I631: Foundation of Computational Geometry
(9) Polytopes I

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Background

- Convex polygons are basic objects in computational geometry
- Convex polytopes are analogues of convex polygons in high dimensions

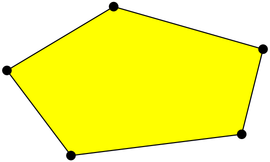
Goal of this lecture

- Learn the relevant notions for convex polytopes
- Acquaint yourself with some intuitions for convex polytopes

V-polytopes

V-polytopes

A set $P \subseteq \mathbb{R}^d$ is a **V-polytope** if
P is the convex hull of some finite point set



V-polytopes: Another example

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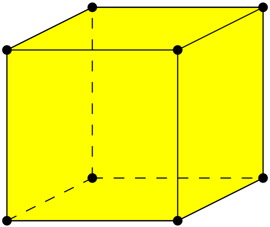
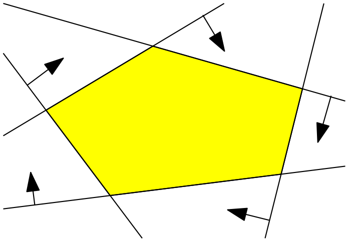


H-polytopes

H-polytopes

A set $P \subseteq \mathbb{R}^d$ is an **H-polytope** if
P is the intersection of a finite number of halfspaces, and *bounded*



H-polytopes: Another example

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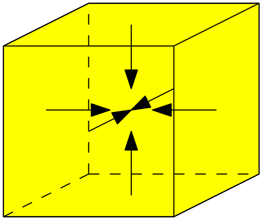
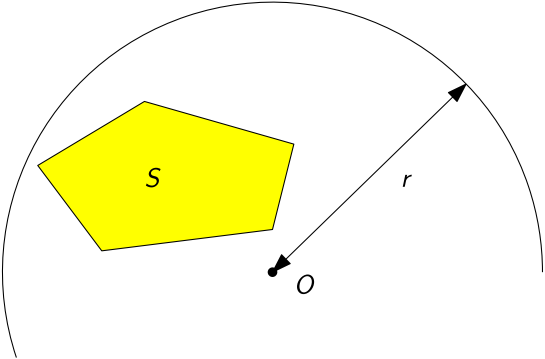


Reminder: Boundedness

Boundedness (reminder)

A set $S \subseteq \mathbb{R}^d$ is **bounded** if $\exists$ a real number $r \in \mathbb{R}$ such that

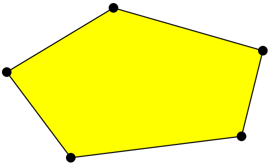
$$\|x\|_2 \leq r \qquad \text{for all } x \in S$$

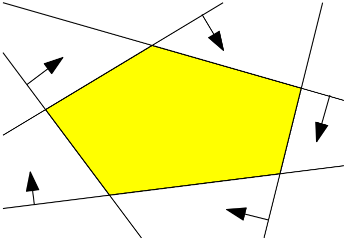


Equivalence of V-polytopes and H-polytopes

Facts

- Every V-polytope is an H-polytope
  - If P is a V-polytope, then there exists a finite number of halfspaces such that P is their intersection
- Every H-polytope is a V-polytope
  - If P is an H-polytope, then there exists a finite point set such that P is its convex hull





Polytopes

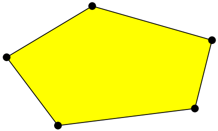
Def.: Polytopes

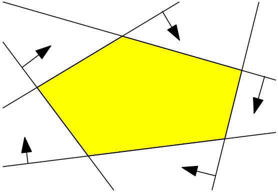
A **polytope** is a V-polytope or an H-polytope

V-representation and H-representation

P a polytope

- A **V-representation** of P is the description of P as the convex hull of a finite point set
- An **H-representation** of P is the description of P as the intersection of a finite number of halfspaces



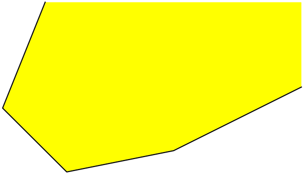


Remark: H-polyhedra

H-polyhedra

A set $P \subseteq \mathbb{R}^d$ is an **H-polyhedron** if
P is the intersection of a finite number of halfspaces

Namely, an H-polyhedron can be *unbounded*



Dimension of an affine subspace

- To define the dimension of a polytope,
  we first define the dimension of an affine subspace
- Let S be an affine subspace of $\mathbb{R}^d$ defined by

$$\{x \in \mathbb{R}^d \mid Ax = Ab'\}$$

for some natural number $k \leq d$, $A \in \mathbb{R}^{k \times d}$ and $b' \in \mathbb{R}^d$

Dimension of an affine subspace

S is ***r*-dimensional** if
the linear subspace $\{x \in \mathbb{R}^d \mid Ax = 0\}$ of $\mathbb{R}^d$ is r-dimensional;
Denote by $\dim(S) = r$

Dimension of an affine subspace: Example

$S = \{x \in \mathbb{R}^d \mid Ax = Ab'\}$ for some $k \leq d$, $A \in \mathbb{R}^{k \times d}$ and $b' \in \mathbb{R}^d$

Dimension of an affine subspace

S is ***r*-dimensional** if
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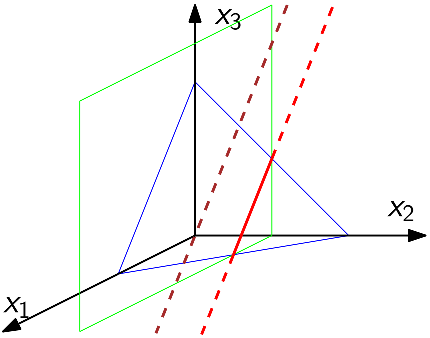


The dimension of a red line is 1

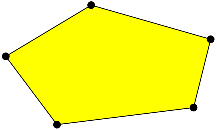
Dimension of a polytope

$P \subseteq \mathbb{R}^d$ a polytope

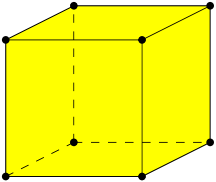
Dimension of a polytope

$$\dim(P) = \dim\Big(\bigcap_{S \text{ affine: } S \supseteq P} S\Big)$$

Namely, P is ***r*-dimensional** if
the minimal affine subspace containing P is r-dimensional



$\dim(P) = 2$



$\dim(P) = 3$

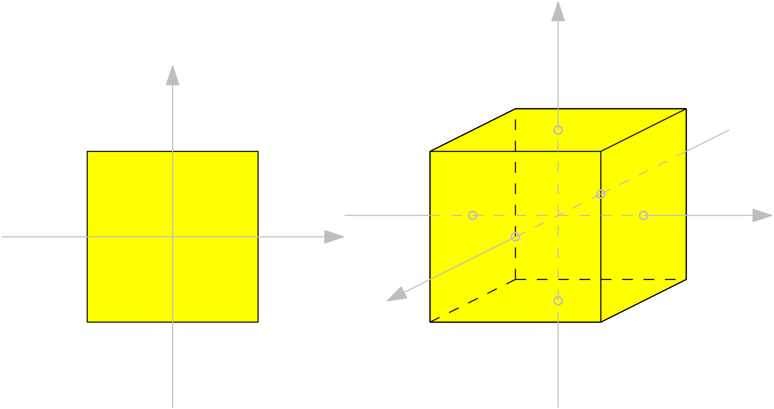
Closed intervals

A closed interval $I = [a, b] \subseteq \mathbb{R}$ is a polytope $(a \leq b)$

- V-representation: $I = \mathrm{conv}(\{a, b\})$
- H-representation: $I = \{x \in \mathbb{R} \mid x \geq a, x \leq b\}$
- $\dim(I) = \begin{cases} 1 & \text{if } a < b, \\ 0 & \text{if } a = b \end{cases}$

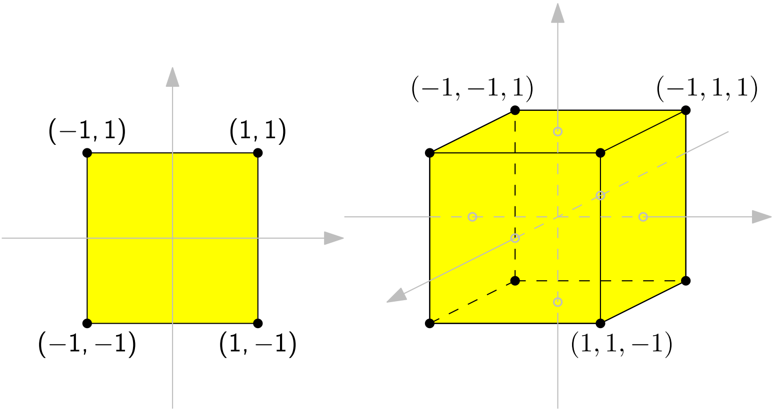
Cubes

A ***d*-dimensional cube** C_d is $[-1,1]^d$



Cubes: V-representations

A ***d*-dimensional cube** C_d is $[-1,1]^d$

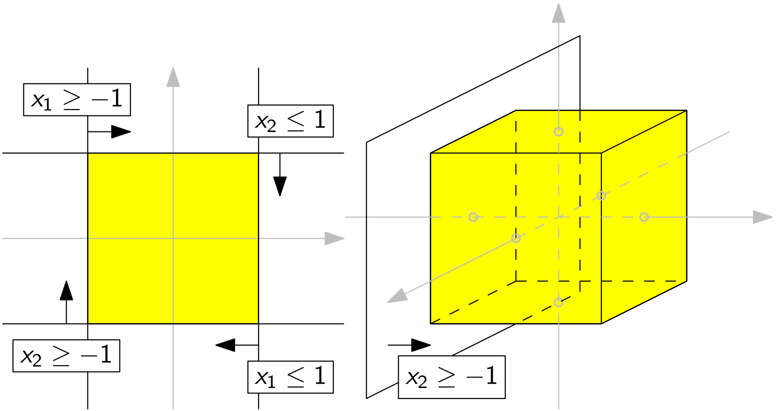


V-representation:

$$C_d = \mathrm{conv}(\{x \in \mathbb{R}^d \mid x_i \in \{-1,1\} \text{ for all } i \in \{1,\ldots,d\}\})$$

Cubes: H-representations

A ***d*-dimensional cube** C_d is $[-1, 1]^d$

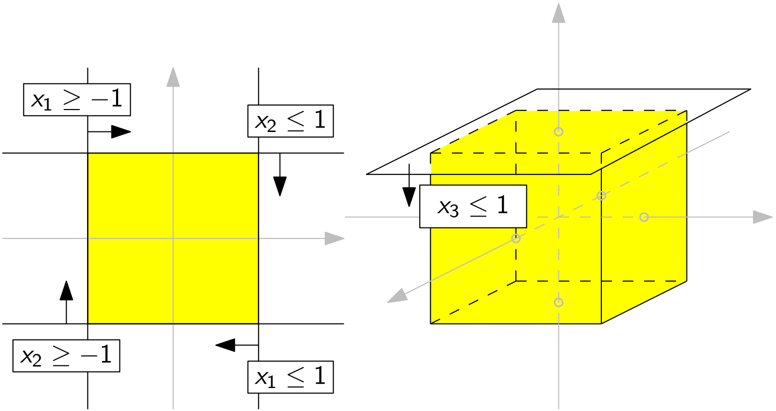


H-representation:

$$C_d = \{x \in \mathbb{R}^d \mid -1 \leq x_i \leq 1 \text{ for all } i \in \{1, \ldots, d\}\}$$

Cubes: H-representations

A ***d*-dimensional cube** C_d is $[-1, 1]^d$

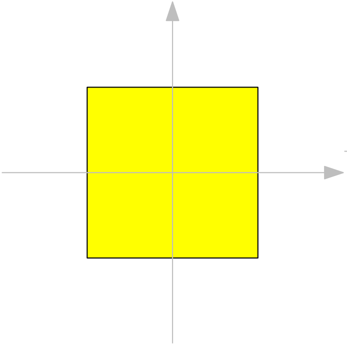


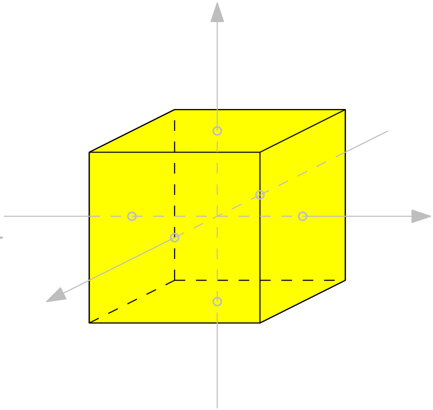
H-representation:

$$C_d = \{x \in \mathbb{R}^d \mid -1 \leq x_i \leq 1 \text{ for all } i \in \{1, \ldots, d\}\}$$

Cubes: Dimensions

A ***d*-dimensional cube** C_d is $[-1, 1]^d$



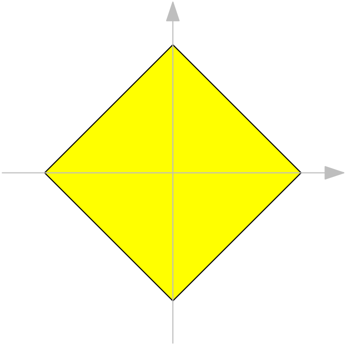


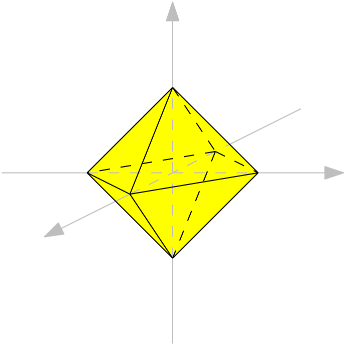
Dimension:

$$\dim(C_d) = d$$

Crosspolytopes

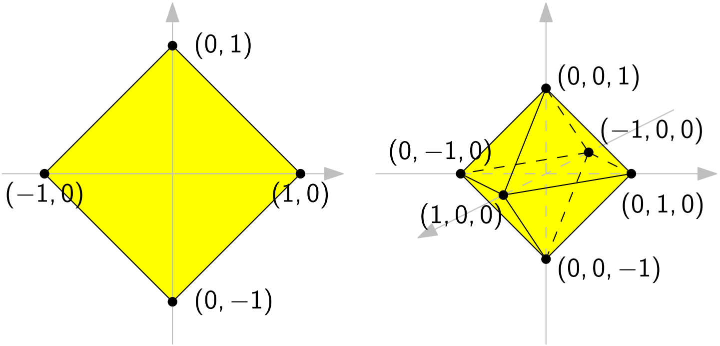
A **d-dimensional crosspolytope** C_d^* is $\left\{ x \in \mathbb{R}^d \mid \sum_{i=1}^{d} |x_i| \leq 1 \right\}$





Crosspolytopes: V-representations

A ***d*-dimensional crosspolytope** C_d^* is $\left\{ x \in \mathbb{R}^d \mid \sum_{i=1}^{d} |x_i| \leq 1 \right\}$

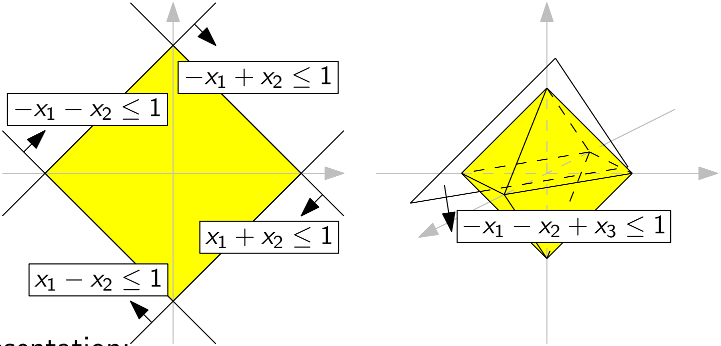


V-representation: If e_i denotes the ith standard basis vector

$$C_d^* = \mathrm{conv}(\{e_i \mid i \in \{1, \ldots, d\}\} \cup \{-e_i \mid i \in \{1, \ldots, d\}\})$$

Crosspolytopes: H-representations

A ***d*-dimensional crosspolytope** C_d^* is $\left\{ x \in \mathbb{R}^d \mid \sum_{i=1}^{d} |x_i| \leq 1 \right\}$

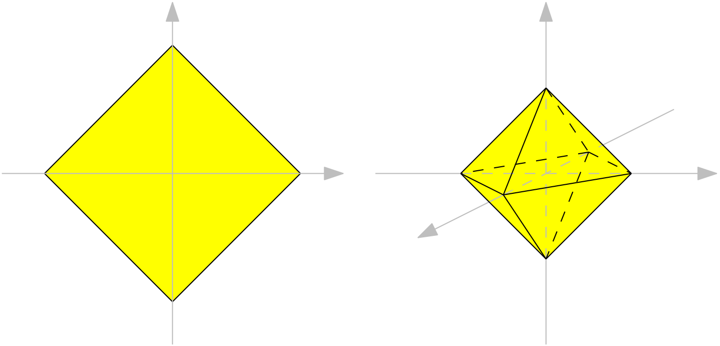


H-representation:

$$C_d^* = \left\{ x \in \mathbb{R}^d \mid \sum_{i=1}^{d} s_i x_i \leq 1, s_i \in \{-1, 1\} \text{ for all } i \in \{1, \ldots, d\} \right\}$$

Crosspolytopes: Dimension

A ***d*-dimensional crosspolytope** C_d^* is $\left\{ x \in \mathbb{R}^d \mid \sum_{i=1}^{d} |x_i| \leq 1 \right\}$



Dimension:

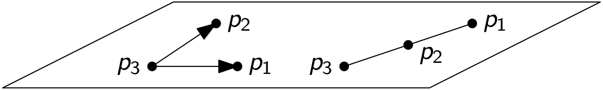
$$\dim(C_d^*) = d$$

Affine independence

To define a simplex, we first define affine independence

Affine independence

A set $P = \{p_1, p_2, \ldots, p_n\} \subseteq \mathbb{R}^d$ of points is **affinely independent** if the vectors $p_1 - p_n, p_2 - p_n, \ldots, p_{n-1} - p_n$ are linearly independent



affinely independent   affinely dependent

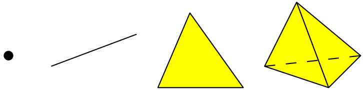
Property (Exercise)

$P \subseteq \mathbb{R}^d$ is affinely independent $\Rightarrow |P| \leq d + 1$

Simplices

Simplex

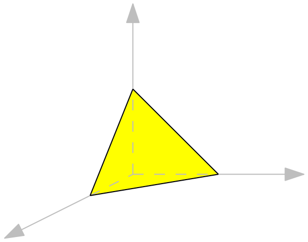
A ***d*-dimensional simplex** is the convex hull of a set of $d+1$ affinely independent points



Simplices: A canonical construction

A d-dimensional regular simplex

$$\Delta_d = \mathrm{conv}(\{e_i \in \mathbb{R}^{d+1} \mid i \in \{1, \ldots, d+1\}\})$$



Note: Δ_d lives in $\mathbb{R}^{d+1}$ but $\dim(\Delta_d) = d$

Valid inequalities

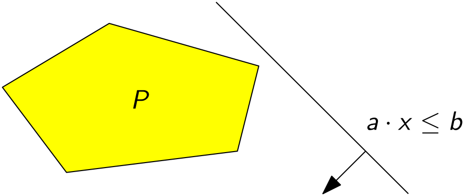
Valid inequalities

A **valid inequality** for a polytope $P \subseteq \mathbb{R}^d$ is an inequality

$$a \cdot x \leq b$$

for some $a \in \mathbb{R}^d$ and $b \in \mathbb{R}$ such that

$$\forall\, z \in P : a \cdot z \leq b$$



The halfspace $\{x \in \mathbb{R}^d \mid a \cdot x \leq b\}$ contains P (when $a \neq 0$)

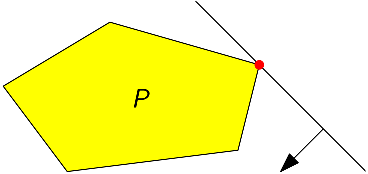
Faces

Faces

A **face** of a polytope $P \subseteq \mathbb{R}^d$ is a set

$$P \cap \{x \mid a \cdot x = b\},$$

where $a \cdot x \leq b$ is a valid inequality for P



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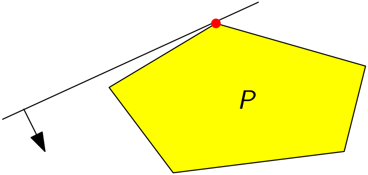

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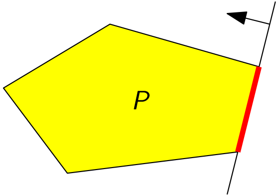


Special faces

$P \subseteq \mathbb{R}^d$ any polytope

- P is a face of P
  - Let $a = 0$ and $b = 0$, then $\{x \mid a \cdot x = b\} = \mathbb{R}^d$, and so
$$P \cap \{x \mid a \cdot x = b\} = P \cap \mathbb{R}^d = P$$
- $\emptyset$ is a face of P
  - Let $a = 0$ and $b = 1$, then $\{x \mid a \cdot x = b\} = \emptyset$, and so
$$P \cap \{x \mid a \cdot x = b\} = P \cap \emptyset = \emptyset$$

Faces of polytopes are polytopes

Observation

A face of a polytope is a polytope

Proof: Let P be a polytope and $F \subseteq P$ a face of P

- Let $F = P \cap \{x \mid a \cdot x = b\}$
  $= P \cap \{x \mid a \cdot x \leq b\} \cap \{x \mid a \cdot x \geq b\}$
- We know: P is the intersection of a finite number of halfspaces
- $\therefore F$ is also the intersection of a finite number of halfspaces
- $\therefore F$ is a polytope $\square$

Vertices, Edges, Ridges, Facets

Since faces are polytopes, the dimension of a face is naturally defined

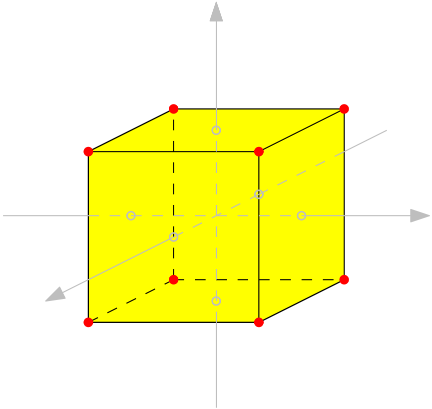
Faces with special names

P a d-dimensional polytope

- $\emptyset$: (-1)-dimensional face
- Vertex: 0-dimensional face
- Edge: 1-dimensional face
- Ridge: $(d-2)$-dimensional face
- Facet: $(d-1)$-dimensional face
- P: d-dimensional face

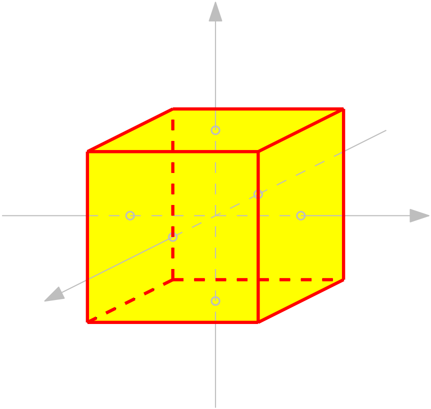
Example: Vertices

The 3-dim cube has eight vertices



Example: Edges

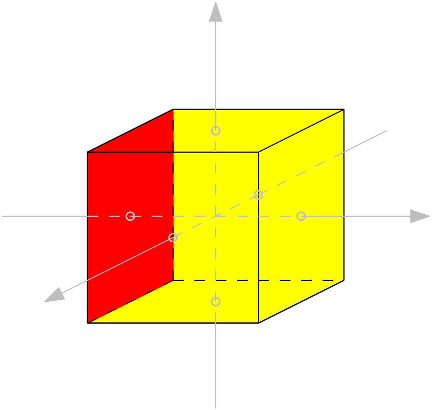
The 3-dim cube has twelve edges



In 3-dimensional polytopes, edges = ridges

Example: Facets

The 3-dim cube has six facets



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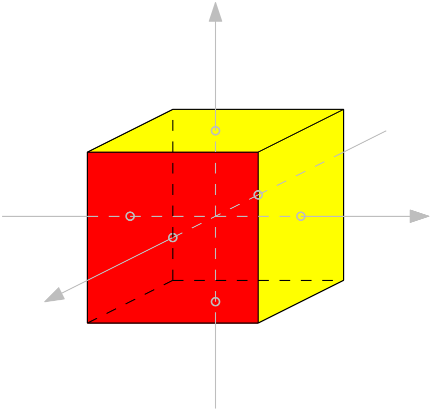

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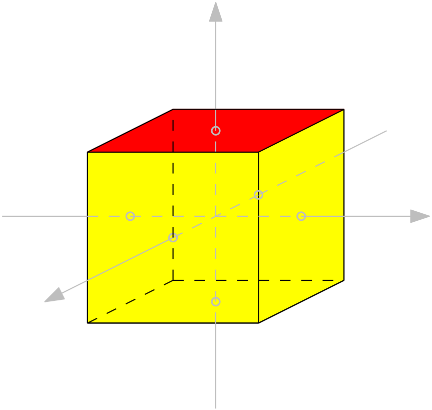

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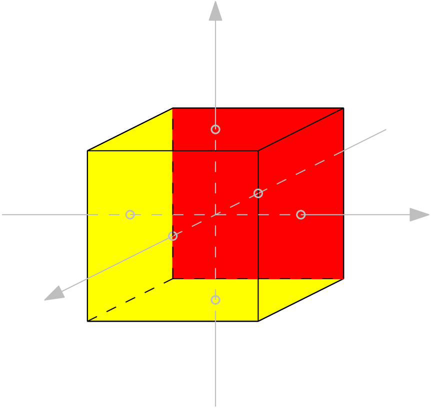

Example: Facets

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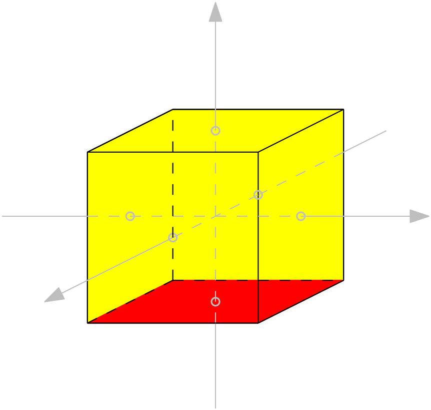

Example: Facets

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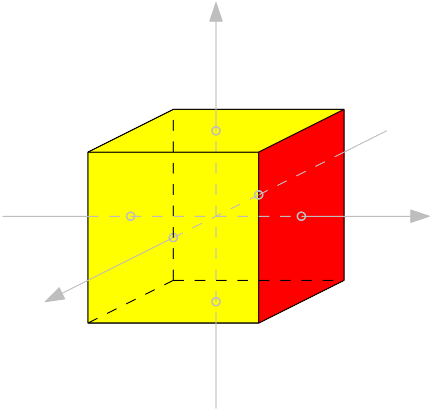
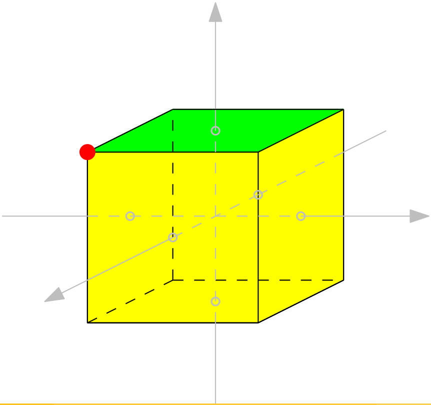


A face of a face of a polytope is a face of the polytope

Fact

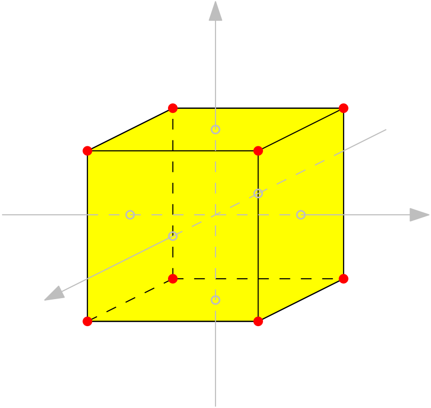
$$\left.\begin{array}{l} P \text{ a polytope} \\ F \subseteq P \text{ a face of } P \\ F' \subseteq F \text{ a face of } F \end{array}\right\} \quad \Rightarrow \quad F' \text{ a face of } P$$



A polytope is the convex hull of its vertices

Fact

$$\left.\begin{array}{l} P \text{ a polytope} \\ V \text{ the set of vertices of } P \end{array}\right\} \quad \Rightarrow \quad P = \mathrm{conv}(V)$$



Relationship of faces

Fact (recap)

$$\left.\begin{array}{l} P \text{ a polytope} \\ F \subseteq P \text{ a face of } P \\ F' \subseteq F \text{ a face of } F \end{array}\right\} \quad \Rightarrow \quad F' \text{ a face of } P$$

Consequence

- Consider the relation "F' is a face of F" on all faces of P
- The fact above implies that this relation is transitive
- This relation is also reflexive
  - P is a face of P
- This relation is also anti-symmetric
  - F is a face of F' and F' is a face of $F \Rightarrow F = F'$ (Exercise)
- $\therefore$ this relation defines a partial order

Face lattices

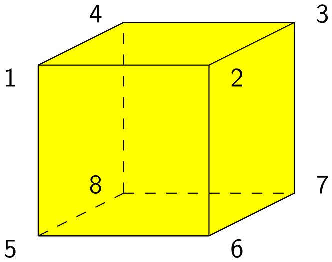
Face lattices

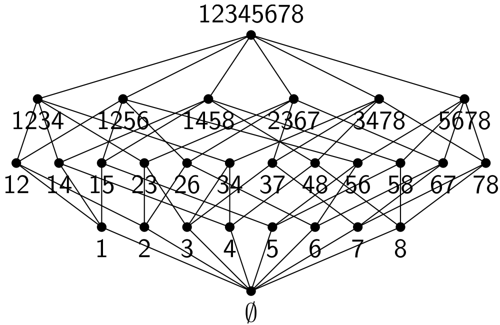
The **face lattice** of a polytope P is the partially ordered set $(\mathcal{F}, \leq)$ where

- $\mathcal{F}$ is the family of all faces of P
- $\forall\ F, F' \in \mathcal{F}$: $F' \leq F$ iff F' is a face of F

As the name suggests, this partially ordered set is actually a lattice, but this is not important in this lecture

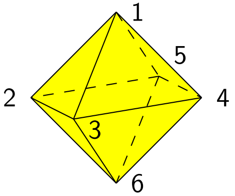
Example: The 3-dimensional cube C_3

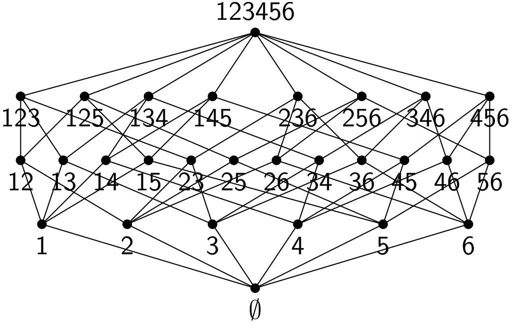




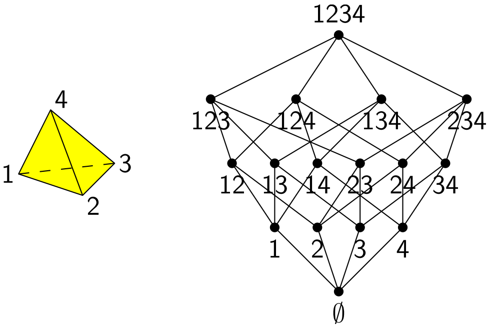
- The right figure shows a **Hasse diagram** of the face lattice
- For example, "1256" means "$\mathrm{conv}(\{1, 2, 5, 6\})$"

Example: The 3-dimensional crosspolytope C_3^*

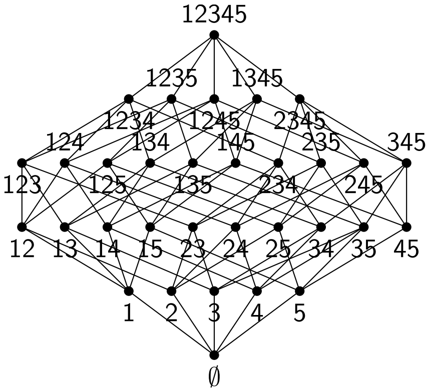




Example: A 3-dimensional simplex



Example: A 4-dimensional simplex



Combinatorial equivalence

Isomorphism of partially ordered sets

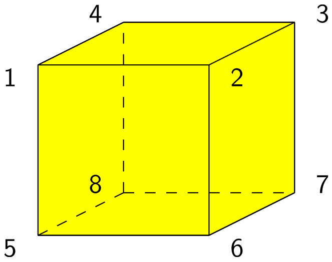
Two partially ordered sets $(X_1, \leq_1)$ and $(X_2, \leq_2)$ are **isomorphic** if $\exists$ a bijection $\varphi\colon X_1 \to X_2$ such that

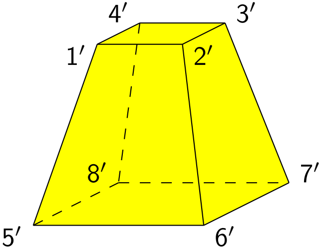
$$x_1 \leq_1 x_1' \qquad \Leftrightarrow \qquad \varphi(x_1) \leq_2 \varphi(x_1')$$

Combinatorial equivalence of polytopes

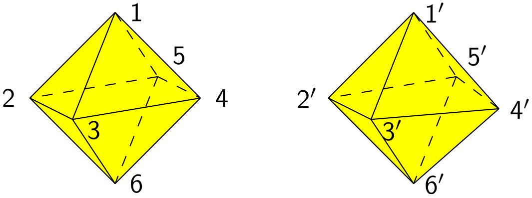
Two polytopes P and Q are **combinatorially equivalent** if their face lattices are isomorphic

Example 1: Combinatorial equivalence





Example 2: Combinatorial equivalence

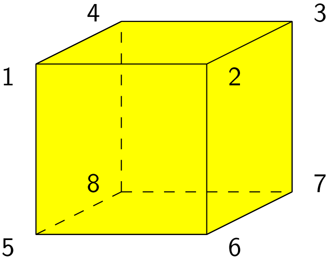


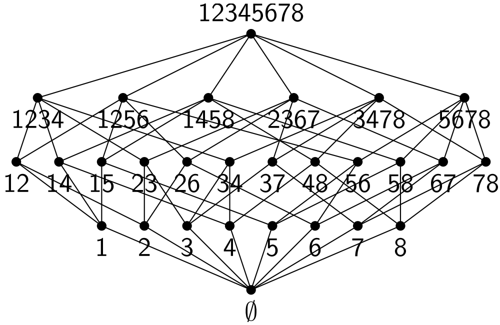
Note: The order types of the vertices are different

Simple polytopes

Simple polytopes

A d-dimensional polytope P is **simple** if every vertex is incident to d edges



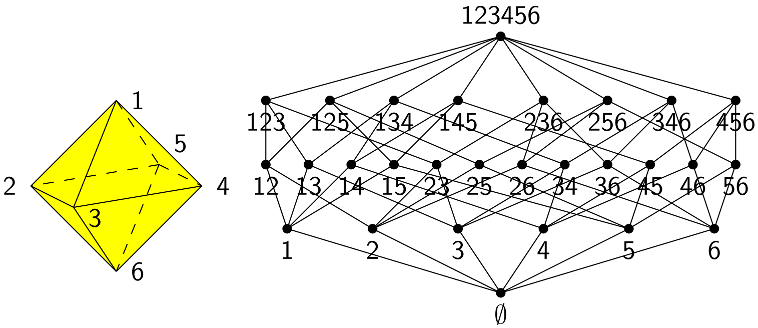


Cubes are simple

Simplicial polytopes

Simplicial polytopes

A d-dimensional polytope P is **simplicial** if every facet is incident to d ridges



Crosspolytopes are simplicial

Summary

Polytopes

- V-polytopes and H-polytopes
- Equivalence of V-polytopes and H-polytopes
- Examples: Cubes, Crosspolytopes, Simplices
- Simple polytopes and simplicial polytopes

Faces

- Def: Intersection of the polytope and a supporting hyperplane
- Vertices, edges, ..., ridges, facets
- Face lattices and combinatorial equivalence

Remarks

Equivalence of V-polytopes and H-polytopes

- They are **mathematically** identical objects
- However, we don't know they are **computationally** (or algorithmically) identical
  - We don't know an efficient algorithm to transform a V-representation of a polytope to an H-representation, and vice versa

Further reading

- Matoušek: *Lectures on Discrete Geometry*
  - Chapter 5
- Ziegler: *Lectures on Polytopes*
  - Lectures 0, 1, 2
- Edelsbrunner: *Algorithms in Combinatorial Geometry*
  - Chapters 1, 8