I631: Foundation of Computational Geometry (11) Hyperplane Arrangements I

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Goal of this lecture

Background

- A hyperplane arrangement is another central concept in discrete and computational geometry (and also in other fields of mathematics)
- It has a close relationship with other objects as finite point sets and polytopes

Goal of this lecture

- Learn the relevant notions for hyperplane arrangements
- Learn connections with finite point sets via duality

Hyperplane arrangements

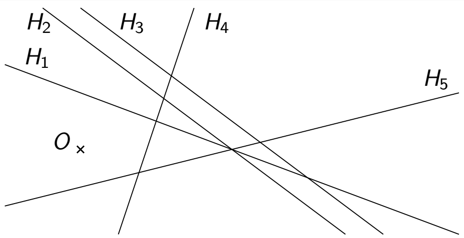
$d \geq 1$ a natural number

Def.: Hyperplane arrangement

A **hyperplane arrangement** is a finite set $\mathcal{A} = \{H_1, \ldots, H_n\}$ of hyperplanes in $\mathbb{R}^d$;

$$H_i = \{x \in \mathbb{R}^d \mid a_i \cdot x = b_i\}$$

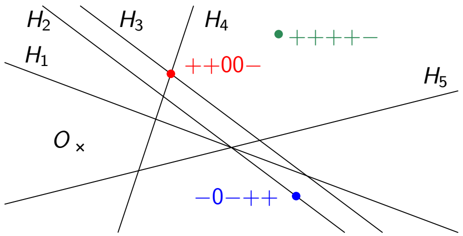
for some $a_i \in \mathbb{R}^d \setminus \{0\}$ and $b_i \in \mathbb{R}$



Assigning a sign vector to a point

- $\mathcal{A} = \{H_1, \dots, H_n\}$ a hyperplane arrangement,
  $H_i = \{x \in \mathbb{R}^d \mid a_i \cdot x = b_i\}$
- To a point $z \in \mathbb{R}^d$, assign the sign vector $\sigma(z) \in \{+, -, 0\}^n$:

$$\sigma(z)_i = \begin{cases} + & \text{if } a_i \cdot z > b_i, \\ 0 & \text{if } a_i \cdot z = b_i, \\ - & \text{if } a_i \cdot z < b_i \end{cases} \quad \text{for all } i \in \{1, \dots, n\}$$



Faces of a hyperplane arrangement

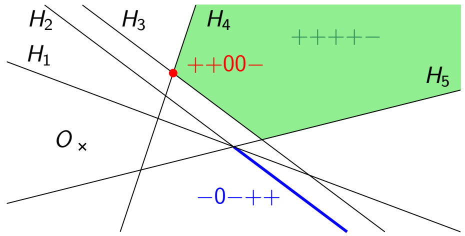
$\mathcal{A} = \{H_1, \dots, H_n\}$ a hyperplane arrangement

Def.: Face

A **face** of $\mathcal{A}$ is a set defined as

$$\{z \in \mathbb{R}^d \mid \sigma(z) = s\}$$

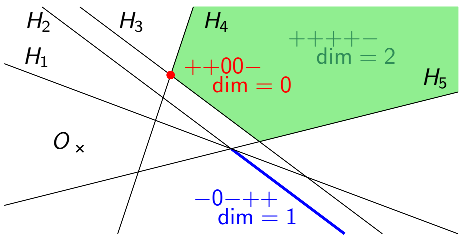
for some sign vector $s \in \{+, -, 0\}^n$



Dimension of a face

Dimension

The **dimension** of a face F of a hyperplane arrangement is the dimension of a minimal affine subspace containing F



Vertices, edges, ridges, facets, cells

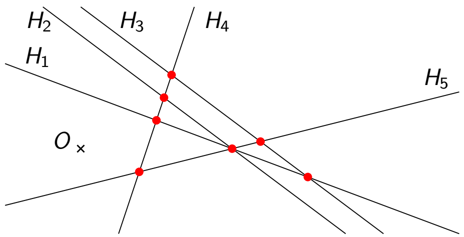
A face has a name according to its dimension

- Vertex: 0-dimensional face
- Edge: 1-dimensional face
- Ridge: $d-2$-dimensional face
- Facet: $d-1$-dimensional face
- Cell: d-dimensional face

- A face (more precisely, the closure of a face) is a polyhedron
- A cell is sometimes called a *region* or a *chamber*

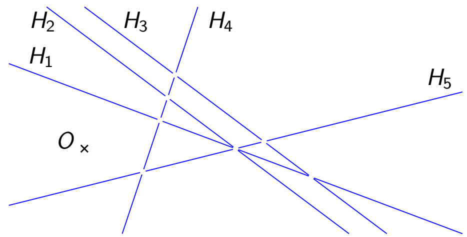
Examples: Vertices

This arrangement has seven vertices



Examples: Edges

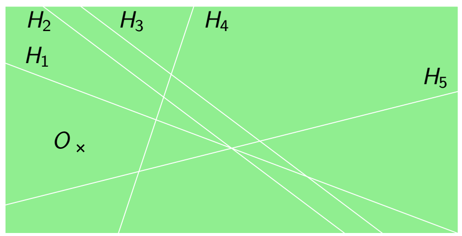
This arrangement has twenty edges;
Among them, ten are bounded and ten are unbounded



Edges are also facets in this arrangement

Examples: Cells

This arrangement has fourteen cells;
Among them, four are bounded and ten are unbounded



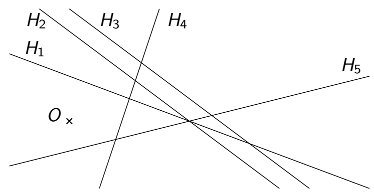
Simple arrangements

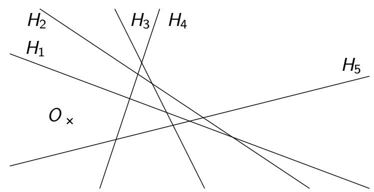
Simple arrangement

A hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^d$ is **simple** if the intersection of k hyperplanes in $\mathcal{A}$ is of dimension $d-k$ for all $k \in \{2, 3, \ldots, d+1\}$

In $\mathbb{R}^2$, the condition says

- The intersection of any two lines is a point, and
- The intersection of any three lines is empty

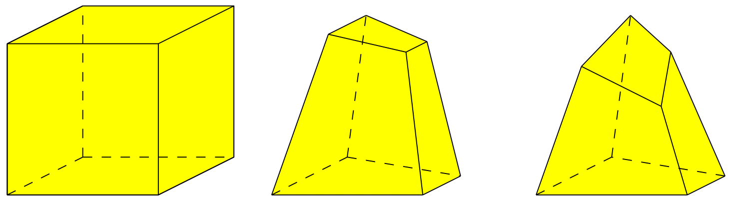




Simple arrangements in $\mathbb{R}^3$

In $\mathbb{R}^3$, the condition says

- The intersection of any two planes is a line,
- The intersection of any three planes is a point, and
- The intersection of any four planes is empty



The number of cells in simple hyperplane arrangements

Proposition

The # of cells of a simple arrangement of n hyperplanes in $\mathbb{R}^d$ is

$$\Phi_d(n) = \sum_{i=0}^{d} \binom{n}{i}$$

Proof: by induction on $n+d$

Base case: $n+d = 1, 2$ (then $n = 0$ or $(n, d) = (1, 1)$)

- When $n = 0$: The # of cells $= 1$
- When $n = 0$: $\Phi_d(n) = \Phi_d(0) = \sum_{i=0}^{d} \binom{0}{i} = 1$
- When $(n, d) = (1, 1)$: The # of cells $= 2$
- When $(n, d) = (1, 1)$: $\Phi_d(n) = \Phi_1(1) = \binom{1}{0} + \binom{1}{1} = 1 + 1 = 2$

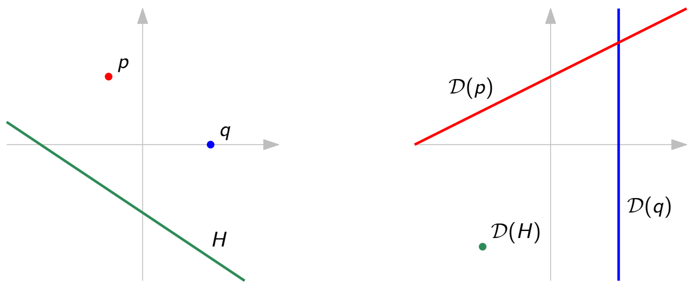
Proof continued

Induction step: Assume the statement holds for all $n' + d' < n + d$

- Consider adding one hyperplane to the arrangement of $n-1$ hyperplanes in $\mathbb{R}^d$
- Addition partitions several cells into two cells
- # partitioned cells $= \Phi_{d-1}(n-1)$ (by simplicity)
- Hence
$$\Phi_d(n) = \Phi_d(n-1) + \Phi_{d-1}(n-1)$$
- This recurrence has a unique solution, and $\sum_{i=0}^{d} \binom{n}{i}$ satisfies the recurrence (exercise) $\square$

Point–hyperplane duality

- For a hyperplane $H = \{x \in \mathbb{R}^d \mid a \cdot x = 1\}$ where $a \in \mathbb{R}^d \setminus \{0\}$
  its **dual** is a point
  $$\mathcal{D}(H) = a \in \mathbb{R}^d$$
- For a point $p \in \mathbb{R}^d \setminus \{0\}$
  its **dual** is a hyperplane
  $$\mathcal{D}(p) = \{x \in \mathbb{R}^d \mid p \cdot x = 1\}$$



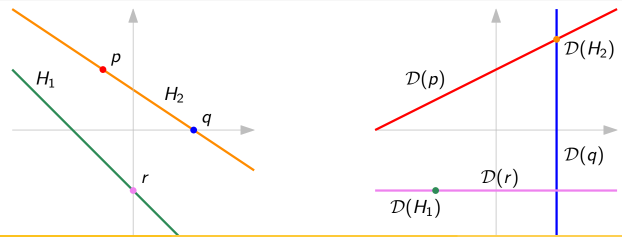
Incidence is preserved under duality

For a hyperplane $H = \{x \in \mathbb{R}^d \mid a \cdot x = 1\}$,
let $H^- = \{x \in \mathbb{R}^d \mid a \cdot x \leq 1\}$

Proposition

For a point $p \in \mathbb{R}^d \setminus \{0\}$ and
a hyperplane $H = \{x \in \mathbb{R}^d \mid a \cdot x = 1\}$ with $a \in \mathbb{R}^d \setminus \{0\}$

1. $p \in H \Leftrightarrow \mathcal{D}(p) \ni \mathcal{D}(H)$
2. $p \in H^- \Leftrightarrow \mathcal{D}(p)^- \ni \mathcal{D}(H)$



Proof of Proposition

Proposition

For a point $p \in \mathbb{R}^d \setminus \{0\}$ and
a hyperplane $H = \{x \in \mathbb{R}^d \mid a \cdot x = 1\}$ with $a \in \mathbb{R}^d \setminus \{0\}$

1. $p \in H \Leftrightarrow \mathcal{D}(p) \ni \mathcal{D}(H)$
2. $p \in H^- \Leftrightarrow \mathcal{D}(p)^- \ni \mathcal{D}(H)$

Proof of (1): (Proof of (2) is left as an exercise)

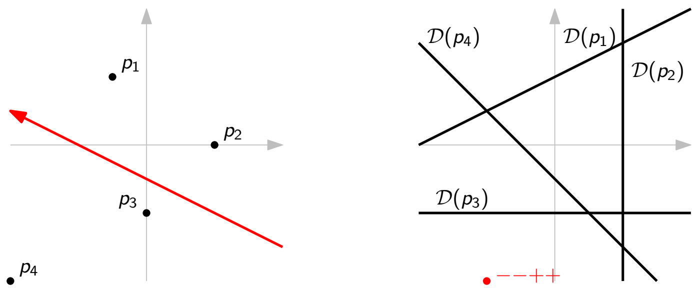
- $p \in H \Leftrightarrow a \cdot p = 1$
- $\mathcal{D}(p) = \{x \in \mathbb{R}^d \mid p \cdot x = 1\}$
- $\mathcal{D}(H) = a$
- $\mathcal{D}(p) \ni \mathcal{D}(H) \Leftrightarrow p \cdot a = 1$ □

The (one-way) correspondence of a signed covector and a face

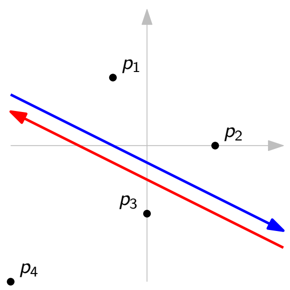
$P = \{p_1, \ldots, p_n\} \subseteq \mathbb{R}^d \setminus \{0\}$ a set of n points

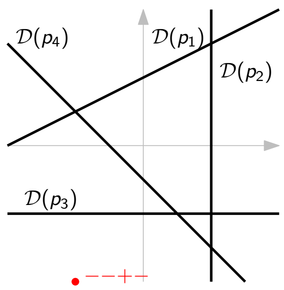
Fact

The arrangement $\mathcal{A} = \{\mathcal{D}(p_i) \mid i \in \{1, \ldots, n\}\}$ has a face
with a sign vector $s \in \{+, -, 0\}^n$
$\Rightarrow s \in \{+, -, 0\}^n$ is a signed covector of P



The converse doesn't hold



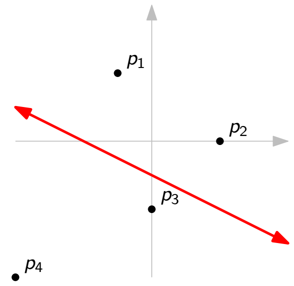


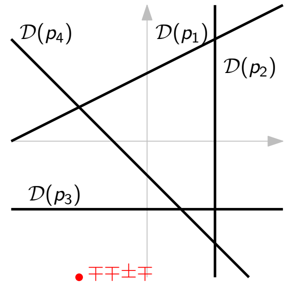
The (two-way) correspondence of a signed covector and a face

$P = \{p_1, \dots, p_n\} \subseteq \mathbb{R}^d \setminus \{0\}$ a set of n points

Fact

The arrangement $\mathcal{A} = \{\mathcal{D}(p_i) \mid i \in \{1, \dots, n\}\}$ has a face with a sign vector $s \in \{+, -, 0\}^n$
$\Leftrightarrow \pm s \in \{+, -, 0\}^n \setminus \{0\}$ are signed covectors of P





Goal of this section

- The facts above propose definitions of signed covectors and signed cocircuits of a hyperplane arrangement
- They encode combinatorial structures of a hyperplane arrangement

Signed covectors of a hyperplane arrangement

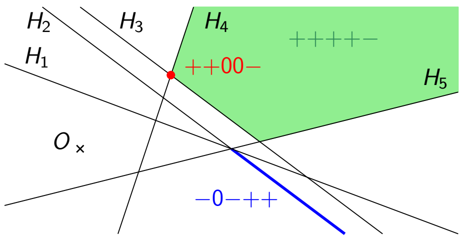
$\mathcal{A} = \{H_1, \ldots, H_n\}$ a hyperplane arrangement in $\mathbb{R}^d$
$H_i = \{x \in \mathbb{R}^d \mid a_i \cdot x = b_i\}$ where $a_i \in \mathbb{R}^d$ and $b_i \in \mathbb{R}$

Signed covectors

The **signed covectors** of $\mathcal{A}$ are the vectors in $\{+, -, 0\}^n$ defined as

$$\mathcal{V}^*(\mathcal{A}) = \{\pm(\mathrm{sgn}(a_1 \cdot x - b_1), \ldots, \mathrm{sgn}(a_n \cdot x - b_n)) \mid x \in \mathbb{R}^d\} \cup \{0\}$$

Each non-zero signed covector corresponds to a face



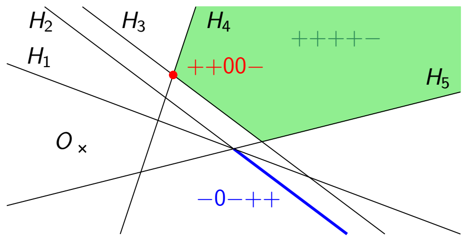
Signed cocircuits of a hyperplane arrangement

$\mathcal{A} = \{H_1, \ldots, H_n\}$ a hyperplane arrangement in $\mathbb{R}^d$
$H_i = \{x \in \mathbb{R}^d \mid a_i \cdot x = b_i\}$ where $a_i \in \mathbb{R}^d$ and $b_i \in \mathbb{R}$

Signed cocircuits

The **signed cocircuits** of $\mathcal{A}$ are the minimal elements in $\mathcal{V}^*(\mathcal{A}) \setminus \{0\}$;
The set of signed cocircuits of $\mathcal{A}$ is denoted by $\mathcal{C}^*(\mathcal{A})$

Each signed cocircuit corresponds to a face of minimum dimension



Duality and signed covectors and cocircuits

$P = \{p_1, \ldots, p_n\} \subseteq \mathbb{R}^d \setminus \{0\}$

Fact

Let $\mathcal{A}$ be the arrangement of n hyperplanes $H_1, \ldots, H_n$,
where $H_i = \mathcal{D}(p_i)$

$$\mathcal{V}^*(P) = \mathcal{V}^*(\mathcal{A}), \qquad \mathcal{C}^*(P) = \mathcal{C}^*(\mathcal{A})$$

$\mathcal{A} = \{H_1, \ldots, H_n\}$ a hyperplane arrangement in $\mathbb{R}^d$,
where $H_i = \{x \in \mathbb{R}^d \mid a_i \cdot x = 1\}$ for $a_i \in \mathbb{R}^d \setminus \{0\}$

Fact

Let $P \subseteq \mathbb{R}^d$ be a set of n points $\mathcal{D}(H_1), \ldots, \mathcal{D}(H_n)$

$$\mathcal{V}^*(\mathcal{A}) = \mathcal{V}^*(P), \qquad \mathcal{C}^*(\mathcal{A}) = \mathcal{C}^*(P)$$

Summary

Hyperplane arrangements

- Def.: a finite set of hyperplanes in $\mathbb{R}^d$
- Concepts: faces, cells, signed covectors, signed cocircuits
- $\#$ cells in a simple arrangement of n hyperplanes in $\mathbb{R}^d$ $= O(n^d)$ (d constant)

Duality

- A point $p \neq 0 \mapsto$ a hyperplane $\{x \mid p \cdot x = 1\}$
- A hyperplane $\{x \mid a \cdot x = 1\}$ with $a \neq 0 \mapsto$ a point a

Further reading

- Matoušek: *Lectures on Discrete Geometry*
  - Chapters 5, 6
- Ziegler: *Lectures on Polytopes*
  - Lecture 7
- Edelsbrunner: *Algorithms in Combinatorial Geometry*
  - Chapters 1, 7