I631: Foundation of Computational Geometry (14) Envelopes and Levels II

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Background

- Determining the maximum number of vertices in the k-level of a line arrangement (in the plane) is a difficult problem
- Some lower bounds and upper bounds are known

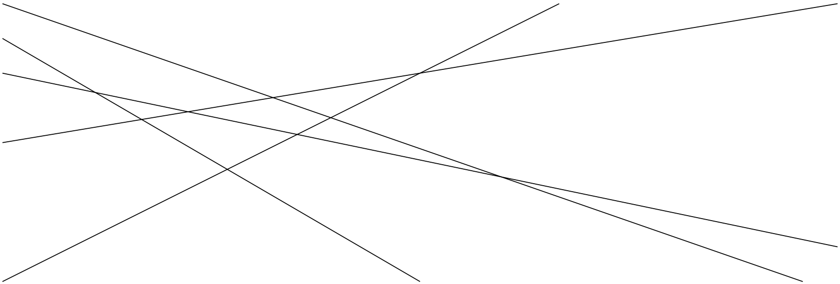
Goal of this lecture

- Learn a typical lower bound argument
- Learn a typical upper bound argument

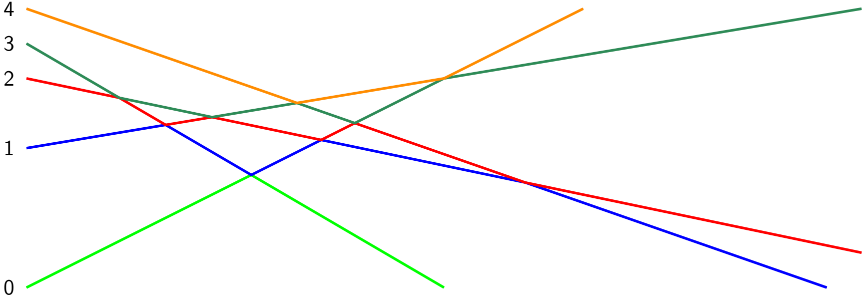
through the k-level problem

1. The k-level problem
2. Lower bound
3. Upper bound

Reminder: The k-level



Reminder: The k-level



The k-level problem

The k-level problem

What is the maximum number of vertices in the k-level of a simple arrangement of n lines in the plane?

- Namely, for a simple line arrangement $\mathcal{A}$ in $\mathbb{R}^2$, let

$$e_k(\mathcal{A}) = \text{ the number of vertices in the } k\text{-level of } \mathcal{A}$$

  and let

$$e_k(n) = \max\{e_k(\mathcal{A}) \mid \mathcal{A} \text{ the arrangement of } n \text{ lines in } \mathbb{R}^2\}$$

- The task is to determine $e_k(n)$

We usually look at the median level

Main target

Determining $e_{\lfloor n/2 \rfloor}(n)$, the number of vertices in the median level

Why?

Fact (Agarwal, Aronov, Chan, Sharir '98)

$e_{\lfloor n/2 \rfloor}(n) = O(n^{\alpha})$ for some constant α
$\Rightarrow e_k(n) = O(n(k+1)^{\alpha-1})$ for all $k \in \{0, \ldots, \lfloor n/2 \rfloor\}$

Fact (Edelsbrunner '87)

$e_{\lfloor n/2 \rfloor}(n) = \Omega(n^{\alpha})$ for some constant α
$\Rightarrow e_k(n) = \Omega(n(k+1)^{\alpha-1})$ for all $k \in \{0, \ldots, \lfloor n/2 \rfloor\}$

Conjectured bound on the number of edges in the median level

Let $e(n) = e_{\lfloor n/2 \rfloor}(n)$, for simplicity

Conjecture (Erdős, Lovász, Simmons, Straus '73)

$$e(n) = o(n^{1+\varepsilon})$$

for any fixed constant $\varepsilon > 0$

We are far from proving/disproving this conjecture

Known upper bounds

- $e(n) = O(n^{3/2})$ (Lovász '71)
  (Erdős, Lovász, Simmons, Straus '73)
  (Agarwal, Aronov, Chan, Sharir '98)
  (Chan '05)
- $e(n) = O(n^{3/2}/\log^* n)$ (Pach, Steiger, Szemerédi '92)
- $e(n) = O(n^{4/3})$ (Dey '98)
  (Andrzejak, Aronov, Har-Peled, Seidel, Welzl '98)

Known lower bounds

- $e(n) = \Omega(n \log n)$ (Erdős, Lovász, Simmons, Straus '73)
- $e(n) = n \exp(\Omega(\sqrt{\log n}))$ (Tóth '01) (Nivasch '08)

What we are going to look at

- Lower bound: $e(n) \geq 2n - 3$ for all $n \geq 2$
  Proof by Erdős, Lovász, Simmons, Straus '73
- Upper bound: $e(n) = O(n^{3/2})$
  Proof by Chan '05

An easy lower bound for $e(n)$

Theorem (Erdős, Lovász, Simmons, Straus '73)

$e(n) \geq 2n - 3$ for all natural numbers $n \geq 2$

Basic strategy for the proof

Prove $e(n+2) \geq e(n) + 4$ for all natural numbers $n \geq 2$

- Then, we can prove $e(n) \geq 2n - 3$ by induction
  - When $n = 2$, we see $e(2) \geq 1$
  - When $n = 3$, we see $e(3) \geq 3$
  - When $n \geq 4$, by the recursion above

$$e(n+2) \geq e(n) + 4 = (2n - 3) + 4 = 2(n+2) - 3$$

How to derive the recursion

Want to prove

$e(n+2) \geq e(n) + 4$ for all natural numbers $n \geq 2$

- Let $\mathcal{A}_n$ be the simple arrangement of n lines that gives the max number of vertices of the median level among all simple arrangements of n lines:

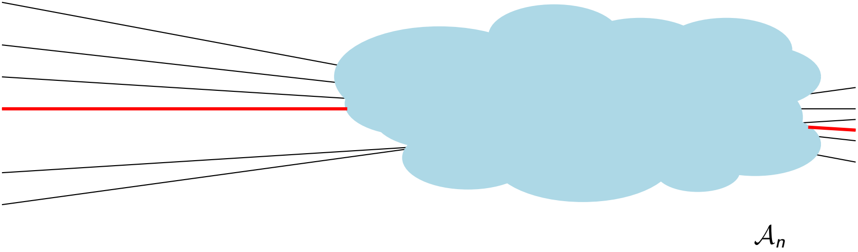
$$e(n) = e_{\lfloor n/2 \rfloor}(\mathcal{A}_n)$$

- We construct a simple arrangement $\mathcal{A}'_{n+2}$ of n+2 lines from $\mathcal{A}_n$ s.t. the number of vertices of the median level is $\geq e(n) + 4$
- Then

$$e(n+2) \geq e_{\lfloor (n+2)/2 \rfloor}(\mathcal{A}'_{n+2}) \geq e(n) + 4$$

Construction of $\mathcal{A}'_{n+2}$

Add two lines to the left of all vertices of $\mathcal{A}_n$ so that their intersection comes below the lines of $\mathcal{A}_n$



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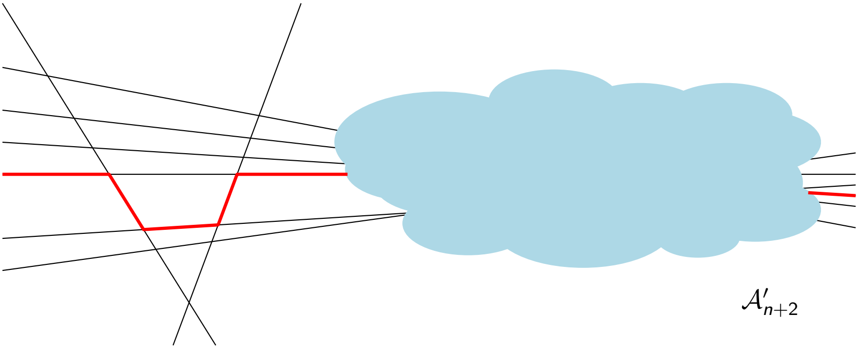


Upper bound for $e(n)$

Theorem (Lovász '71)

$e(n) = O(n^{3/2})$

Namely, for any simple arrangement $\mathcal{A}$ of n lines in $\mathbb{R}^2$

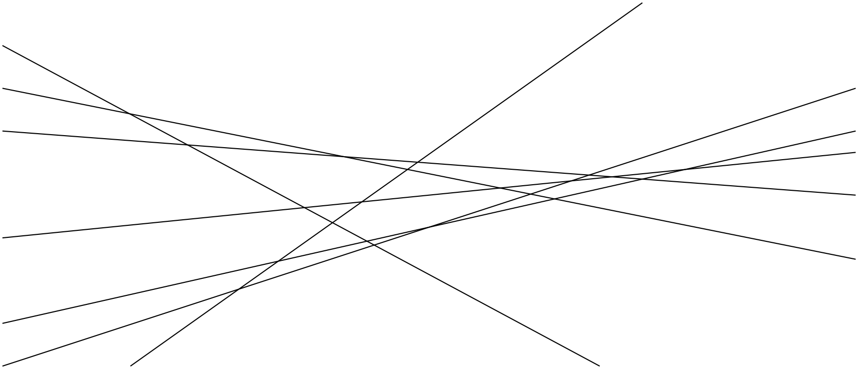
$$e_{\lfloor n/2 \rfloor}(\mathcal{A}) = O(n^{3/2})$$

- We look at the proof by Chan '05
- In his proof, we study a more general problem

A more general problem: Setup (1)

$\mathcal{A}$ a simple arrangement of n lines in $\mathbb{R}^2$

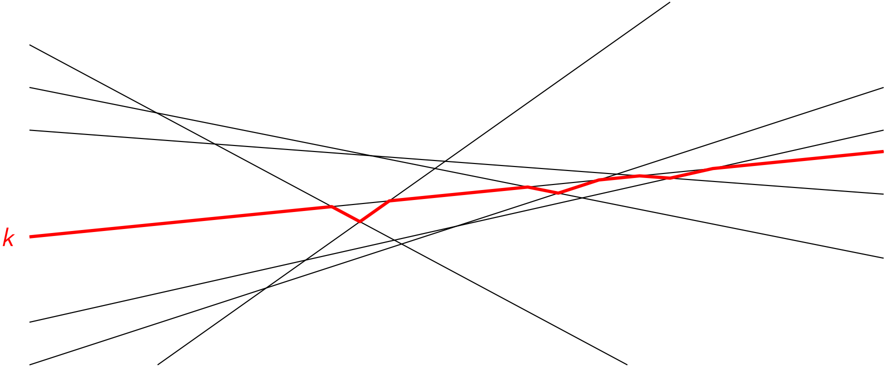
- Let $k \in \{0, \ldots, n-1\}$ fixed
- Let $i \geq 1$ be a natural number
- Let $V_i(\mathcal{A})$ = the set of vertices of the i-level of $\mathcal{A}$
- (Let $V_i(\mathcal{A}) = \emptyset$ when $i < 0$ or $i \geq n$)



A more general problem: Setup (2)

Let

- $B_i = (V_{k-i}(\mathcal{A}) \setminus V_{k-i-1}(\mathcal{A})) \cup (V_{k+i}(\mathcal{A}) \setminus V_{k+i+1}(\mathcal{A}))$
- $I_i = (V_{k-i+1} \cup V_{k-i+2} \cup \cdots \cup V_{k+i-2} \cup V_{k+i-1}) \setminus (V_{k-i} \cup V_{k+i})$
- Note: $|I_2| = e_k(\mathcal{A})$ for any k



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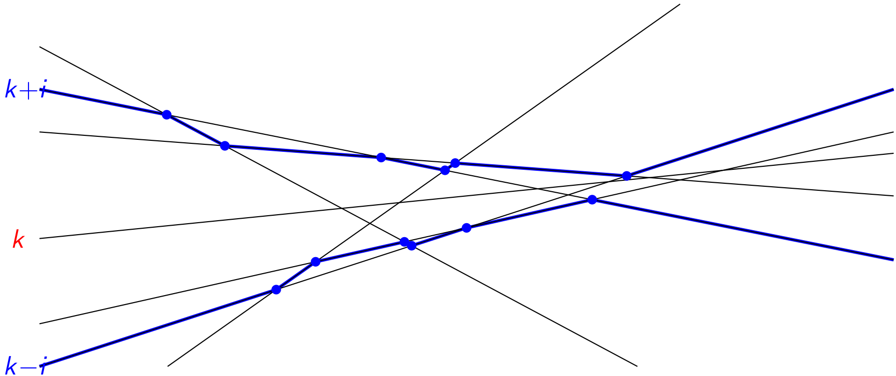

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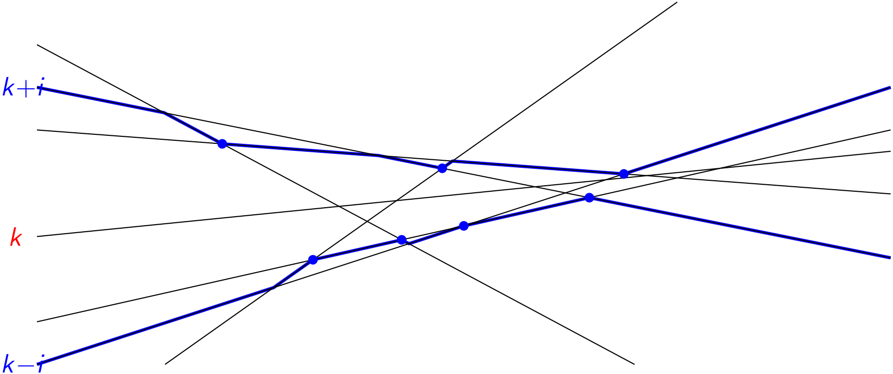

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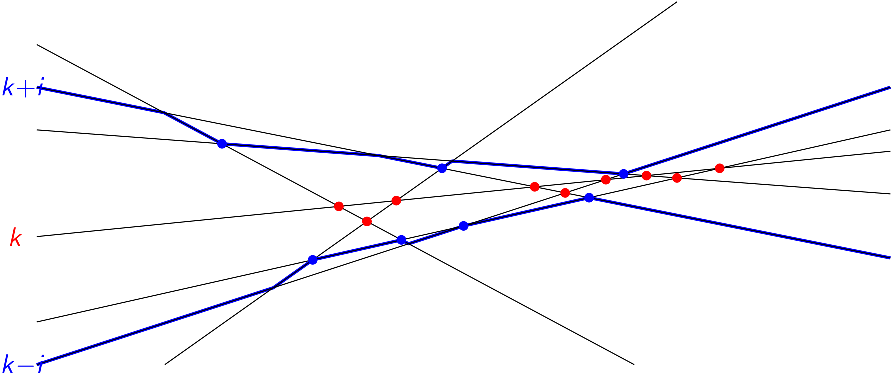
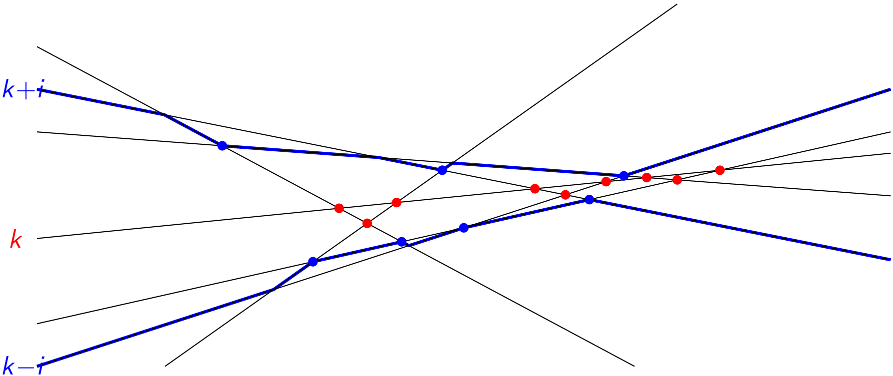


A more general problem: Claim

Claim

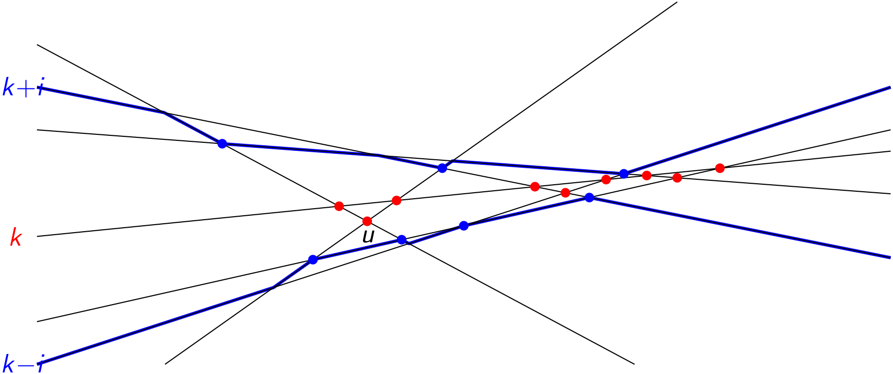
$$|I_i| \leq 2i \cdot |B_i| + 2i^2$$



Proof of the claim (1)

We employ the following *charging scheme*

- Fix $u \in I_i$
- Let ℓ_1, ℓ_2 intersect at u
- Walk to the right along ℓ_1, ℓ_2 from u
- You reach vertices v_1, v_2 in B_i or go to the $+x$-infinity



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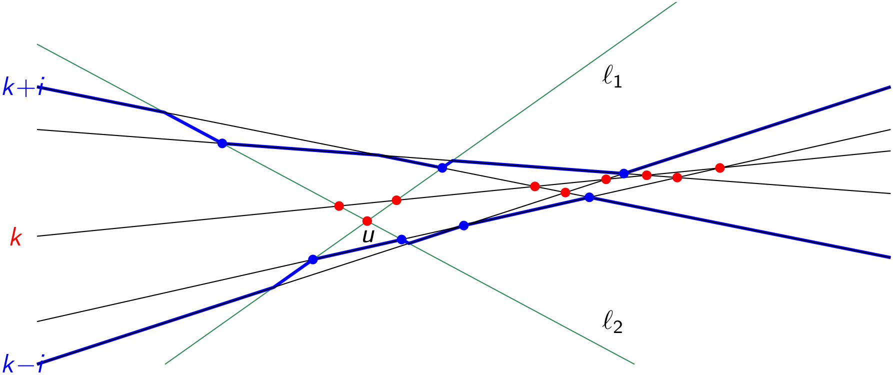

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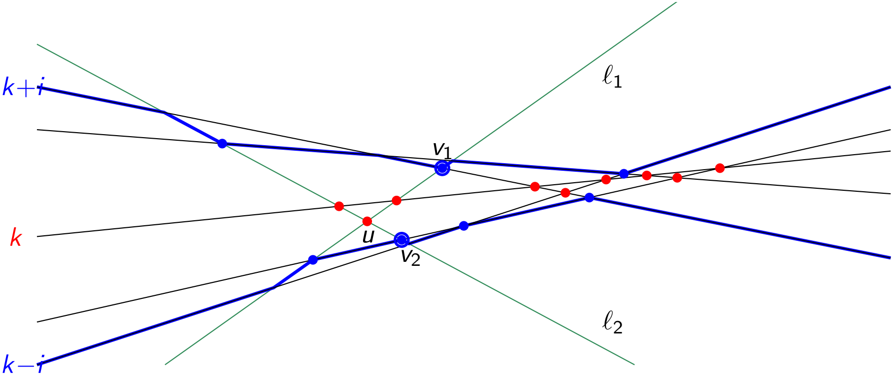

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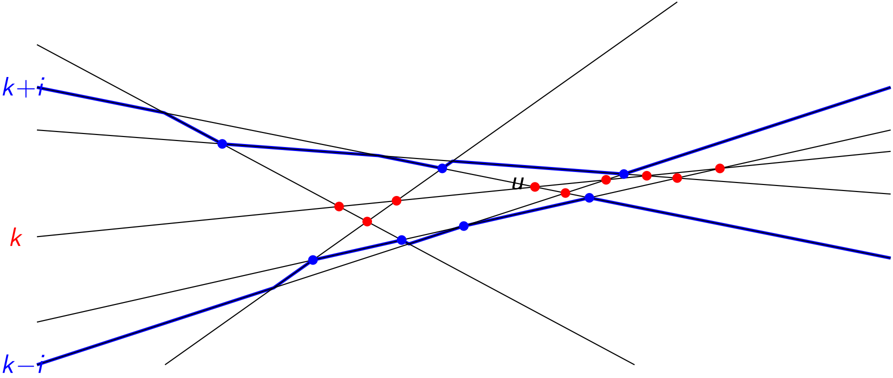

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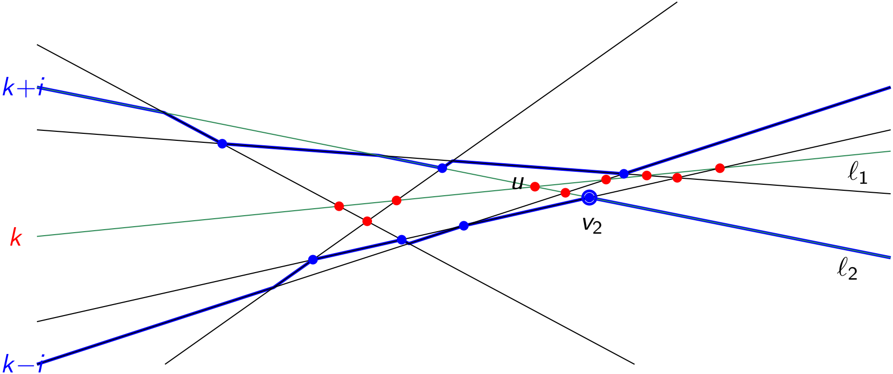

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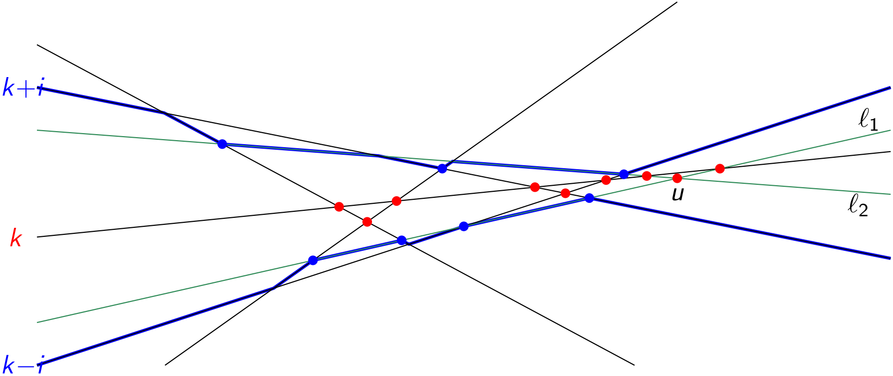
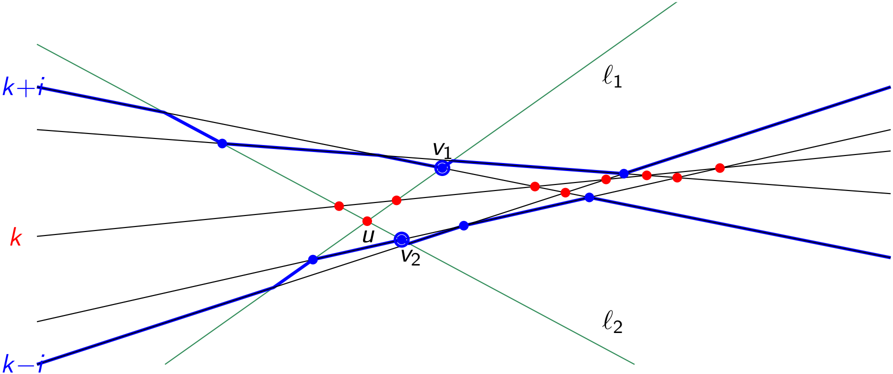


Proof of the claim (2)

We employ the following *charging scheme*

- If v_1 lies to the left of v_2, we charge u to v_1
- If v_2 lies to the left of v_1, we charge u to v_2
- If both go to the $+x$-infinity, we charge u to $+\infty$



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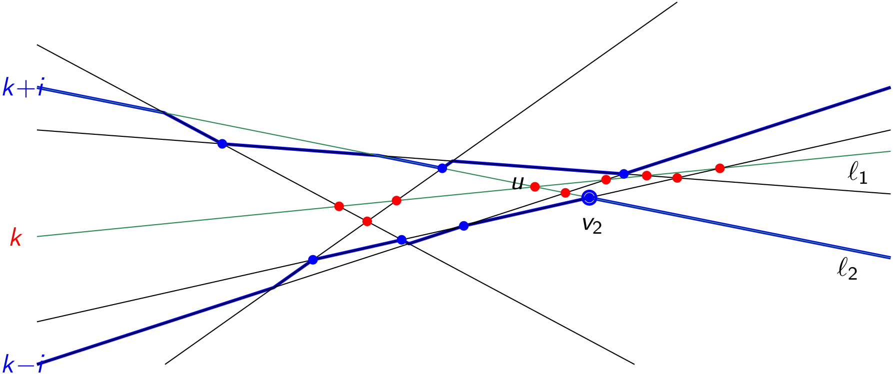

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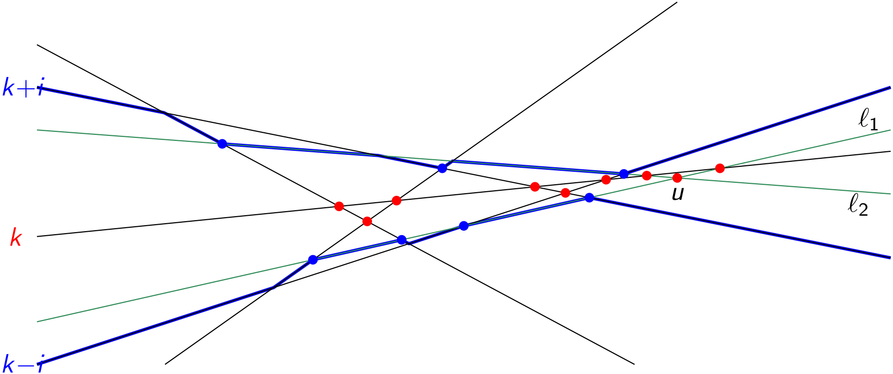
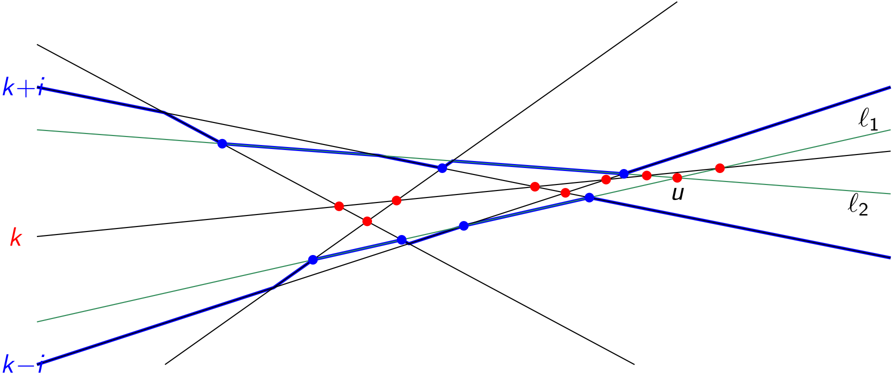


Proof of the claim (2)

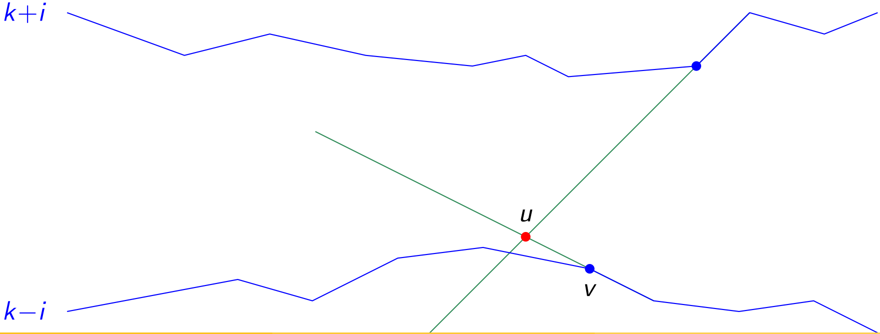
- Each $u \in I_i$ pays one unit
- Q.: How much does each $v \in B_i \cup \{+\infty\}$ receive?



Proof of the claim (3)

For $v \in V_{k-i}(\mathcal{A}) \setminus V_{k-i-1}(\mathcal{A}) \subseteq B_i$, receiving from $u \in I_i$

- u is the intersection of two lines, say ℓ_1 and ℓ_2, ℓ_1 goes through v, ℓ_2 goes above v
- $\therefore$ Each such u is associated with a line strictly between the $(k-i)$-level and the $(k+i)$-level of $\mathcal{A}$ at the x-coordinate of v
- $\therefore$ # of such u is $< 2i$



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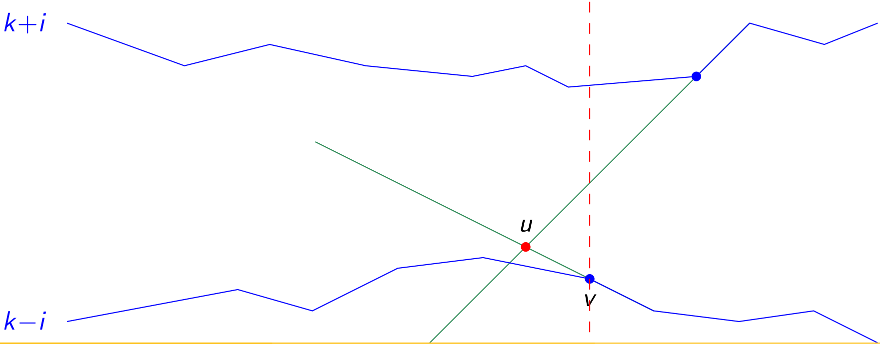

Proof of the claim (4)

For $v \in V_{k+i}(\mathcal{A}) \setminus V_{k+i+1}(\mathcal{A}) \subseteq B_i$, receiving from $u \in I_i$

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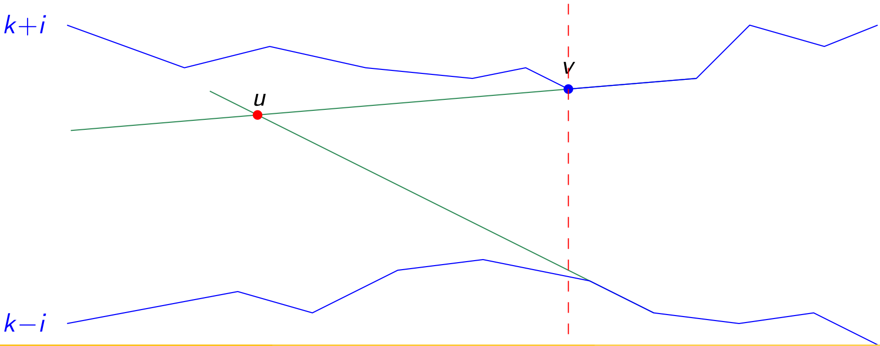
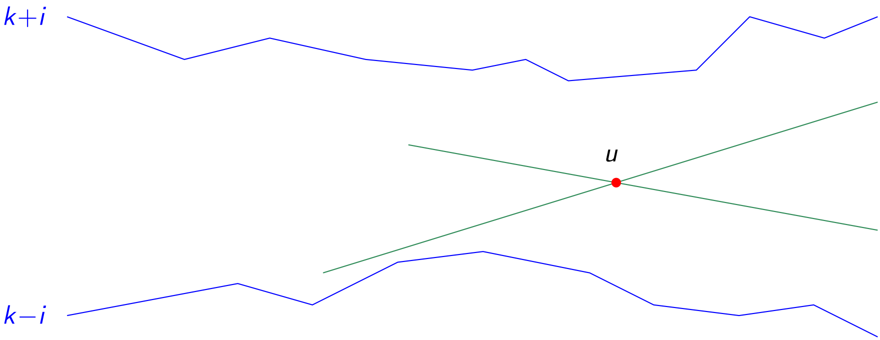


Proof of the claim (5)

For $v = +\infty$

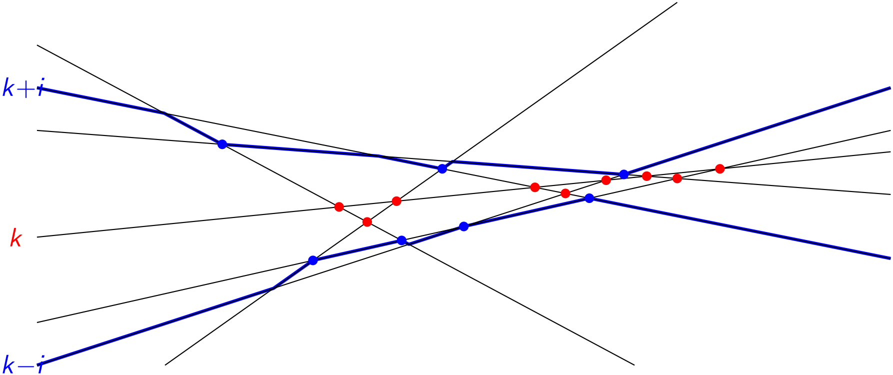
- $\#$ of lines ℓ leading to $+\infty < 2i$
- When v receives from u, u is the intersection of two such lines
- $\#$ of pairs $< 2i^2$



Proof of the claim (6)

Summarizing

- $|I_i|$ = transferred units $\leq 2i \cdot |B_i| + 2i^2$
- $\therefore |I_i| \leq 2i \cdot |B_i| + 2i^2$



The next step

- We have proved
$$|I_i| \le 2i|B_i| + 2i^2$$
- Observe: $|B_i| = |I_{i+1}| - |I_i|$
- Therefore
$$\begin{aligned} |I_i| &\le 2i(|I_{i+1}| - |I_i|) + 2i^2 \\ \therefore \quad |I_i| &\le \frac{2i}{2i+1} \cdot |I_{i+1}| + i \end{aligned}$$
- Enough to solve this recursion with $|I_n| = \binom{n}{2}$

Solving the recursion (1)

$$
\begin{aligned}
|I_i| &\leq \frac{2i}{2i+1}|I_{i+1}| + i \\
&\leq \frac{2i}{2i+1}\left(\frac{2(i+1)}{2(i+1)+1}|I_{i+2}| + (i+1)\right) + i \\
&= \frac{2i}{2i+1}\frac{2(i+1)}{2(i+1)+1}|I_{i+2}| + \frac{2i}{2i+1}(i+1) + i
\end{aligned}
$$

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\end{aligned}
$$

Solving the recursion (2)

$$
\begin{aligned}
|I_i| &\le \frac{2i}{2i+1}\frac{2(i+1)}{2(i+1)+1}|I_{i+2}| + \frac{2i}{2i+1}(i+1)+i \\
&\le \frac{2i}{2i+1}\frac{2(i+1)}{2(i+1)+1}\left(\frac{2(i+2)}{2(i+2)+1}|I_{i+3}| + (i+2)\right) \\
&\quad + \frac{2i}{2i+1}(i+1)+i \\
&= \frac{2i}{2i+1}\frac{2(i+1)}{2(i+1)+1}\frac{2(i+2)}{2(i+2)+1}|I_{i+3}| \\
&\quad + \frac{2i}{2i+1}\frac{2(i+1)}{2(i+1)+1}(i+2) + \frac{2i}{2i+1}(i+1)+i \\
&= \prod_{j=i}^{i+2}\frac{2j}{2j+1}|I_{i+3}| + \sum_{j=i}^{i+1}\prod_{h=i}^{j}\frac{2h}{2h+1}(j+1)+i
\end{aligned}
$$

Solving the recursion (2)

$$\begin{aligned}
|I_i| &\le \frac{2i}{2i+1}\frac{2(i+1)}{2(i+1)+1}|I_{i+2}| + \frac{2i}{2i+1}(i+1)+i \\
&\le \frac{2i}{2i+1}\frac{2(i+1)}{2(i+1)+1}\left(\frac{2(i+2)}{2(i+2)+1}|I_{i+3}| + (i+2)\right) \\
&\quad + \frac{2i}{2i+1}(i+1)+i \\
&= \frac{2i}{2i+1}\frac{2(i+1)}{2(i+1)+1}\frac{2(i+2)}{2(i+2)+1}|I_{i+3}| \\
&\quad + \frac{2i}{2i+1}\frac{2(i+1)}{2(i+1)+1}(i+2) + \frac{2i}{2i+1}(i+1)+i \\
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\end{aligned}$$

Solving the recursion (2)

$$
\begin{aligned}
|I_i| &\leq \frac{2i}{2i+1}\frac{2(i+1)}{2(i+1)+1}|I_{i+2}| + \frac{2i}{2i+1}(i+1)+i \\
&\leq \frac{2i}{2i+1}\frac{2(i+1)}{2(i+1)+1}\left(\frac{2(i+2)}{2(i+2)+1}|I_{i+3}| + (i+2)\right) \\
&\quad + \frac{2i}{2i+1}(i+1)+i \\
&= \frac{2i}{2i+1}\frac{2(i+1)}{2(i+1)+1}\frac{2(i+2)}{2(i+2)+1}|I_{i+3}| \\
&\quad + \frac{2i}{2i+1}\frac{2(i+1)}{2(i+1)+1}(i+2) + \frac{2i}{2i+1}(i+1)+i \\
&= \prod_{j=i}^{i+2}\frac{2j}{2j+1}|I_{i+3}| + \sum_{j=i}^{i+1}\prod_{h=i}^{j}\frac{2h}{2h+1}(j+1)+i
\end{aligned}
$$

Solving the recursion (2)

$$
\begin{aligned}
|I_i| &\le \frac{2i}{2i+1}\frac{2(i+1)}{2(i+1)+1}|I_{i+2}| + \frac{2i}{2i+1}(i+1) + i \\
&\le \frac{2i}{2i+1}\frac{2(i+1)}{2(i+1)+1}\left(\frac{2(i+2)}{2(i+2)+1}|I_{i+3}| + (i+2)\right) \\
&\quad + \frac{2i}{2i+1}(i+1) + i \\
&= \frac{2i}{2i+1}\frac{2(i+1)}{2(i+1)+1}\frac{2(i+2)}{2(i+2)+1}|I_{i+3}| \\
&\quad + \frac{2i}{2i+1}\frac{2(i+1)}{2(i+1)+1}(i+2) + \frac{2i}{2i+1}(i+1) + i \\
&= \prod_{j=i}^{i+2}\frac{2j}{2j+1}|I_{i+3}| + \sum_{j=i}^{i+1}\prod_{h=i}^{j}\frac{2h}{2h+1}(j+1) + i
\end{aligned}
$$

Solving the recursion (2)

$$
\begin{aligned}
|I_i| &\le \frac{2i}{2i+1}\frac{2(i+1)}{2(i+1)+1}|I_{i+2}| + \frac{2i}{2i+1}(i+1) + i \\
&\le \frac{2i}{2i+1}\frac{2(i+1)}{2(i+1)+1}\left(\frac{2(i+2)}{2(i+2)+1}|I_{i+3}| + (i+2)\right) \\
&\quad + \frac{2i}{2i+1}(i+1) + i \\
&= \frac{2i}{2i+1}\frac{2(i+1)}{2(i+1)+1}\frac{2(i+2)}{2(i+2)+1}|I_{i+3}| \\
&\quad + \frac{2i}{2i+1}\frac{2(i+1)}{2(i+1)+1}(i+2) + \frac{2i}{2i+1}(i+1) + i \\
&= \prod_{j=i}^{i+2}\frac{2j}{2j+1}|I_{i+3}| + \sum_{j=i}^{i+1}\prod_{h=i}^{j}\frac{2h}{2h+1}(j+1) + i
\end{aligned}
$$

Solving the recursion (2)

$$
\begin{aligned}
|I_i| &\le \frac{2i}{2i+1}\frac{2(i+1)}{2(i+1)+1}|I_{i+2}| + \frac{2i}{2i+1}(i+1) + i \\
&\le \frac{2i}{2i+1}\frac{2(i+1)}{2(i+1)+1}\left(\frac{2(i+2)}{2(i+2)+1}|I_{i+3}| + (i+2)\right) \\
&\quad + \frac{2i}{2i+1}(i+1) + i \\
&= \frac{2i}{2i+1}\frac{2(i+1)}{2(i+1)+1}\frac{2(i+2)}{2(i+2)+1}|I_{i+3}| \\
&\quad + \frac{2i}{2i+1}\frac{2(i+1)}{2(i+1)+1}(i+2) + \frac{2i}{2i+1}(i+1) + i \\
&= \prod_{j=i}^{i+2}\frac{2j}{2j+1}|I_{i+3}| + \sum_{j=i}^{i+1}\prod_{h=i}^{j}\frac{2h}{2h+1}(j+1) + i
\end{aligned}
$$

Solving the recursion (3)

$$
\begin{aligned}
|I_i| &\le \prod_{j=i}^{i+2} \frac{2j}{2j+1} |I_{i+3}| + \sum_{j=i}^{i+1} \prod_{h=i}^{j} \frac{2h}{2h+1} (j+1) + i \\
&\vdots \\
&= \prod_{j=i}^{n-1} \frac{2j}{2j+1} |I_n| + \sum_{j=i}^{n-2} \prod_{h=i}^{j} \frac{2h}{2h+1} (j+1) + i \\
&\le \sqrt{\frac{i}{n-1}} \frac{n^2}{2} + \sum_{j=i}^{n-2} \sqrt{\frac{i}{j}} (j+1) + i
\end{aligned}
$$

Exercise

$$
\prod_{j=i}^{m} \frac{2j}{2j+1} \le \sqrt{\frac{i}{m}}
$$

Solving the recursion (4)

$$
\begin{aligned}
|I_i| &\leq \sqrt{\frac{i}{n-1}\frac{n^2}{2}} + \sum_{j=i}^{n-2} \sqrt{\frac{i}{j}}(j+1) + i \\
&\leq O(n^{3/2} i^{1/2}) + \sum_{j=i}^{n-2} O(\sqrt{j}\sqrt{i}) \\
&\leq O(n^{3/2} i^{1/2}) + \sum_{j=1}^{n} O(\sqrt{n}\sqrt{i}) \\
&\leq O(n^{3/2} i^{1/2}) + O(n^{3/2} i^{1/2}) \\
&= O(n^{3/2} i^{1/2})
\end{aligned}
$$

Finalizing the proof

For any k

$$e_k(\mathcal{A}) = |I_2| = O(n^{3/2});$$

In particular

$$e_{\lfloor n/2 \rfloor}(\mathcal{A}) = O(n^{3/2})$$

$\square$

Summary

$e(n) = \max \#$ of vert's in the med level of the arrangement of n lines

We have proved

- $e(n) \geq 2n - 3$
- $e(n) = O(n^{3/2})$

State of the art

- $e(n) = n \exp(\Omega(\sqrt{\log n}))$
- $e(n) = O(n^{4/3})$

Determining $e(n)$ is a notorious problem

- Matoušek: *Lectures on Discrete Geometry*
  - Chapter 11
- Edelsbrunner: *Algorithms in Combinatorial Geometry*
  - Chapter 3
- Chan: On levels in arrangements of curves II: A simple inequality and its consequences
  - Discrete & Computational Geometry **34** (2005) 11–24

Today, please submit the survey sheet for the latter half of this course

1. The k-level problem

2. Lower bound

3. Upper bound