I631: Foundation of Computational Geometry (12) Hyperplane Arrangements II

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Background

- A hyperplane arrangement is closely related to zonotopes
- Zonotopes are important polytopes that play important roles in many fields of mathematics

Goal of this lecture

- Learn the relevant notions for zonotopes
- Learn connections with hyperplane arrangements

1. Zonotopes

2. Relationship with hyperplane arrangements

Zonotopes

$d \geq 1$ a natural number

Def.: Zonotope

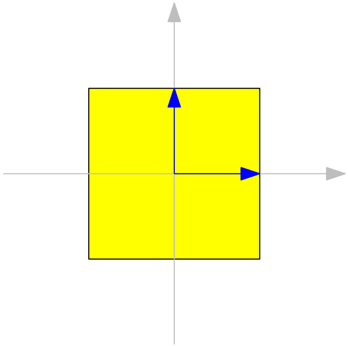
A **zonotope** is a polytope constructed as

$$Z = \left\{ \sum_{i=1}^{m} \lambda_i v_i \mid \lambda_i \in [-1, 1] \text{ for all } i \in \{1, \ldots, m\} \right\},$$

where $v_1, \ldots, v_m \in \mathbb{R}^d \setminus \{0\}$ are non-zero vectors

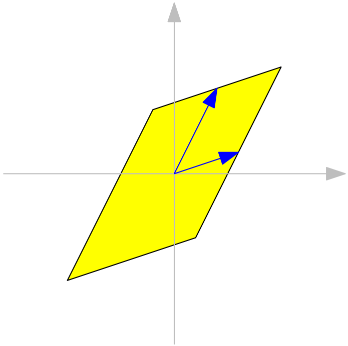
The vectors $v_1, \ldots, v_m$ are called the **generators** of Z

Example: Square



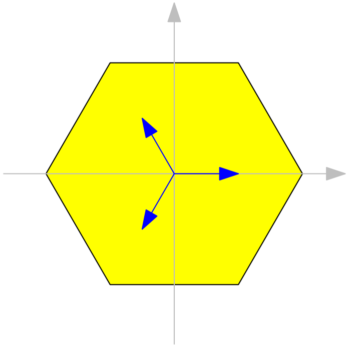
Generators $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \in \mathbb{R}^2$

Example: Parallelogram



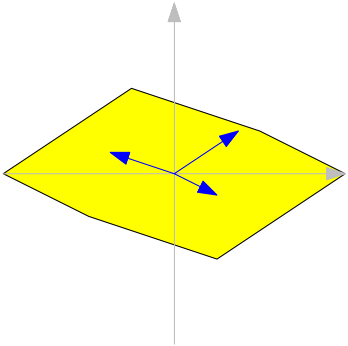
Generators $\begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \in \mathbb{R}^2$

Example: Regular hexagon



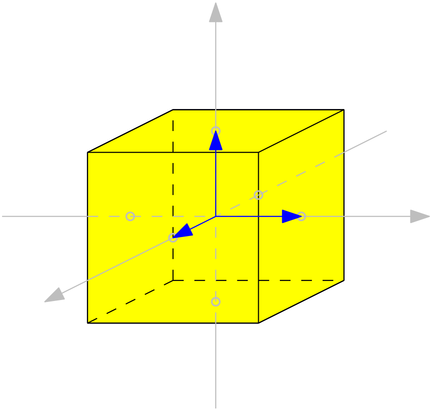
Generators $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}, \begin{pmatrix} -1/2 \\ -\sqrt{3}/2 \end{pmatrix} \in \mathbb{R}^2$

Example: A hexagon



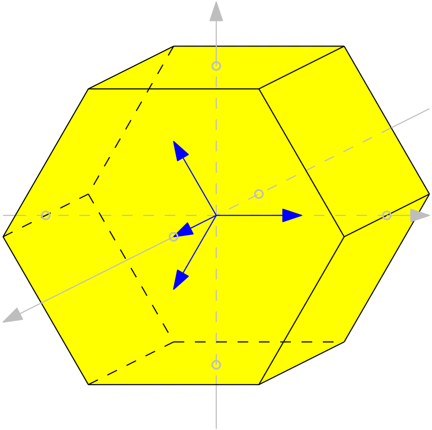
Generators $\begin{pmatrix} 3/4 \\ 1/2 \end{pmatrix}, \begin{pmatrix} -1/2 \\ -1/4 \end{pmatrix}, \begin{pmatrix} -3/4 \\ 1/4 \end{pmatrix} \in \mathbb{R}^2$

Example: Cube



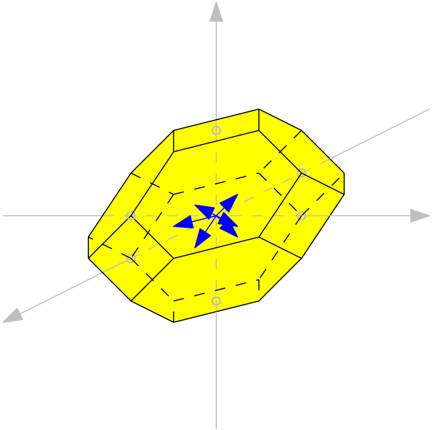
Generators $e_1, e_2, \ldots, e_d \in \mathbb{R}^d$

Example: Hexagonal prism



Generators $\begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\-1/2\\\sqrt{3}/2\end{pmatrix}, \begin{pmatrix}0\\-1/2\\-\sqrt{3}/2\end{pmatrix}, \begin{pmatrix}1/2\\0\\0\end{pmatrix} \in \mathbb{R}^3$

Example: Truncated octahedron



Generators $\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix} \in \mathbb{R}^3$

An equivalent definition of a zonotope

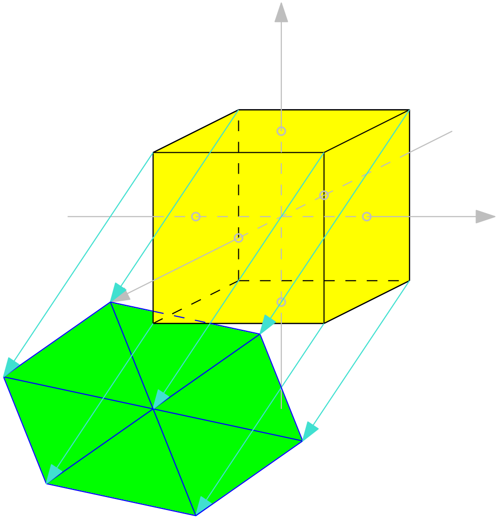
Reminder: k-dimensional cube $C^k = \left\{ \sum_{i=1}^{k} \lambda_i e_i \mid \lambda_i \in [-1, 1] \ \forall \ i \right\}$

Proposition

A polytope $P \subseteq \mathbb{R}^d$ is a zonotope if and only if
$\exists$ a natural number $k \geq 0$ and a matrix $A \in \mathbb{R}^{d \times k}$ s.t.

$$P = \{Ax \mid x \in C^k\}$$

An equivalent definition of a zonotope: Example



Proof

Proof of "only if": Let $v_1, \ldots, v_m \in \mathbb{R}^d$ be the generators of P

- Let $A = [v_1, \ldots, v_m] \in \mathbb{R}^{d \times m}$
- Then, $v_i = Ae_i$
- Therefore,

$$
\begin{aligned}
&\left\{ \sum_{i=1}^{m} \lambda_i v_i \mid \lambda_i \in [-1, 1] \text{ for all } i \in \{1, \ldots, m\} \right\} \\
&= \left\{ \sum_{i=1}^{m} \lambda_i A e_i \mid \lambda_i \in [-1, 1] \text{ for all } i \in \{1, \ldots, m\} \right\} \\
&= \left\{ A \sum_{i=1}^{m} \lambda_i e_i \mid \lambda_i \in [-1, 1] \text{ for all } i \in \{1, \ldots, m\} \right\} \\
&= \{ Ax \mid x \in C^m \}
\end{aligned}
$$

Proof, continued

Proof of "if":

$$\left\{Ax \mid x \in C^k\right\}$$

$$= \left\{A\sum_{i=1}^{k} \lambda_i e_i \mid \lambda_i \in [-1,1] \text{ for all } i \in \{1,\dots,k\}\right\}$$

$$= \left\{\sum_{i=1}^{k} \lambda_i A e_i \mid \lambda_i \in [-1,1] \text{ for all } i \in \{1,\dots,k\}\right\}$$

Hence, it is a zonotope with generators $Ae_1, \dots, Ae_k$ □

Projections

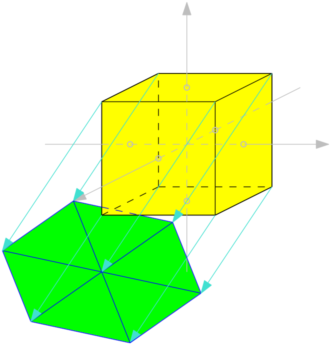
Namely, *a zonotope is a projection of a cube*

Projection

For a matrix $A \in \mathbb{R}^{d \times k}$,
let $\pi_A \colon \mathbb{R}^k \to \mathbb{R}^d$ be

$$\pi_A(x) = Ax;$$

A polytope $P \subseteq \mathbb{R}^d$ is a **projection** of a polytope $Q \subseteq \mathbb{R}^k$ if $P = \pi_A(Q)$ for some $A \in \mathbb{R}^{k \times d}$

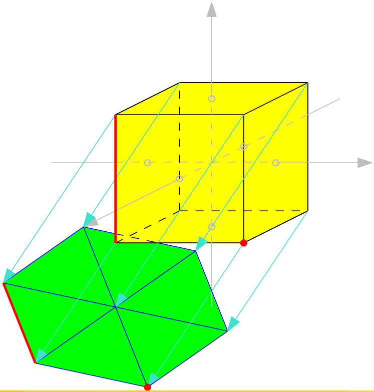


Faces of a projection

$P \subseteq \mathbb{R}^d, Q \subseteq \mathbb{R}^k, \pi_A(Q) = P$

Fact

- F a face of $P \Leftrightarrow \pi_A^{-1}(F)$ a face of Q
- For faces F, F' of P: $F \subseteq F' \Leftrightarrow \pi_A^{-1}(F) \subseteq \pi_A^{-1}(F')$



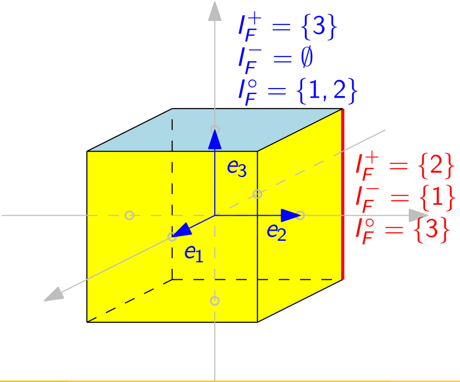
1. Zonotopes

2. Relationship with hyperplane arrangements

Assigning sign vectors to faces of a cube (1)

A face F of a cube C^k can be determined by partitioning $\{1,\dots,k\}$ into three parts I_F^+, I_F^-, I_F° so that

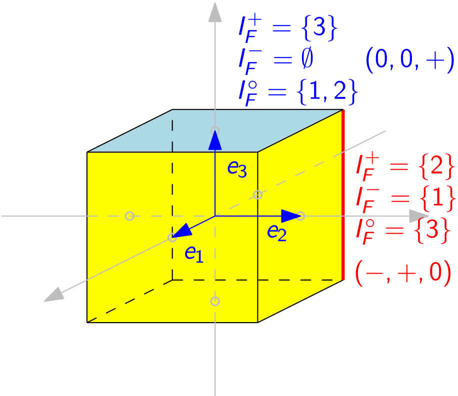
$$F = \left\{ \sum_{i=1}^{k} \lambda_i e_i \;\middle|\; \begin{array}{rcll} & \lambda_i & = 1 & \forall\, i \in I_F^+, \\ & \lambda_i & = -1 & \forall\, i \in I_F^-, \\ -1 \leq & \lambda_i & \leq 1 & \forall\, i \in I_F^\circ \end{array} \right\}$$



Assigning sign vectors to faces of a cube (2)

For each face F, we assign a sign vector $\sigma(F) \in \{+, -, 0\}$ by

$$\sigma(F)_i = \begin{cases} + & \text{if } i \in I_F^+, \\ - & \text{if } i \in I_F^-, \\ 0 & \text{if } i \in I_F^\circ \end{cases}$$

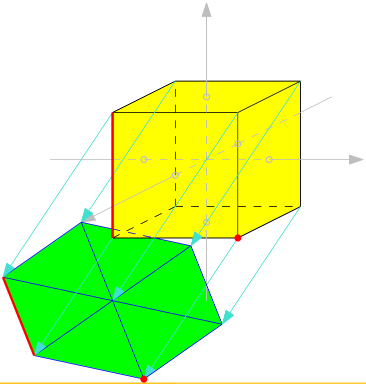


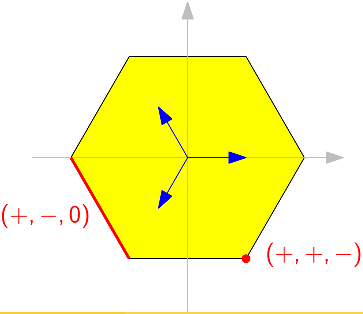
Assigning sign vectors to faces of a zonotope

$Z \subseteq \mathbb{R}^d$ a zonotope, as a projected image by π of the cube C^k

- By a fact before, each face F of Z has the corresponding face $\pi_A^{-1}(F)$ of C^k
- We assign a sign vector $\sigma(F) \in \{+,-,0\}^k$ by

$$\sigma(F) = \sigma(\pi_A^{-1}(F))$$



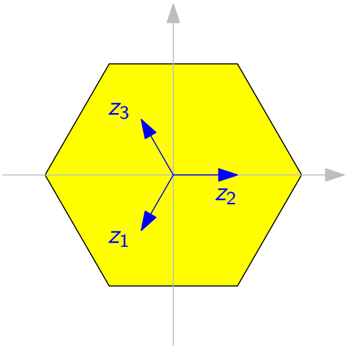


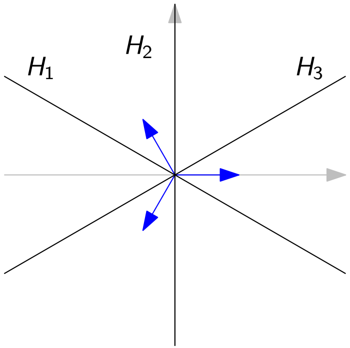
From zonotopes to hyperplane arrangements

$Z \subseteq \mathbb{R}^d$ a zonotope with generators $v_1, \ldots, v_n$

- Consider the hyperplane arrangement $\mathcal{A} = \{H_1, \ldots, H_n\}$ where

$$H_i = \{x \in \mathbb{R}^d \mid v_i \cdot x = 0\}$$





Correspondence of zonotopes and hyperplane arrangements (1)

- $Z \subseteq \mathbb{R}^d$ a zonotope with generators $v_1, \ldots, v_n$
- $\mathcal{A} = \{H_1, \ldots, H_n\}$ a hyperplane arrangement in $\mathbb{R}^d$, where $H_i = \{x \in \mathbb{R}^d \mid v_i \cdot x = 0\}$

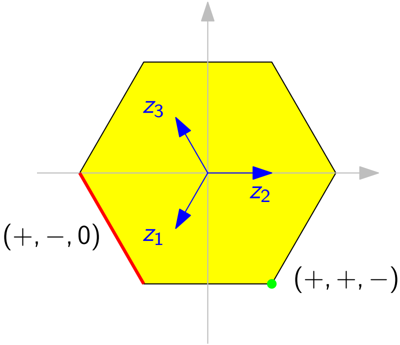
Fact

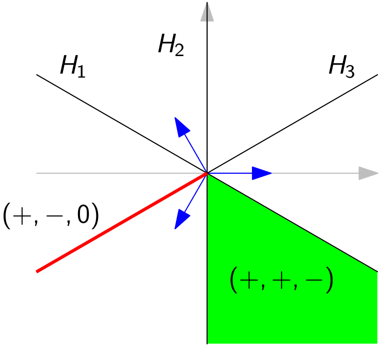
The following sets are identical

- $\{\sigma(F) \mid F \text{ a non-empty face of } Z, F \neq Z\} \cup \{0\}$
- $\mathcal{V}^*(\mathcal{A})$

Correspondence of zonotopes and hyperplane arrangements (1)

Example:





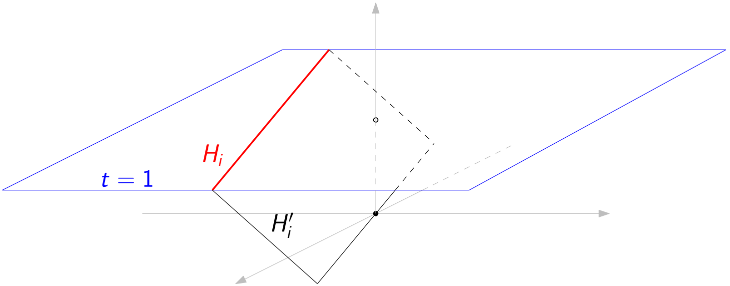
From hyperplane arrangements to zonotopes

$\mathcal{A} = \{H_1, \dots, H_n\}$ a hyperplane arrangement in $\mathbb{R}^d$, where
$H_i = \{x \in \mathbb{R}^d \mid a_i \cdot x = b_i\}$ with $a_i \in \mathbb{R}^d \setminus \{0\}$, $b_i \in \mathbb{R}$

- (**Homogenization**) For every $i \in \{1, \dots, n\}$

$$H'_i = \left\{ \begin{pmatrix} x \\ t \end{pmatrix} \in \mathbb{R}^{d+1} \mid a_i \cdot x - b_i t = 0 \right\}$$

- The zonotope Z generated by $\begin{pmatrix} a_1 \\ -b_1 \end{pmatrix}, \dots, \begin{pmatrix} a_n \\ -b_n \end{pmatrix} \in \mathbb{R}^{d+1}$



Correspondence of zonotopes and hyperplane arrangements (2)

- $\mathcal{A} = \{H_1, \ldots, H_n\}$ a hyperplane arrangement in $\mathbb{R}^d$, where $H_i = \{x \in \mathbb{R}^d \mid a_i \cdot x = b_i\}$
- $Z \subseteq \mathbb{R}^{d+1}$ a zonotope with generators $\begin{pmatrix} a_1 \\ -b_1 \end{pmatrix}, \ldots, \begin{pmatrix} a_n \\ -b_n \end{pmatrix}$

Fact

The following sets are identical

- $\mathcal{V}^*(\mathcal{A})$
- $\{\sigma(F) \mid F \text{ a non-empty face of } Z, F \neq Z\} \cup \{0\}$

Three "equivalent" geometric objects in terms of signed covectors

Finite point set

duality $\uparrow\downarrow$ duality

Hyperplane arrangement

generators to normal vectors $\uparrow\downarrow$ normal vectors to generators ($+$ homogenization)

Zonotope

Zonotopes

- Def.: the set of linear combinations of generators with coefficients bounded in $[-1, 1]$
- Equiv. def.: projection of a cube
- Natural assignment of sign vectors to faces of zonotopes

Relationship with hyperplane arrangements

- Natural correspondence to a hyperplane arrangement (thru 0)
- The sign vectors assigned to faces of a zonotope = The signed covectors of the hyperplane arrangement
- For the other direction: homogenize

Further reading

- Ziegler: *Lectures on Polytopes*
  - Lecture 7
- Edelsbrunner: *Algorithms in Combinatorial Geometry*
  - Chapters 1, 7

1. Zonotopes

2. Relationship with hyperplane arrangements