1 + \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2 \\ \text{s.t.} \quad & A_{11}\boldsymbol{x}_1 + A_{12}\boldsymbol{x}_2 \leq \boldsymbol{b}_1, \\ & A_{21}\boldsymbol{x}_1 + A_{22}\boldsymbol{x}_2 = \boldsymbol{b}_2, \\ & \boldsymbol{x}_1 \geq \boldsymbol{0} \end{aligned}$$

双対問題

$$\begin{aligned} \text{min.} \quad & \boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{y}_1 + \boldsymbol{b}_2^{\mathrm{T}}\boldsymbol{y}_2 \\ \text{s.t.} \quad & A_{11}^{\mathrm{T}}\boldsymbol{y}_1 + A_{21}^{\mathrm{T}}\boldsymbol{y}_2 \geq \boldsymbol{c}_1, \\ & A_{12}^{\mathrm{T}}\boldsymbol{y}_1 + A_{22}^{\mathrm{T}}\boldsymbol{y}_2 = \boldsymbol{c}_2, \\ & \boldsymbol{y}_1 \geq \boldsymbol{0} \end{aligned}$$

$\boldsymbol{x}_1, \boldsymbol{x}_2$：主最適解
$\boldsymbol{y}_1, \boldsymbol{y}_2$：双対最適解

$\Leftrightarrow$

$\boldsymbol{x}_1, \boldsymbol{x}_2$：主許容解，$\boldsymbol{y}_1, \boldsymbol{y}_2$：双対許容解
$\boldsymbol{x}_1^{\mathrm{T}}(A_{11}^{\mathrm{T}}\boldsymbol{y}_1 + A_{21}^{\mathrm{T}}\boldsymbol{y}_2 - \boldsymbol{c}_1) = 0$
$\boldsymbol{y}_1^{\mathrm{T}}(\boldsymbol{b}_1 - A_{11}\boldsymbol{x}_1 - A_{12}\boldsymbol{x}_2) = 0$

$\Rightarrow$ の証明　最適解は許容解なので，最初の 2 つの条件は満たされる

$$0 \leq \boldsymbol{x}_1^{\mathrm{T}}(A_{11}^{\mathrm{T}}\boldsymbol{y}_1 + A_{21}^{\mathrm{T}}\boldsymbol{y}_2 - \boldsymbol{c}_1)$$

相補性定理：一般論 — 証明 (1)

主問題.

$$\begin{aligned} \text{max.} \quad & \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_1 + \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2 \\ \text{s.t.} \quad & A_{11}\boldsymbol{x}_1 + A_{12}\boldsymbol{x}_2 \leq \boldsymbol{b}_1, \\ & A_{21}\boldsymbol{x}_1 + A_{22}\boldsymbol{x}_2 = \boldsymbol{b}_2, \\ & \boldsymbol{x}_1 \geq \boldsymbol{0} \end{aligned}$$

双対問題

$$\begin{aligned} \text{min.} \quad & \boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{y}_1 + \boldsymbol{b}_2^{\mathrm{T}}\boldsymbol{y}_2 \\ \text{s.t.} \quad & A_{11}^{\mathrm{T}}\boldsymbol{y}_1 + A_{21}^{\mathrm{T}}\boldsymbol{y}_2 \geq \boldsymbol{c}_1, \\ & A_{12}^{\mathrm{T}}\boldsymbol{y}_1 + A_{22}^{\mathrm{T}}\boldsymbol{y}_2 = \boldsymbol{c}_2, \\ & \boldsymbol{y}_1 \geq \boldsymbol{0} \end{aligned}$$

$\boldsymbol{x}_1, \boldsymbol{x}_2$：主最適解
$\boldsymbol{y}_1, \boldsymbol{y}_2$：双対最適解

$\Leftrightarrow$

$\boldsymbol{x}_1, \boldsymbol{x}_2$：主許容解，$\boldsymbol{y}_1, \boldsymbol{y}_2$：双対許容解
$\boldsymbol{x}_1^{\mathrm{T}}(A_{11}^{\mathrm{T}}\boldsymbol{y}_1 + A_{21}^{\mathrm{T}}\boldsymbol{y}_2 - \boldsymbol{c}_1) = 0$
$\boldsymbol{y}_1^{\mathrm{T}}(\boldsymbol{b}_1 - A_{11}\boldsymbol{x}_1 - A_{12}\boldsymbol{x}_2) = 0$

⇒ の証明　最適解は許容解なので，最初の 2 つの条件は満たされる

$$\begin{aligned} 0 &\leq \boldsymbol{x}_1^{\mathrm{T}}(A_{11}^{\mathrm{T}}\boldsymbol{y}_1 + A_{21}^{\mathrm{T}}\boldsymbol{y}_2 - \boldsymbol{c}_1) \\ &= \boldsymbol{x}_1^{\mathrm{T}}(A_{11}^{\mathrm{T}}\boldsymbol{y}_1 + A_{21}^{\mathrm{T}}\boldsymbol{y}_2 - \boldsymbol{c}_1) + \boldsymbol{x}_2^{\mathrm{T}}(A_{12}^{\mathrm{T}}\boldsymbol{y}_1 + A_{22}^{\mathrm{T}}\boldsymbol{y}_2 - \boldsymbol{c}_2) \end{aligned}$$

相補性定理：一般論 ― 証明 (1)

主問題.

$$\begin{aligned} \text{max.}\quad & \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_1 + \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2 \\ \text{s.t.}\quad & A_{11}\boldsymbol{x}_1 + A_{12}\boldsymbol{x}_2 \le \boldsymbol{b}_1, \\ & A_{21}\boldsymbol{x}_1 + A_{22}\boldsymbol{x}_2 = \boldsymbol{b}_2, \\ & \boldsymbol{x}_1 \ge \boldsymbol{0} \end{aligned}$$

双対問題

$$\begin{aligned} \text{min.}\quad & \boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{y}_1 + \boldsymbol{b}_2^{\mathrm{T}}\boldsymbol{y}_2 \\ \text{s.t.}\quad & A_{11}^{\mathrm{T}}\boldsymbol{y}_1 + A_{21}^{\mathrm{T}}\boldsymbol{y}_2 \ge \boldsymbol{c}_1, \\ & A_{12}^{\mathrm{T}}\boldsymbol{y}_1 + A_{22}^{\mathrm{T}}\boldsymbol{y}_2 = \boldsymbol{c}_2, \\ & \boldsymbol{y}_1 \ge \boldsymbol{0} \end{aligned}$$

$$\begin{array}{l} \boldsymbol{x}_1, \boldsymbol{x}_2：\text{主最適解} \\ \boldsymbol{y}_1, \boldsymbol{y}_2：\text{双対最適解} \end{array} \quad \Leftrightarrow \quad \begin{array}{l} \boldsymbol{x}_1, \boldsymbol{x}_2：\text{主許容解，}\ \boldsymbol{y}_1, \boldsymbol{y}_2：\text{双対許容解} \\ \boldsymbol{x}_1^{\mathrm{T}}(A_{11}^{\mathrm{T}}\boldsymbol{y}_1 + A_{21}^{\mathrm{T}}\boldsymbol{y}_2 - \boldsymbol{c}_1) = 0 \\ \boldsymbol{y}_1^{\mathrm{T}}(\boldsymbol{b}_1 - A_{11}\boldsymbol{x}_1 - A_{12}\boldsymbol{x}_2) = 0 \end{array}$$

⇒ の証明　最適解は許容解なので，最初の 2 つの条件は満たされる

$$\begin{aligned} 0 &\le \boldsymbol{x}_1^{\mathrm{T}}(A_{11}^{\mathrm{T}}\boldsymbol{y}_1 + A_{21}^{\mathrm{T}}\boldsymbol{y}_2 - \boldsymbol{c}_1) \\ &= \boldsymbol{x}_1^{\mathrm{T}}(A_{11}^{\mathrm{T}}\boldsymbol{y}_1 + A_{21}^{\mathrm{T}}\boldsymbol{y}_2 - \boldsymbol{c}_1) + \boldsymbol{x}_2^{\mathrm{T}}(A_{12}^{\mathrm{T}}\boldsymbol{y}_1 + A_{22}^{\mathrm{T}}\boldsymbol{y}_2 - \boldsymbol{c}_2) \\ &= (\boldsymbol{x}_1^{\mathrm{T}}A_{11}^{\mathrm{T}} + \boldsymbol{x}_2^{\mathrm{T}}A_{12}^{\mathrm{T}})\boldsymbol{y}_1 + (\boldsymbol{x}_1^{\mathrm{T}}A_{21}^{\mathrm{T}} + \boldsymbol{x}_2^{\mathrm{T}}A_{22}^{\mathrm{T}})\boldsymbol{y}_2 - \boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{c}_1 - \boldsymbol{x}_2^{\mathrm{T}}\boldsymbol{c}_2 \end{aligned}$$

相補性定理：一般論 ― 証明 (1)

主問題.

$$\begin{array}{ll} \text{max.} & \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_1 + \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2 \\ \text{s.t.} & A_{11}\boldsymbol{x}_1 + A_{12}\boldsymbol{x}_2 \leq \boldsymbol{b}_1, \\ & A_{21}\boldsymbol{x}_1 + A_{22}\boldsymbol{x}_2 = \boldsymbol{b}_2, \\ & \boldsymbol{x}_1 \geq \boldsymbol{0} \end{array}$$

双対問題

$$\begin{array}{ll} \text{min.} & \boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{y}_1 + \boldsymbol{b}_2^{\mathrm{T}}\boldsymbol{y}_2 \\ \text{s.t.} & A_{11}^{\mathrm{T}}\boldsymbol{y}_1 + A_{21}^{\mathrm{T}}\boldsymbol{y}_2 \geq \boldsymbol{c}_1, \\ & A_{12}^{\mathrm{T}}\boldsymbol{y}_1 + A_{22}^{\mathrm{T}}\boldsymbol{y}_2 = \boldsymbol{c}_2, \\ & \boldsymbol{y}_1 \geq \boldsymbol{0} \end{array}$$

$\boldsymbol{x}_1, \boldsymbol{x}_2$：主最適解
$\boldsymbol{y}_1, \boldsymbol{y}_2$：双対最適解

$\Leftrightarrow$

$\boldsymbol{x}_1, \boldsymbol{x}_2$：主許容解，$\boldsymbol{y}_1, \boldsymbol{y}_2$：双対許容解
$\boldsymbol{x}_1^{\mathrm{T}}(A_{11}^{\mathrm{T}}\boldsymbol{y}_1 + A_{21}^{\mathrm{T}}\boldsymbol{y}_2 - \boldsymbol{c}_1) = 0$
$\boldsymbol{y}_1^{\mathrm{T}}(\boldsymbol{b}_1 - A_{11}\boldsymbol{x}_1 - A_{12}\boldsymbol{x}_2) = 0$

⇒ の証明　最適解は許容解なので，最初の 2 つの条件は満たされる

$$\begin{aligned} 0 &\leq \boldsymbol{x}_1^{\mathrm{T}}(A_{11}^{\mathrm{T}}\boldsymbol{y}_1 + A_{21}^{\mathrm{T}}\boldsymbol{y}_2 - \boldsymbol{c}_1) \\ &= \boldsymbol{x}_1^{\mathrm{T}}(A_{11}^{\mathrm{T}}\boldsymbol{y}_1 + A_{21}^{\mathrm{T}}\boldsymbol{y}_2 - \boldsymbol{c}_1) + \boldsymbol{x}_2^{\mathrm{T}}(A_{12}^{\mathrm{T}}\boldsymbol{y}_1 + A_{22}^{\mathrm{T}}\boldsymbol{y}_2 - \boldsymbol{c}_2) \\ &= (\boldsymbol{x}_1^{\mathrm{T}}A_{11}^{\mathrm{T}} + \boldsymbol{x}_2^{\mathrm{T}}A_{12}^{\mathrm{T}})\boldsymbol{y}_1 + (\boldsymbol{x}_1^{\mathrm{T}}A_{21}^{\mathrm{T}} + \boldsymbol{x}_2^{\mathrm{T}}A_{22}^{\mathrm{T}})\boldsymbol{y}_2 - \boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{c}_1 - \boldsymbol{x}_2^{\mathrm{T}}\boldsymbol{c}_2 \\ &\leq \boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{y}_1 + \boldsymbol{b}_2^{\mathrm{T}}\boldsymbol{y}_2 - \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_1 - \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2 \end{aligned}$$

相補性定理：一般論 — 証明 (1)

主問題.

$$\begin{array}{ll} \text{max.} & c_1^{\mathrm{T}} x_1 + c_2^{\mathrm{T}} x_2 \\ \text{s.t.} & A_{11} x_1 + A_{12} x_2 \le b_1, \\ & A_{21} x_1 + A_{22} x_2 = b_2, \\ & x_1 \ge 0 \end{array}$$

双対問題

$$\begin{array}{ll} \text{min.} & b_1^{\mathrm{T}} y_1 + b_2^{\mathrm{T}} y_2 \\ \text{s.t.} & A_{11}^{\mathrm{T}} y_1 + A_{21}^{\mathrm{T}} y_2 \ge c_1, \\ & A_{12}^{\mathrm{T}} y_1 + A_{22}^{\mathrm{T}} y_2 = c_2, \\ & y_1 \ge 0 \end{array}$$

x_1, x_2：主最適解
y_1, y_2：双対最適解

⇔

x_1, x_2：主許容解，y_1, y_2：双対許容解
$x_1^{\mathrm{T}}(A_{11}^{\mathrm{T}} y_1 + A_{21}^{\mathrm{T}} y_2 - c_1) = 0$
$y_1^{\mathrm{T}}(b_1 - A_{11} x_1 - A_{12} x_2) = 0$

⇒ の証明　最適解は許容解なので，最初の 2 つの条件は満たされる

$$\begin{aligned} 0 &\le x_1^{\mathrm{T}}(A_{11}^{\mathrm{T}} y_1 + A_{21}^{\mathrm{T}} y_2 - c_1) \\ &= x_1^{\mathrm{T}}(A_{11}^{\mathrm{T}} y_1 + A_{21}^{\mathrm{T}} y_2 - c_1) + x_2^{\mathrm{T}}(A_{12}^{\mathrm{T}} y_1 + A_{22}^{\mathrm{T}} y_2 - c_2) \\ &= (x_1^{\mathrm{T}} A_{11}^{\mathrm{T}} + x_2^{\mathrm{T}} A_{12}^{\mathrm{T}}) y_1 + (x_1^{\mathrm{T}} A_{21}^{\mathrm{T}} + x_2^{\mathrm{T}} A_{22}^{\mathrm{T}}) y_2 - x_1^{\mathrm{T}} c_1 - x_2^{\mathrm{T}} c_2 \\ &\le b_1^{\mathrm{T}} y_1 + b_2^{\mathrm{T}} y_2 - c_1^{\mathrm{T}} x_1 - c_2^{\mathrm{T}} x_2 \underset{\substack{\uparrow \\ \text{強双対定理}}}{=} 0 \end{aligned}$$

相補性定理：一般論 — 証明 (2)

主問題.

$$\begin{array}{ll} \text{max.} & \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_1 + \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2 \\ \text{s.t.} & A_{11}\boldsymbol{x}_1 + A_{12}\boldsymbol{x}_2 \leq \boldsymbol{b}_1, \\ & A_{21}\boldsymbol{x}_1 + A_{22}\boldsymbol{x}_2 = \boldsymbol{b}_2, \\ & \boldsymbol{x}_1 \geq \boldsymbol{0} \end{array}$$

双対問題

$$\begin{array}{ll} \text{min.} & \boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{y}_1 + \boldsymbol{b}_2^{\mathrm{T}}\boldsymbol{y}_2 \\ \text{s.t.} & A_{11}^{\mathrm{T}}\boldsymbol{y}_1 + A_{21}^{\mathrm{T}}\boldsymbol{y}_2 \geq \boldsymbol{c}_1, \\ & A_{12}^{\mathrm{T}}\boldsymbol{y}_1 + A_{22}^{\mathrm{T}}\boldsymbol{y}_2 = \boldsymbol{c}_2, \\ & \boldsymbol{y}_1 \geq \boldsymbol{0} \end{array}$$

$\boldsymbol{x}_1, \boldsymbol{x}_2$：主最適解
$\boldsymbol{y}_1, \boldsymbol{y}_2$：双対最適解

$\Leftrightarrow$

$\boldsymbol{x}_1, \boldsymbol{x}_2$：主許容解，$\boldsymbol{y}_1, \boldsymbol{y}_2$：双対許容解
$\boldsymbol{x}_1^{\mathrm{T}}(A_{11}^{\mathrm{T}}\boldsymbol{y}_1 + A_{21}^{\mathrm{T}}\boldsymbol{y}_2 - \boldsymbol{c}_1) = 0$
$\boldsymbol{y}_1^{\mathrm{T}}(\boldsymbol{b}_1 - A_{11}\boldsymbol{x}_1 - A_{12}\boldsymbol{x}_2) = 0$

$\Rightarrow$ の証明 (続き)

$$0 \leq \boldsymbol{y}_1^{\mathrm{T}}(\boldsymbol{b}_1 - A_{11}\boldsymbol{x}_1 - A_{12}\boldsymbol{x}_2)$$

相補性定理：一般論 — 証明 (2)

主問題.

$$\begin{array}{ll} \text{max.} & \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_1 + \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2 \\ \text{s.t.} & A_{11}\boldsymbol{x}_1 + A_{12}\boldsymbol{x}_2 \leq \boldsymbol{b}_1, \\ & A_{21}\boldsymbol{x}_1 + A_{22}\boldsymbol{x}_2 = \boldsymbol{b}_2, \\ & \boldsymbol{x}_1 \geq \boldsymbol{0} \end{array}$$

双対問題

$$\begin{array}{ll} \text{min.} & \boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{y}_1 + \boldsymbol{b}_2^{\mathrm{T}}\boldsymbol{y}_2 \\ \text{s.t.} & A_{11}^{\mathrm{T}}\boldsymbol{y}_1 + A_{21}^{\mathrm{T}}\boldsymbol{y}_2 \geq \boldsymbol{c}_1, \\ & A_{12}^{\mathrm{T}}\boldsymbol{y}_1 + A_{22}^{\mathrm{T}}\boldsymbol{y}_2 = \boldsymbol{c}_2, \\ & \boldsymbol{y}_1 \geq \boldsymbol{0} \end{array}$$

$\boldsymbol{x}_1, \boldsymbol{x}_2$：主最適解
$\boldsymbol{y}_1, \boldsymbol{y}_2$：双対最適解
$\Leftrightarrow$
$\boldsymbol{x}_1, \boldsymbol{x}_2$：主許容解，$\boldsymbol{y}_1, \boldsymbol{y}_2$：双対許容解
$\boldsymbol{x}_1^{\mathrm{T}}(A_{11}^{\mathrm{T}}\boldsymbol{y}_1 + A_{21}^{\mathrm{T}}\boldsymbol{y}_2 - \boldsymbol{c}_1) = 0$
$\boldsymbol{y}_1^{\mathrm{T}}(\boldsymbol{b}_1 - A_{11}\boldsymbol{x}_1 - A_{12}\boldsymbol{x}_2) = 0$

$\Rightarrow$ の証明 (続き)

$$\begin{aligned} 0 &\leq \boldsymbol{y}_1^{\mathrm{T}}(\boldsymbol{b}_1 - A_{11}\boldsymbol{x}_1 - A_{12}\boldsymbol{x}_2) \\ &= \boldsymbol{y}_1^{\mathrm{T}}(\boldsymbol{b}_1 - A_{11}\boldsymbol{x}_1 - A_{12}\boldsymbol{x}_2) + \boldsymbol{y}_2^{\mathrm{T}}(\boldsymbol{b}_2 - A_{21}\boldsymbol{x}_1 - A_{22}\boldsymbol{x}_2) \end{aligned}$$

相補性定理：一般論 — 証明 (2)

主問題.

$$\begin{array}{ll} \text{max.} & \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_1 + \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2 \\ \text{s.t.} & A_{11}\boldsymbol{x}_1 + A_{12}\boldsymbol{x}_2 \le \boldsymbol{b}_1, \\ & A_{21}\boldsymbol{x}_1 + A_{22}\boldsymbol{x}_2 = \boldsymbol{b}_2, \\ & \boldsymbol{x}_1 \ge \boldsymbol{0} \end{array}$$

双対問題

$$\begin{array}{ll} \text{min.} & \boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{y}_1 + \boldsymbol{b}_2^{\mathrm{T}}\boldsymbol{y}_2 \\ \text{s.t.} & A_{11}^{\mathrm{T}}\boldsymbol{y}_1 + A_{21}^{\mathrm{T}}\boldsymbol{y}_2 \ge \boldsymbol{c}_1, \\ & A_{12}^{\mathrm{T}}\boldsymbol{y}_1 + A_{22}^{\mathrm{T}}\boldsymbol{y}_2 = \boldsymbol{c}_2, \\ & \boldsymbol{y}_1 \ge \boldsymbol{0} \end{array}$$

$\boldsymbol{x}_1, \boldsymbol{x}_2$：主最適解
$\boldsymbol{y}_1, \boldsymbol{y}_2$：双対最適解

$\Leftrightarrow$

$\boldsymbol{x}_1, \boldsymbol{x}_2$：主許容解，$\boldsymbol{y}_1, \boldsymbol{y}_2$：双対許容解
$\boldsymbol{x}_1^{\mathrm{T}}(A_{11}^{\mathrm{T}}\boldsymbol{y}_1 + A_{21}^{\mathrm{T}}\boldsymbol{y}_2 - \boldsymbol{c}_1) = 0$
$\boldsymbol{y}_1^{\mathrm{T}}(\boldsymbol{b}_1 - A_{11}\boldsymbol{x}_1 - A_{12}\boldsymbol{x}_2) = 0$

$\Rightarrow$ の証明 (続き)

$$\begin{aligned} 0 &\le \boldsymbol{y}_1^{\mathrm{T}}(\boldsymbol{b}_1 - A_{11}\boldsymbol{x}_1 - A_{12}\boldsymbol{x}_2) \\ &= \boldsymbol{y}_1^{\mathrm{T}}(\boldsymbol{b}_1 - A_{11}\boldsymbol{x}_1 - A_{12}\boldsymbol{x}_2) + \boldsymbol{y}_2^{\mathrm{T}}(\boldsymbol{b}_2 - A_{21}\boldsymbol{x}_1 - A_{22}\boldsymbol{x}_2) \\ &= \boldsymbol{y}_1^{\mathrm{T}}\boldsymbol{b}_1 + \boldsymbol{y}_2^{\mathrm{T}}\boldsymbol{b}_2 - (\boldsymbol{y}_1^{\mathrm{T}}A_{11} + \boldsymbol{y}_2^{\mathrm{T}}A_{21})\boldsymbol{x}_1 - (\boldsymbol{y}_1^{\mathrm{T}}A_{21} + \boldsymbol{y}_2^{\mathrm{T}}A_{22})\boldsymbol{x}_2 \end{aligned}$$

相補性定理：一般論 ― 証明 (2)

主問題.

$$\begin{aligned} \text{max.} \quad & \boldsymbol{c}_1^{\mathrm{T}} \boldsymbol{x}_1 + \boldsymbol{c}_2^{\mathrm{T}} \boldsymbol{x}_2 \\ \text{s.t.} \quad & A_{11}\boldsymbol{x}_1 + A_{12}\boldsymbol{x}_2 \leq \boldsymbol{b}_1, \\ & A_{21}\boldsymbol{x}_1 + A_{22}\boldsymbol{x}_2 = \boldsymbol{b}_2, \\ & \boldsymbol{x}_1 \geq \boldsymbol{0} \end{aligned}$$

双対問題

$$\begin{aligned} \text{min.} \quad & \boldsymbol{b}_1^{\mathrm{T}} \boldsymbol{y}_1 + \boldsymbol{b}_2^{\mathrm{T}} \boldsymbol{y}_2 \\ \text{s.t.} \quad & A_{11}^{\mathrm{T}}\boldsymbol{y}_1 + A_{21}^{\mathrm{T}}\boldsymbol{y}_2 \geq \boldsymbol{c}_1, \\ & A_{12}^{\mathrm{T}}\boldsymbol{y}_1 + A_{22}^{\mathrm{T}}\boldsymbol{y}_2 = \boldsymbol{c}_2, \\ & \boldsymbol{y}_1 \geq \boldsymbol{0} \end{aligned}$$

$\boldsymbol{x}_1, \boldsymbol{x}_2$：主最適解
$\boldsymbol{y}_1, \boldsymbol{y}_2$：双対最適解

$\Leftrightarrow$

$\boldsymbol{x}_1, \boldsymbol{x}_2$：主許容解，$\boldsymbol{y}_1, \boldsymbol{y}_2$：双対許容解
$\boldsymbol{x}_1^{\mathrm{T}}(A_{11}^{\mathrm{T}}\boldsymbol{y}_1 + A_{21}^{\mathrm{T}}\boldsymbol{y}_2 - \boldsymbol{c}_1) = 0$
$\boldsymbol{y}_1^{\mathrm{T}}(\boldsymbol{b}_1 - A_{11}\boldsymbol{x}_1 - A_{12}\boldsymbol{x}_2) = 0$

⇒ の証明 (続き)

$$\begin{aligned} 0 &\leq \boldsymbol{y}_1^{\mathrm{T}}(\boldsymbol{b}_1 - A_{11}\boldsymbol{x}_1 - A_{12}\boldsymbol{x}_2) \\ &= \boldsymbol{y}_1^{\mathrm{T}}(\boldsymbol{b}_1 - A_{11}\boldsymbol{x}_1 - A_{12}\boldsymbol{x}_2) + \boldsymbol{y}_2^{\mathrm{T}}(\boldsymbol{b}_2 - A_{21}\boldsymbol{x}_1 - A_{22}\boldsymbol{x}_2) \\ &= \boldsymbol{y}_1^{\mathrm{T}}\boldsymbol{b}_1 + \boldsymbol{y}_2^{\mathrm{T}}\boldsymbol{b}_2 - (\boldsymbol{y}_1^{\mathrm{T}}A_{11} + \boldsymbol{y}_2^{\mathrm{T}}A_{21})\boldsymbol{x}_1 - (\boldsymbol{y}_1^{\mathrm{T}}A_{21} + \boldsymbol{y}_2^{\mathrm{T}}A_{22})\boldsymbol{x}_2 \\ &\leq \boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{y}_1 + \boldsymbol{b}_2^{\mathrm{T}}\boldsymbol{y}_2 - \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_1 - \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2 \end{aligned}$$

相補性定理：一般論 — 証明 (2)

主問題.

$$\begin{aligned} \text{max.}\quad & \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_1 + \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2 \\ \text{s.t.}\quad & A_{11}\boldsymbol{x}_1 + A_{12}\boldsymbol{x}_2 \le \boldsymbol{b}_1, \\ & A_{21}\boldsymbol{x}_1 + A_{22}\boldsymbol{x}_2 = \boldsymbol{b}_2, \\ & \boldsymbol{x}_1 \ge \boldsymbol{0} \end{aligned}$$

双対問題

$$\begin{aligned} \text{min.}\quad & \boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{y}_1 + \boldsymbol{b}_2^{\mathrm{T}}\boldsymbol{y}_2 \\ \text{s.t.}\quad & A_{11}^{\mathrm{T}}\boldsymbol{y}_1 + A_{21}^{\mathrm{T}}\boldsymbol{y}_2 \ge \boldsymbol{c}_1, \\ & A_{12}^{\mathrm{T}}\boldsymbol{y}_1 + A_{22}^{\mathrm{T}}\boldsymbol{y}_2 = \boldsymbol{c}_2, \\ & \boldsymbol{y}_1 \ge \boldsymbol{0} \end{aligned}$$

$\boldsymbol{x}_1, \boldsymbol{x}_2$ ：主最適解
$\boldsymbol{y}_1, \boldsymbol{y}_2$ ：双対最適解

$\Leftrightarrow$

$\boldsymbol{x}_1, \boldsymbol{x}_2$ ：主許容解，$\boldsymbol{y}_1, \boldsymbol{y}_2$ ：双対許容解
$\boldsymbol{x}_1^{\mathrm{T}}(A_{11}^{\mathrm{T}}\boldsymbol{y}_1 + A_{21}^{\mathrm{T}}\boldsymbol{y}_2 - \boldsymbol{c}_1) = 0$
$\boldsymbol{y}_1^{\mathrm{T}}(\boldsymbol{b}_1 - A_{11}\boldsymbol{x}_1 - A_{12}\boldsymbol{x}_2) = 0$

⇒ の証明 (続き)

$$\begin{aligned} 0 &\le \boldsymbol{y}_1^{\mathrm{T}}(\boldsymbol{b}_1 - A_{11}\boldsymbol{x}_1 - A_{12}\boldsymbol{x}_2) \\ &= \boldsymbol{y}_1^{\mathrm{T}}(\boldsymbol{b}_1 - A_{11}\boldsymbol{x}_1 - A_{12}\boldsymbol{x}_2) + \boldsymbol{y}_2^{\mathrm{T}}(\boldsymbol{b}_2 - A_{21}\boldsymbol{x}_1 - A_{22}\boldsymbol{x}_2) \\ &= \boldsymbol{y}_1^{\mathrm{T}}\boldsymbol{b}_1 + \boldsymbol{y}_2^{\mathrm{T}}\boldsymbol{b}_2 - (\boldsymbol{y}_1^{\mathrm{T}}A_{11} + \boldsymbol{y}_2^{\mathrm{T}}A_{21})\boldsymbol{x}_1 - (\boldsymbol{y}_1^{\mathrm{T}}A_{21} + \boldsymbol{y}_2^{\mathrm{T}}A_{22})\boldsymbol{x}_2 \\ &\le \boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{y}_1 + \boldsymbol{b}_2^{\mathrm{T}}\boldsymbol{y}_2 - \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_1 - \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2 \underset{\uparrow}{=} 0 \end{aligned}$$

強双対定理

相補性定理：一般論 — 証明 (3)

主問題.

$$\begin{array}{ll} \text{max.} & c_1^{\mathrm{T}} x_1 + c_2^{\mathrm{T}} x_2 \\ \text{s.t.} & A_{11} x_1 + A_{12} x_2 \leq b_1, \\ & A_{21} x_1 + A_{22} x_2 = b_2, \\ & x_1 \geq 0 \end{array}$$

双対問題

$$\begin{array}{ll} \text{min.} & b_1^{\mathrm{T}} y_1 + b_2^{\mathrm{T}} y_2 \\ \text{s.t.} & A_{11}^{\mathrm{T}} y_1 + A_{21}^{\mathrm{T}} y_2 \geq c_1, \\ & A_{12}^{\mathrm{T}} y_1 + A_{22}^{\mathrm{T}} y_2 = c_2, \\ & y_1 \geq 0 \end{array}$$

x_1, x_2：主最適解
y_1, y_2：双対最適解

$\Leftrightarrow$

x_1, x_2：主許容解，y_1, y_2：双対許容解
$x_1^{\mathrm{T}}(A_{11}^{\mathrm{T}} y_1 + A_{21}^{\mathrm{T}} y_2 - c_1) = 0$
$y_1^{\mathrm{T}}(b_1 - A_{11} x_1 - A_{12} x_2) = 0$

$\Leftarrow$ の証明 $c_1^{\mathrm{T}} x_1 + c_2^{\mathrm{T}} x_2 = b_1^{\mathrm{T}} y_1 + b_2^{\mathrm{T}} y_2$ を示せばよい ($\because$ 双対定理)

相補性定理：一般論 — 証明 (3)

主問題.

$$\begin{array}{ll} \text{max.} & c_1^{\mathrm{T}} x_1 + c_2^{\mathrm{T}} x_2 \\ \text{s.t.} & A_{11} x_1 + A_{12} x_2 \leq b_1, \\ & A_{21} x_1 + A_{22} x_2 = b_2, \\ & \qquad\qquad x_1 \geq 0 \end{array}$$

双対問題

$$\begin{array}{ll} \text{min.} & b_1^{\mathrm{T}} y_1 + b_2^{\mathrm{T}} y_2 \\ \text{s.t.} & A_{11}^{\mathrm{T}} y_1 + A_{21}^{\mathrm{T}} y_2 \geq c_1, \\ & A_{12}^{\mathrm{T}} y_1 + A_{22}^{\mathrm{T}} y_2 = c_2, \\ & \qquad\qquad y_1 \geq 0 \end{array}$$

x_1, x_2：主最適解
y_1, y_2：双対最適解

$\Leftrightarrow$

x_1, x_2：主許容解，y_1, y_2：双対許容解
$x_1^{\mathrm{T}}(A_{11}^{\mathrm{T}} y_1 + A_{21}^{\mathrm{T}} y_2 - c_1) = 0$
$y_1^{\mathrm{T}}(b_1 - A_{11} x_1 - A_{12} x_2) = 0$

$\underline{\Leftarrow \text{の証明}}$ $c_1^{\mathrm{T}} x_1 + c_2^{\mathrm{T}} x_2 = b_1^{\mathrm{T}} y_1 + b_2^{\mathrm{T}} y_2$ を示せばよい ($\because$ 双対定理)

$$0 \leq b_1^{\mathrm{T}} y_1 + b_2^{\mathrm{T}} y_2 - c_1^{\mathrm{T}} x_1 - c_2^{\mathrm{T}} x_2$$

弱双対定理

相補性定理：一般論 — 証明 (3)

主問題.

$$\begin{array}{ll} \text{max.} & \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_1 + \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2 \\ \text{s.t.} & A_{11}\boldsymbol{x}_1 + A_{12}\boldsymbol{x}_2 \le \boldsymbol{b}_1, \\ & A_{21}\boldsymbol{x}_1 + A_{22}\boldsymbol{x}_2 = \boldsymbol{b}_2, \\ & \boldsymbol{x}_1 \ge \boldsymbol{0} \end{array}$$

双対問題

$$\begin{array}{ll} \text{min.} & \boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{y}_1 + \boldsymbol{b}_2^{\mathrm{T}}\boldsymbol{y}_2 \\ \text{s.t.} & A_{11}^{\mathrm{T}}\boldsymbol{y}_1 + A_{21}^{\mathrm{T}}\boldsymbol{y}_2 \ge \boldsymbol{c}_1, \\ & A_{12}^{\mathrm{T}}\boldsymbol{y}_1 + A_{22}^{\mathrm{T}}\boldsymbol{y}_2 = \boldsymbol{c}_2, \\ & \boldsymbol{y}_1 \ge \boldsymbol{0} \end{array}$$

$\boldsymbol{x}_1, \boldsymbol{x}_2$：主最適解
$\boldsymbol{y}_1, \boldsymbol{y}_2$：双対最適解

$\Leftrightarrow$

$\boldsymbol{x}_1, \boldsymbol{x}_2$：主許容解，$\boldsymbol{y}_1, \boldsymbol{y}_2$：双対許容解
$\boldsymbol{x}_1^{\mathrm{T}}(A_{11}^{\mathrm{T}}\boldsymbol{y}_1 + A_{21}^{\mathrm{T}}\boldsymbol{y}_2 - \boldsymbol{c}_1) = 0$
$\boldsymbol{y}_1^{\mathrm{T}}(\boldsymbol{b}_1 - A_{11}\boldsymbol{x}_1 - A_{12}\boldsymbol{x}_2) = 0$

$\Leftarrow$ の証明 $\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_1 + \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2 = \boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{y}_1 + \boldsymbol{b}_2^{\mathrm{T}}\boldsymbol{y}_2$ を示せばよい (∵ 双対定理)

$$\begin{aligned} 0 &\le \boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{y}_1 + \boldsymbol{b}_2^{\mathrm{T}}\boldsymbol{y}_2 - \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_1 - \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2 \\ &\le (\boldsymbol{y}_1^{\mathrm{T}}A_{11}\boldsymbol{x}_1 + \boldsymbol{y}_1^{\mathrm{T}}A_{12}\boldsymbol{x}_2) + \boldsymbol{y}_2^{\mathrm{T}}\boldsymbol{b}_2 - (\boldsymbol{y}_1^{\mathrm{T}}A_{11}\boldsymbol{x}_1 + \boldsymbol{y}_2^{\mathrm{T}}A_{21}\boldsymbol{x}_1) - \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2 \end{aligned}$$

相補性定理：一般論 — 証明 (3)

主問題.

max. $\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_1 + \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2$

s.t. $A_{11}\boldsymbol{x}_1 + A_{12}\boldsymbol{x}_2 \leq \boldsymbol{b}_1,$
$A_{21}\boldsymbol{x}_1 + A_{22}\boldsymbol{x}_2 = \boldsymbol{b}_2,$
$\boldsymbol{x}_1 \geq \boldsymbol{0}$

双対問題

min. $\boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{y}_1 + \boldsymbol{b}_2^{\mathrm{T}}\boldsymbol{y}_2$

s.t. $A_{11}^{\mathrm{T}}\boldsymbol{y}_1 + A_{21}^{\mathrm{T}}\boldsymbol{y}_2 \geq \boldsymbol{c}_1,$
$A_{12}^{\mathrm{T}}\boldsymbol{y}_1 + A_{22}^{\mathrm{T}}\boldsymbol{y}_2 = \boldsymbol{c}_2,$
$\boldsymbol{y}_1 \geq \boldsymbol{0}$

$\boldsymbol{x}_1, \boldsymbol{x}_2$：主最適解
$\boldsymbol{y}_1, \boldsymbol{y}_2$：双対最適解
$\Leftrightarrow$
$\boldsymbol{x}_1, \boldsymbol{x}_2$：主許容解，$\boldsymbol{y}_1, \boldsymbol{y}_2$：双対許容解
$\boldsymbol{x}_1^{\mathrm{T}}(A_{11}^{\mathrm{T}}\boldsymbol{y}_1 + A_{21}^{\mathrm{T}}\boldsymbol{y}_2 - \boldsymbol{c}_1) = 0$
$\boldsymbol{y}_1^{\mathrm{T}}(\boldsymbol{b}_1 - A_{11}\boldsymbol{x}_1 - A_{12}\boldsymbol{x}_2) = 0$

$\underline{\Leftarrow \text{の証明}}$ $\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_1 + \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2 = \boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{y}_1 + \boldsymbol{b}_2^{\mathrm{T}}\boldsymbol{y}_2$ を示せばよい ($\because$ 双対定理)

$$\begin{aligned}0 &\leq \boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{y}_1 + \boldsymbol{b}_2^{\mathrm{T}}\boldsymbol{y}_2 - \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_1 - \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2 \\ &\leq (\boldsymbol{y}_1^{\mathrm{T}}A_{11}\boldsymbol{x}_1 + \boldsymbol{y}_1^{\mathrm{T}}A_{12}\boldsymbol{x}_2) + \boldsymbol{y}_2^{\mathrm{T}}\boldsymbol{b}_2 - (\boldsymbol{y}_1^{\mathrm{T}}A_{11}\boldsymbol{x}_1 + \boldsymbol{y}_2^{\mathrm{T}}A_{21}\boldsymbol{x}_1) - \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2 \\ &= \boldsymbol{y}_1^{\mathrm{T}}A_{12}\boldsymbol{x}_2 + \boldsymbol{y}_2^{\mathrm{T}}\boldsymbol{b}_2 - \boldsymbol{y}_2^{\mathrm{T}}A_{21}\boldsymbol{x}_1 - \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2\end{aligned}$$

相補性定理：一般論 — 証明 (3)

主問題.

$$\begin{array}{ll} \text{max.} & \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_1 + \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2 \\ \text{s.t.} & A_{11}\boldsymbol{x}_1 + A_{12}\boldsymbol{x}_2 \leq \boldsymbol{b}_1, \\ & A_{21}\boldsymbol{x}_1 + A_{22}\boldsymbol{x}_2 = \boldsymbol{b}_2, \\ & \boldsymbol{x}_1 \geq \boldsymbol{0} \end{array}$$

双対問題

$$\begin{array}{ll} \text{min.} & \boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{y}_1 + \boldsymbol{b}_2^{\mathrm{T}}\boldsymbol{y}_2 \\ \text{s.t.} & A_{11}^{\mathrm{T}}\boldsymbol{y}_1 + A_{21}^{\mathrm{T}}\boldsymbol{y}_2 \geq \boldsymbol{c}_1, \\ & A_{12}^{\mathrm{T}}\boldsymbol{y}_1 + A_{22}^{\mathrm{T}}\boldsymbol{y}_2 = \boldsymbol{c}_2, \\ & \boldsymbol{y}_1 \geq \boldsymbol{0} \end{array}$$

$$\boxed{\begin{array}{lcl} \boldsymbol{x}_1, \boldsymbol{x}_2 : \text{主最適解} & & \boldsymbol{x}_1, \boldsymbol{x}_2 : \text{主許容解，} \boldsymbol{y}_1, \boldsymbol{y}_2 : \text{双対許容解} \\ \boldsymbol{y}_1, \boldsymbol{y}_2 : \text{双対最適解} & \Leftrightarrow & \boldsymbol{x}_1^{\mathrm{T}}(A_{11}^{\mathrm{T}}\boldsymbol{y}_1 + A_{21}^{\mathrm{T}}\boldsymbol{y}_2 - \boldsymbol{c}_1) = 0 \\ & & \boldsymbol{y}_1^{\mathrm{T}}(\boldsymbol{b}_1 - A_{11}\boldsymbol{x}_1 - A_{12}\boldsymbol{x}_2) = 0 \end{array}}$$

⇐ の証明 $\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_1 + \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2 = \boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{y}_1 + \boldsymbol{b}_2^{\mathrm{T}}\boldsymbol{y}_2$ を示せばよい (∵ 双対定理)

$$\begin{aligned} 0 &\leq \boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{y}_1 + \boldsymbol{b}_2^{\mathrm{T}}\boldsymbol{y}_2 - \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_1 - \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2 \\ &\leq (\boldsymbol{y}_1^{\mathrm{T}}A_{11}\boldsymbol{x}_1 + \boldsymbol{y}_1^{\mathrm{T}}A_{12}\boldsymbol{x}_2) + \boldsymbol{y}_2^{\mathrm{T}}\boldsymbol{b}_2 - (\boldsymbol{y}_1^{\mathrm{T}}A_{11}\boldsymbol{x}_1 + \boldsymbol{y}_2^{\mathrm{T}}A_{21}\boldsymbol{x}_1) - \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2 \\ &= \boldsymbol{y}_1^{\mathrm{T}}A_{12}\boldsymbol{x}_2 + \boldsymbol{y}_2^{\mathrm{T}}\boldsymbol{b}_2 - \boldsymbol{y}_2^{\mathrm{T}}A_{21}\boldsymbol{x}_1 - \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2 \\ &= \boldsymbol{y}_1^{\mathrm{T}}A_{12}\boldsymbol{x}_2 + \boldsymbol{y}_2^{\mathrm{T}}(A_{21}\boldsymbol{x}_1 + A_{22}\boldsymbol{x}_2) - \boldsymbol{y}_2^{\mathrm{T}}A_{21}\boldsymbol{x}_1 - (\boldsymbol{y}_1^{\mathrm{T}}A_{12} + \boldsymbol{y}_2^{\mathrm{T}}A_{22})\boldsymbol{x}_2 \end{aligned}$$

相補性定理：一般論 — 証明 (3)

主問題.

$$\begin{array}{lrl} \text{max.} & \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_1+\boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2 & \\ \text{s.t.} & \boldsymbol{A}_{11}\boldsymbol{x}_1+\boldsymbol{A}_{12}\boldsymbol{x}_2 \leq \boldsymbol{b}_1, & \\ & \boldsymbol{A}_{21}\boldsymbol{x}_1+\boldsymbol{A}_{22}\boldsymbol{x}_2 = \boldsymbol{b}_2, & \\ & \boldsymbol{x}_1 \geq \boldsymbol{0} & \end{array}$$

双対問題

$$\begin{array}{lrl} \text{min.} & \boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{y}_1+\boldsymbol{b}_2^{\mathrm{T}}\boldsymbol{y}_2 & \\ \text{s.t.} & \boldsymbol{A}_{11}^{\mathrm{T}}\boldsymbol{y}_1+\boldsymbol{A}_{21}^{\mathrm{T}}\boldsymbol{y}_2 \geq \boldsymbol{c}_1, & \\ & \boldsymbol{A}_{12}^{\mathrm{T}}\boldsymbol{y}_1+\boldsymbol{A}_{22}^{\mathrm{T}}\boldsymbol{y}_2 = \boldsymbol{c}_2, & \\ & \boldsymbol{y}_1 \geq \boldsymbol{0} & \end{array}$$

$$\boxed{\begin{array}{lcl} \boldsymbol{x}_1,\boldsymbol{x}_2：\text{主最適解} & & \boldsymbol{x}_1,\boldsymbol{x}_2：\text{主許容解，}\ \boldsymbol{y}_1,\boldsymbol{y}_2：\text{双対許容解} \\ \boldsymbol{y}_1,\boldsymbol{y}_2：\text{双対最適解} & \Leftrightarrow & \boldsymbol{x}_1^{\mathrm{T}}(\boldsymbol{A}_{11}^{\mathrm{T}}\boldsymbol{y}_1+\boldsymbol{A}_{21}^{\mathrm{T}}\boldsymbol{y}_2-\boldsymbol{c}_1)=0 \\ & & \boldsymbol{y}_1^{\mathrm{T}}(\boldsymbol{b}_1-\boldsymbol{A}_{11}\boldsymbol{x}_1-\boldsymbol{A}_{12}\boldsymbol{x}_2)=0 \end{array}}$$

$\Leftarrow$ の証明 $\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_1+\boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2=\boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{y}_1+\boldsymbol{b}_2^{\mathrm{T}}\boldsymbol{y}_2$ を示せばよい ($\because$ 双対定理)

$$\begin{aligned} 0 &\leq \boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{y}_1+\boldsymbol{b}_2^{\mathrm{T}}\boldsymbol{y}_2-\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_1-\boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2 \\ &\leq (\boldsymbol{y}_1^{\mathrm{T}}\boldsymbol{A}_{11}\boldsymbol{x}_1+\boldsymbol{y}_1^{\mathrm{T}}\boldsymbol{A}_{12}\boldsymbol{x}_2)+\boldsymbol{y}_2^{\mathrm{T}}\boldsymbol{b}_2-(\boldsymbol{y}_1^{\mathrm{T}}\boldsymbol{A}_{11}\boldsymbol{x}_1+\boldsymbol{y}_2^{\mathrm{T}}\boldsymbol{A}_{21}\boldsymbol{x}_1)-\boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2 \\ &= \boldsymbol{y}_1^{\mathrm{T}}\boldsymbol{A}_{12}\boldsymbol{x}_2+\boldsymbol{y}_2^{\mathrm{T}}\boldsymbol{b}_2-\boldsymbol{y}_2^{\mathrm{T}}\boldsymbol{A}_{21}\boldsymbol{x}_1-\boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2 \\ &= \boldsymbol{y}_1^{\mathrm{T}}\boldsymbol{A}_{12}\boldsymbol{x}_2+\boldsymbol{y}_2^{\mathrm{T}}(\boldsymbol{A}_{21}\boldsymbol{x}_1+\boldsymbol{A}_{22}\boldsymbol{x}_2)-\boldsymbol{y}_2^{\mathrm{T}}\boldsymbol{A}_{21}\boldsymbol{x}_1-(\boldsymbol{y}_1^{\mathrm{T}}\boldsymbol{A}_{12}+\boldsymbol{y}_2^{\mathrm{T}}\boldsymbol{A}_{22})\boldsymbol{x}_2 \\ &= 0 \end{aligned}$$

$\square$

相補性定理：まとめ

主問題.

$$\begin{array}{ll} \text{max.} & \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_1 + \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_2 \\ \text{s.t.} & A_{11}\boldsymbol{x}_1 + A_{12}\boldsymbol{x}_2 \leq \boldsymbol{b}_1, \\ & A_{21}\boldsymbol{x}_1 + A_{22}\boldsymbol{x}_2 = \boldsymbol{b}_2, \\ & \boldsymbol{x}_1 \geq \boldsymbol{0} \end{array}$$

双対問題

$$\begin{array}{ll} \text{min.} & \boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{y}_1 + \boldsymbol{b}_2^{\mathrm{T}}\boldsymbol{y}_2 \\ \text{s.t.} & A_{11}^{\mathrm{T}}\boldsymbol{y}_1 + A_{21}^{\mathrm{T}}\boldsymbol{y}_2 \geq \boldsymbol{c}_1, \\ & A_{12}^{\mathrm{T}}\boldsymbol{y}_1 + A_{22}^{\mathrm{T}}\boldsymbol{y}_2 = \boldsymbol{c}_2, \\ & \boldsymbol{y}_1 \geq \boldsymbol{0} \end{array}$$

$\boldsymbol{x}_1, \boldsymbol{x}_2$：主最適解
$\boldsymbol{y}_1, \boldsymbol{y}_2$：双対最適解

⇔

$\boldsymbol{x}_1, \boldsymbol{x}_2$：主許容解，$\boldsymbol{y}_1, \boldsymbol{y}_2$：双対許容解

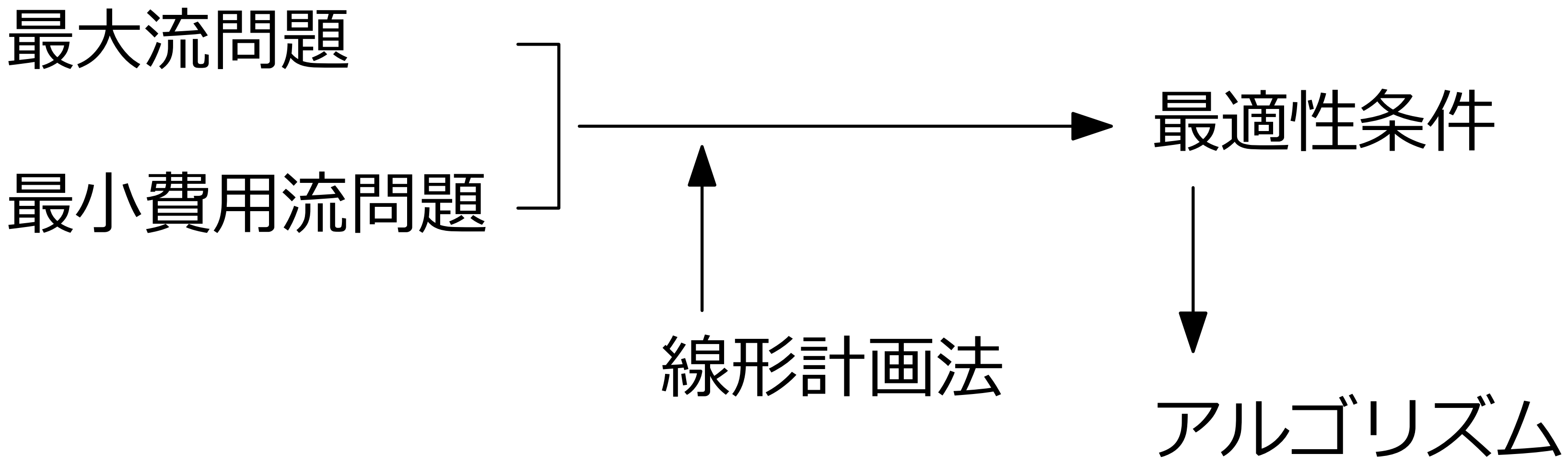
$$\boldsymbol{x}_1^{\mathrm{T}}(A_{11}^{\mathrm{T}}\boldsymbol{y}_1 + A_{21}^{\mathrm{T}}\boldsymbol{y}_2 - \boldsymbol{c}_1) = 0$$

$$\boldsymbol{y}_1^{\mathrm{T}}(\boldsymbol{b}_1 - A_{11}\boldsymbol{x}_1 - A_{12}\boldsymbol{x}_2) = 0$$

最適性 ⇔

- 主許容性
- 双対許容性
- 相補性

次回予告



次回の予告

- **最大流問題** を線形計画問題として記述する
- 双対定理を使って **最大流問題** の性質を調べる