I631: Foundation of Computational Geometry (13) Envelopes and Levels I

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November 16, 2011

"Last updated: 2011/11/16 10:15"

Goal of this lecture

Background

- A hyperplane arrangement has abundant information
- Often, we're interested in substructures of a hyperplane arrangement
- Envelopes and levels are examples of such structures

Goal of this lecture

- Learn the relevant notions for envelopes and levels
- Learn the connection with Voronoi diagrams

Nearest neighbors

$P = \{p_1, \dots, p_n\} \subseteq \mathbb{R}^d$ a finite point set

Def.: Nearest neighbor

A **nearest neighbor** of a point $q \in \mathbb{R}^d$ in P is a point $p \in P$ such that

$$d(p, q) \leq d(p', q) \quad \forall\, p' \in P$$

Voronoi diagrams

$P = \{p_1, \ldots, p_n\} \subseteq \mathbb{R}^d$ a finite point set, $n \geq 2$

Def.: Voronoi diagram

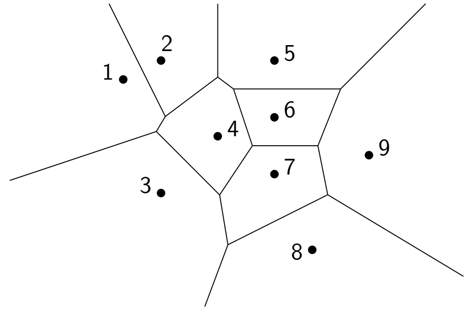
The **Voronoi diagram** of P is a partition of $\mathbb{R}^d$ by the regions

$$\mathrm{vor}(S) = \{q \in \mathbb{R}^d \mid S = \text{ the nearest neighbors of } q \text{ in } P\}$$

for all $S \subseteq P$, $|S| \geq 1$

- Denote the Voronoi diagram of P by $\mathrm{Vor}(P)$
- Each non-empty $\mathrm{vor}(S)$ is called the **Voronoi region** of S

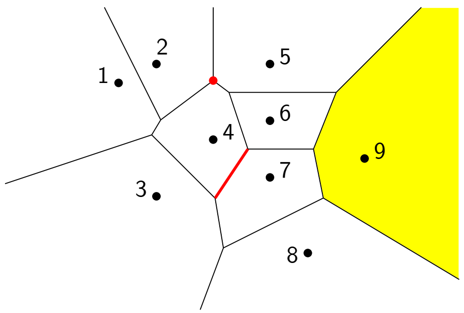
Example: Voronoi diagrams



Dimension of Voronoi regions

Def.: Dimension of a Voronoi region

The **dimension** of $\mathrm{vor}(S)$ is
the dimension of a minimal affine subspace containing $\mathrm{vor}(S)$



Special Voronoi regions

Voronoi regions have names according to their dimensions

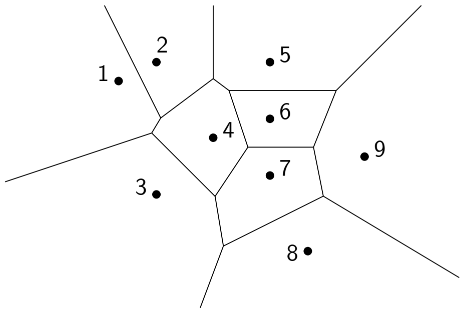
- Voronoi vertex: 0-dimensional Voronoi region
- Voronoi edge: 1-dimensional Voronoi region
- Voronoi ridge: $d-2$-dimensional Voronoi region
- Voronoi facet: $d-1$-dimensional Voronoi region
- Voronoi cell: d-dimensional Voronoi region

The number of Voronoi cells

Question

How many Voronoi cells can there be in the Voronoi diagram of a set of n points in $\mathbb{R}^d$?

This determines the intrinsic difficulty of the problem of computing the Voronoi diagram of a given point set



2-Nearest neighbors

$P = \{p_1, \ldots, p_n\} \subseteq \mathbb{R}^d$ a finite point set

Def.: 2-Nearest neighbor

A **2-nearest neighbor** of a point $q \in \mathbb{R}^d$ in P is a point $p \in P$ such that there exists a set X with $|X| \leq 1$ and

$$d(p, q) \leq d(p', q) \quad \forall\, p' \in P \setminus X$$

Order-2 Voronoi diagrams

$P = \{p_1, \ldots, p_n\} \subseteq \mathbb{R}^d$ a finite point set, $n \geq 3$

Def.: Order-2 Voronoi diagram

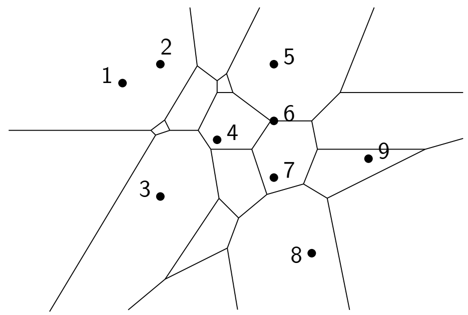
The **order-2 Voronoi diagram** of P is a partition of $\mathbb{R}^d$ by the regions

$$\mathrm{vor}^{(2)}(S) = \{q \in \mathbb{R}^d \mid S = \text{ the 2-nearest neighbors of } q \text{ in } P\}$$

for all $S \subseteq P$, $|S| \geq 2$

- Denote the order-2 Voronoi diagram of P by $\mathrm{Vor}^{(2)}(P)$
- Each non-empty $\mathrm{vor}^{(2)}(S)$ is called the **Voronoi region** of S
  - Dimensions and special names are defined similarly

Example: Order-2 Voronoi diagrams



k-Nearest neighbors

$P = \{p_1, \dots, p_n\} \subseteq \mathbb{R}^d$ a finite point set

Def.: k-Nearest neighbor

A **k-nearest neighbor** of a point $q \in \mathbb{R}^d$ in P is a point $p \in P$ such that there exist a set $X \subseteq P$ with $|X| \leq k-1$ and

$$d(p, q) \leq d(p', q) \quad \forall\, p' \in P \setminus X$$

Order-k Voronoi diagrams

$P = \{p_1, \dots, p_n\} \subseteq \mathbb{R}^d$ a finite point set, $n \geq k+1$

Def.: Order-k Voronoi diagram

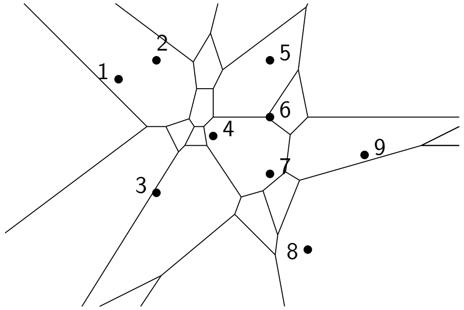
The **order-k Voronoi diagram** of P is a partition of $\mathbb{R}^d$ by the regions

$$\mathrm{vor}^{(k)}(S) = \{q \in \mathbb{R}^d \mid S = \text{ the } k\text{-nearest neighbors of } q \text{ in } P\}$$

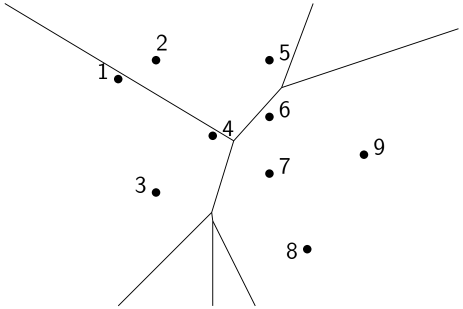
for all $S \subseteq P$, $|S| \geq k$

- Denote the Voronoi diagram of P by $\mathrm{Vor}^{(k)}(P)$
- Each non-empty $\mathrm{vor}^{(k)}(S)$ is called the **Voronoi region** of S
  - Dimensions and special names are defined similarly

Example: Order-3 Voronoi diagrams



Example: Farthest-point Voronoi diagrams



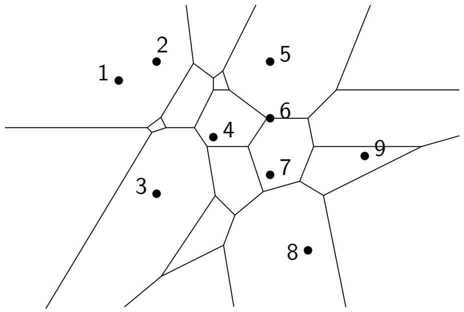
The order-$(n-1)$ Voronoi diagram is usually called the **farthest-point Voronoi diagram**

The number of Voronoi cells in $\mathrm{Vor}^{(k)}(P)$

Question

How many can Voronoi cells there be in the order-k Voronoi diagram of a set of n points in $\mathbb{R}^d$?

This determines the intrinsic difficulty of the problem of computing the order-k Voronoi diagram of a given point set



The rest of today's lecture

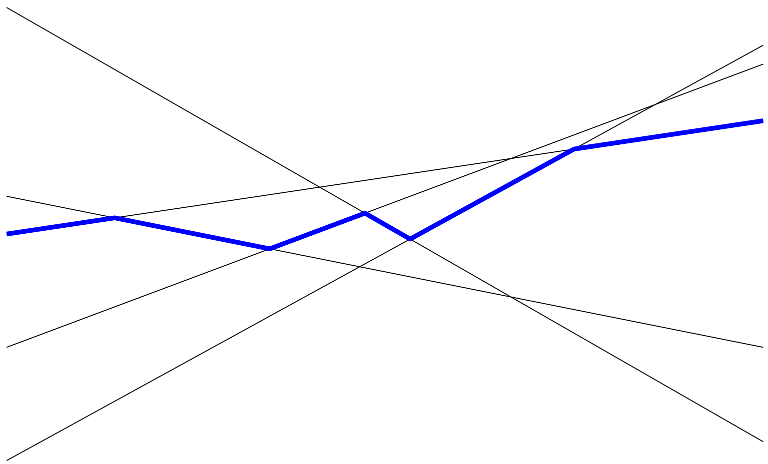
We will see

Voronoi diagrams and order-k Voronoi diagrams are closely related to envelopes and levels of hyperplane arrangements

The rest of the lecture

- Definition of envelopes and levels
- Relationship with Voronoi diagrams

Example: Levels of a hyperplane arrangement



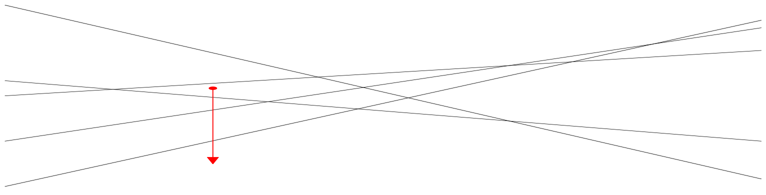
Levels of a point in a hyperplane arrangement

$\mathcal{A} = \{H_1, \ldots, H_n\}$ a hyperplane arrangement in $\mathbb{R}^d$ where
$H_i = \{x \in \mathbb{R}^d \mid a_i \cdot x = b_i\}$, $a_i \in \mathbb{R}^d \setminus \{0\}$ $(a_i \cdot e_d > 0)$, $b_i \in \mathbb{R}$

Level of a point

The **level** of a point $p \in \mathbb{R}^d$ in $\mathcal{A}$ is
the number of hyperplanes in $\mathcal{A}$ below p;
Alternatively, the level of p is k if

$$k = |\{i \in \{1, \ldots, n\} \mid a_i \cdot p < b_i\}|$$



Levels of a point in a hyperplane arrangement

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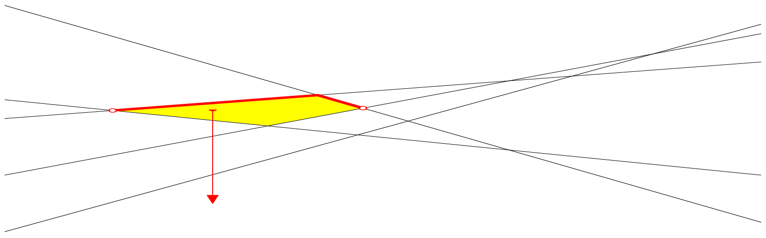
Observation

Every point in the same face in the arrangement has the same level



Levels of a hyperplane arrangement

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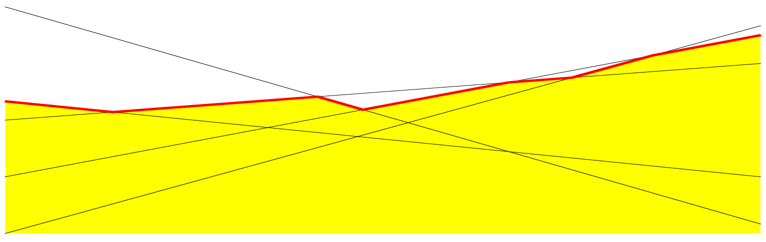
Def.: Levels of a hyperplane arrangement

The ***k*-level** of $\mathcal{A}$ is the boundary of the set of all points of level at most k

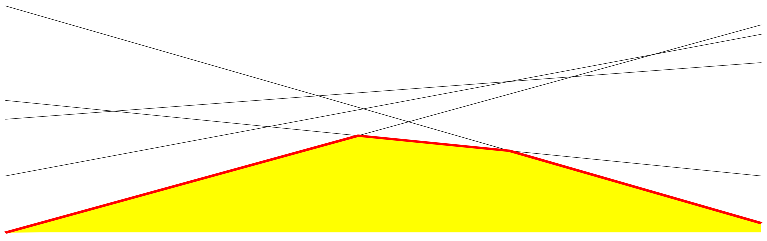


Lower envelope of a hyperplane arrangement

$\mathcal{A} = \{H_1, \ldots, H_n\}$ a hyperplane arrangement in $\mathbb{R}^d$ where
$H_i = \{x \in \mathbb{R}^d \mid a_i \cdot x = b_i\}$, $a_i \in \mathbb{R}^d \setminus \{0\}$ $(a_i \cdot e_d > 0)$, $b_i \in \mathbb{R}$

Def.: Lower envelope of a hyperplane arrangement

The **lower envelope** of $\mathcal{A}$ is the 0-level of $\mathcal{A}$

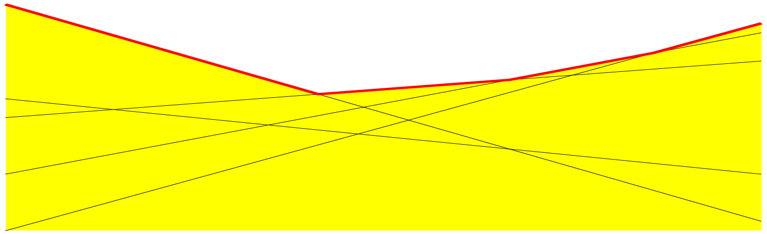


Upper envelope of a hyperplane arrangement

$\mathcal{A} = \{H_1, \ldots, H_n\}$ a hyperplane arrangement in $\mathbb{R}^d$ where
$H_i = \{x \in \mathbb{R}^d \mid a_i \cdot x = b_i\}$, $a_i \in \mathbb{R}^d \setminus \{0\}$ $(a_i \cdot e_d > 0)$, $b_i \in \mathbb{R}$

Def.: Upper envelope of a hyperplane arrangement

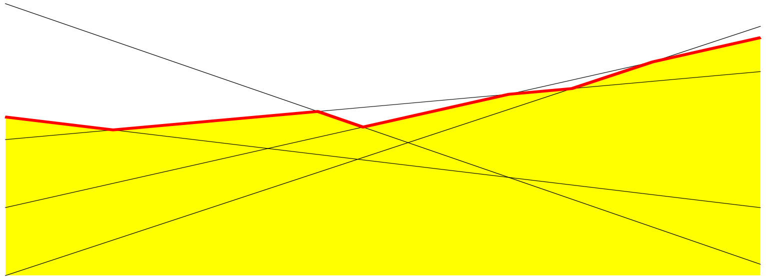
The **upper envelope** of $\mathcal{A}$ is the $(n-1)$-level of $\mathcal{A}$



Structures of levels

The k-level of a hyperplane arrangement is a collection of polyhedra

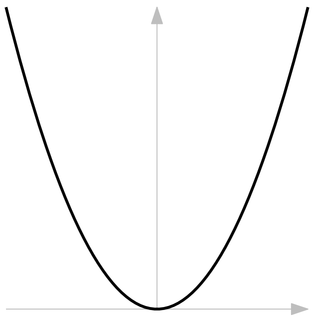
- They are faces of the hyperplane arrangements
- In particular, they are convex

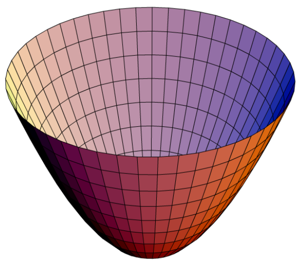


Unit paraboloids

Define the **unit paraboloid** in $\mathbb{R}^{d+1}$ as

$$U = \{x \in \mathbb{R}^{d+1} \mid x_{d+1} = x_1^2 + x_2^2 + \cdots + x_d^2\}$$

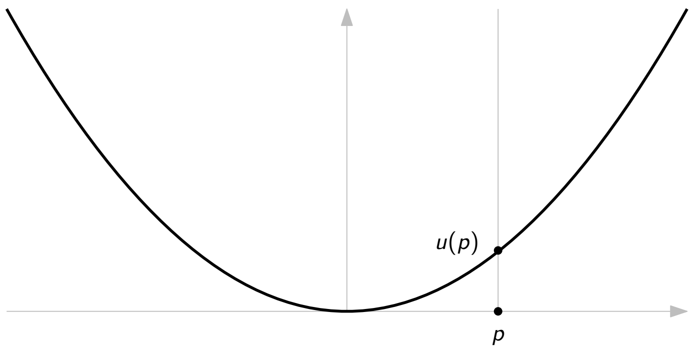




http://en.wikipedia.org/wiki/Paraboloid

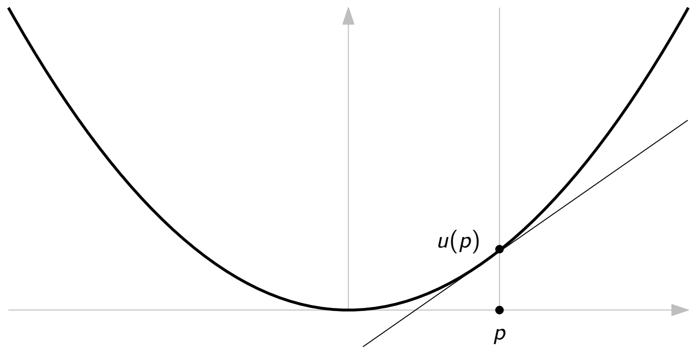
Lifting up a point set to the unit paraboloid

For $p \in \mathbb{R}^d$, let $u(p) = (p, p_1^2 + p_2^2 + \cdots + p_d^2) \in \mathbb{R}^{d+1}$



Tangent hyperplanes

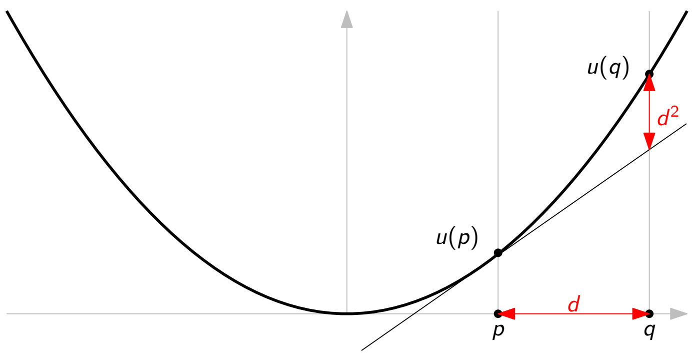
The hyperplane $H(p) = \{x \in \mathbb{R}^{d+1} \mid x_{d+1} = \sum_{i=1}^{d}(2p_i x_i - p_i^2)\}$ is tangent to U at $u(p)$ (Exercise)



Relationship with distances

Observation

For points $p, q \in \mathbb{R}^d$,
the vertical distance from q to $H(p)$ is $d(p,q)^2$



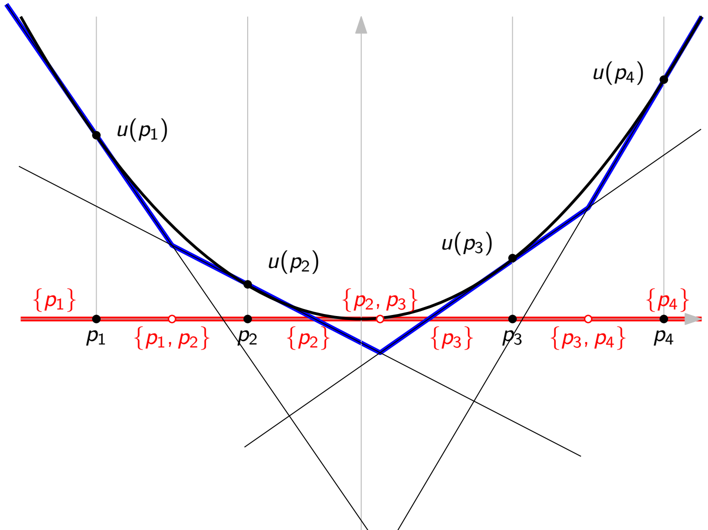
Proof of the observation

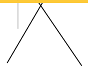
- $d(p,q)^2 = \sum_{i=1}^{d}(p_i - q_i)^2$
- The vertical distance from q to $H(p)$ is

$$
\begin{aligned}
&= u(q) - \sum_{i=1}^{d}(2p_i q_i - p_i^2) \\
&= \sum_{i=1}^{d} q_i^2 - \sum_{i=1}^{d}(2p_i q_i - p_i^2) \\
&= \sum_{i=1}^{d}(q_i^2 - 2p_i q_i + p_i^2) = \sum_{i=1}^{d}(p_i - q_i)^2
\end{aligned}
$$

- These two are equal □

Comparing distances

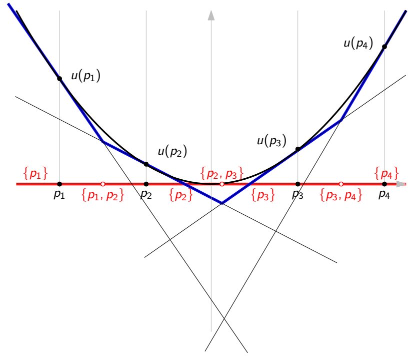




Relationship with Voronoi diagrams

Proposition

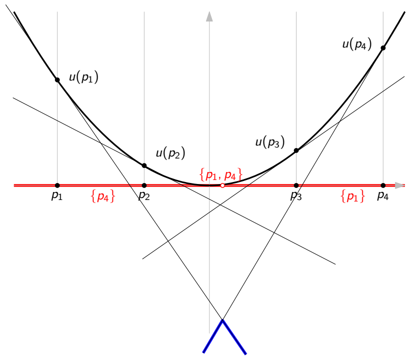
The Voronoi diagram of P is the projection of the upper envelope of the hyperplane arrangement $\{H(p) \mid p \in P\}$ to the hyperplane $\{x \in \mathbb{R}^{d+1} \mid x_{d+1} = 0\}$



Relationship with farthest-point Voronoi diagrams

Proposition

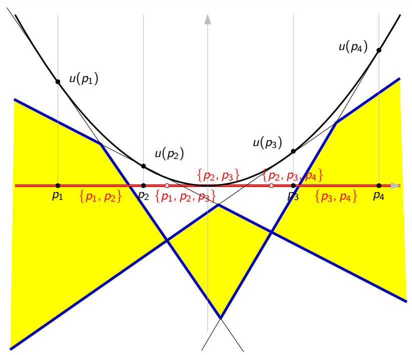
The farthest-point Voronoi diagram of P is the projection of the lower envelope of the hyperplane arrangement $\{H(p) \mid p \in P\}$ to the hyperplane $\{x \in \mathbb{R}^{d+1} \mid x_{d+1} = 0\}$



Relationship with order-2 Voronoi diagrams

Fact

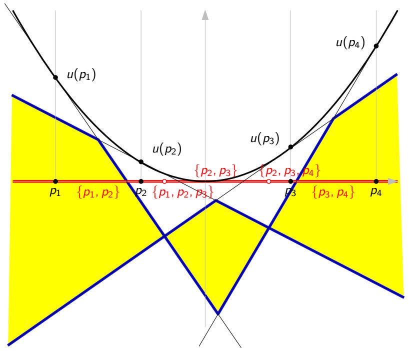
The order-2 Voronoi diagram of P is the projection of the part of the hyperplane arrangement $\{H(p) \mid p \in P\}$ between the $(n-2)$-level and the $(n-3)$-level to the hyperplane $\{x \in \mathbb{R}^{d+1} \mid x_{d+1} = 0\}$



Relationship with order-k Voronoi diagrams

Fact

The order-k Voronoi diagram of P is the projection of the part of the hyperplane arrangement $\{H(p) \mid p \in P\}$ between the $(n-k)$-level and the $(n-k-1)$-level to the hyperplane $\{x \in \mathbb{R}^{d+1} \mid x_{d+1} = 0\}$



The number of Voronoi cells in Voronoi diagrams

Consequence

The Voronoi diagram of a set of n points in $\mathbb{R}^d$ has

- at most n Voronoi cells
- at most $O(n^{\lceil d/2 \rceil})$ vertices

This is a consequence of the facts that

- The upper envelope is the boundary of a polyhedron with at most n facets
- The number of vertices of a d-dim polytope with n facets is $O(n^{\lfloor d/2 \rfloor})$ (Upper bound theorem)

The number of Voronoi cells in farthest-point Voronoi diagrams

Consequence

The farthest-point Voronoi diagram of a set of n points in $\mathbb{R}^d$ has

- at most n Voronoi cells
- at most $O(n^{\lceil d/2 \rceil})$ vertices

This is a consequence of the facts that

- The lower envelope is the boundary of a polyhedron with at most n facets
- The number of vertices of a d-dim polytope with n facets is $O(n^{\lfloor d/2 \rfloor})$ (Upper bound theorem)

Convexity of Voronoi regions

Consequence

Every Voronoi region of the order-k Voronoi diagram of a point set in $\mathbb{R}^d$ is convex

This is a consequence of the facts that

- Each face of the hyperplane arrangement is convex
- The projection of a convex set is convex (Exercise)

Summary

Voronoi diagrams

- Def.: A partition of $\mathbb{R}^d$ with respect to the distances to a given point set
- Terms: Voronoi regions, Voronoi cells, ...

Levels and Envelopes

- Level of a point: the number of hyperplanes below the point
- k-Level: the boundary of the set of points with level at most k
- Lower envelope: 0-level
- Upper envelope: $(n-1)$-level

Connection to Voronoi diagrams: Lift up to the unit paraboloid, and consider the hyperplane arrangement induced by tangents

A remark

Fact (Lee '82)

The order-k Voronoi diagram of a set of n points in $\mathbb{R}^2$ has $O(k(n-k))$ Voronoi cells; This bound is tight

Not much is known for the number of Voronoi cells in the order-k Voronoi diagrams when $d \geq 3$

Next lecture

- Even in $\mathbb{R}^2$, determining the maximum number of edges in the k-level of a line arrangement is a difficult problem
- We'll look at an argument that gives some upper bound

Further reading

- Matoušek: *Lectures on Discrete Geometry*
  - 4, 5, 11
- Edelsbrunner: *Algorithms in Combinatorial Geometry*
  - Chapters 1, 13