I631: Foundation of Computational Geometry (10) Polytopes II

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Goal of this lecture

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- Look at the intrinsic difficulty of the convex hull computation
- Look at important concepts: Polarity, Cyclic polytopes, ...

Polar sets

The polar of a set

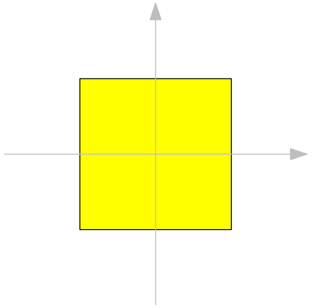
The **polar** of a set $S \subseteq \mathbb{R}^d$ is defined as

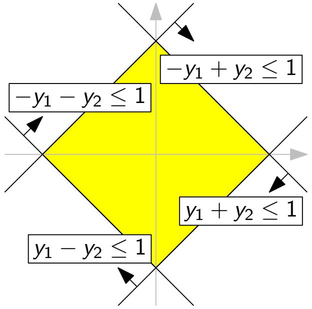
$$S^* = \{y \in \mathbb{R}^d \mid x \cdot y \leq 1 \quad \forall\, x \in S\}$$

Example

Let $S = [-1, 1]^2 \subseteq \mathbb{R}^2$, then

$$S^* = \{y \mid y_1 + y_2 \leq 1, y_1 - y_2 \leq 1, -y_1 + y_2 \leq 1, -y_1 - y_2 \leq 1\}$$





Proof of $\subseteq$:

- $S' = \{(1, 1), (1, -1), (-1, 1), (-1, -1)\} \subseteq S$
- RHS $= \{y \mid x \cdot y \leq 1 \; \forall\, x \in S'\} \supseteq \{y \mid x \cdot y \leq 1 \; \forall\, x \in S\}$ □

Example

Let $S = [-1, 1]^2 \subseteq \mathbb{R}^2$, then

$$S^* = \{y \mid y_1 + y_2 \leq 1, y_1 - y_2 \leq 1, -y_1 + y_2 \leq 1, -y_1 - y_2 \leq 1\}$$

Proof of $\supseteq$:

- Let $y \in$ RHS and $x \in S$
- To prove: $x \cdot y = x_1 y_1 + x_2 y_2 \leq 1$
- Case 1: $y_1 \geq 0$ and $y_2 \geq 0$
  - $x_1 y_1 + x_2 y_2 \leq y_1 + y_2$ $(\because x_1 \leq 1, x_2 \leq 1)$
  - $y_1 + y_2 \leq 1$ $(\because y \in \text{RHS})$
- Case 2: $y_1 \geq 0$ and $y_2 \leq 0$
- Case 3: $y_1 \leq 0$ and $y_2 \geq 0$
- Case 4: $y_1 \leq 0$ and $y_2 \leq 0$ (Exercise)

□

The polar of any set is convex

Proposition

$S \subseteq \mathbb{R}^d \Rightarrow S^*$ convex

Proof: Check S^* satisfies the condition in the def of convex sets

- Let $y_1, y_2 \in S^*$ and $\lambda \in [0, 1]$
- To prove: $\lambda y_1 + (1 - \lambda) y_2 \in S^*$
- A calculation follows...

The polar of a convex set is convex (cont'd)

- $x \cdot y_1 \leq 1$ for all $x \in S$ $(\because y_1 \in S^*)$
- $x \cdot y_2 \leq 1$ for all $x \in S$ $(\because y_2 \in S^*)$
- Hence, for all $x \in S$

$$\begin{aligned} x \cdot (\lambda y_1 + (1 - \lambda) y_2) &= \lambda x \cdot y_1 + (1 - \lambda) x \cdot y_2 \\ &\leq \lambda + (1 - \lambda) \\ &= 1 \end{aligned}$$

- $\therefore \lambda y_1 + (1 - \lambda) y_2 \in S^*$ □

The polar of a polytope is a polyhedron

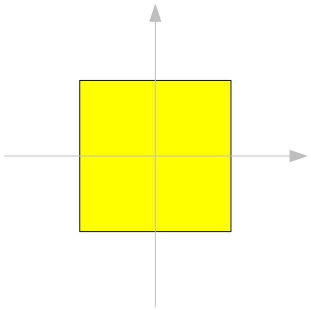
Fact

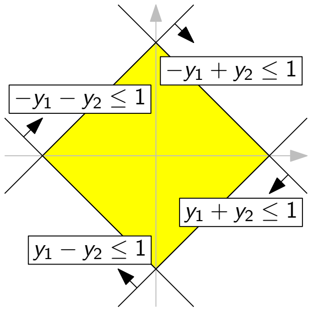
$P \subseteq \mathbb{R}^d$ a d-dimensional polytope

$\Rightarrow$ $P^* \subseteq \mathbb{R}^d$ a d-dimensional *polyhedron* (not necessarily bounded)

Moreover, $0 \in P$ in its interior

$\Rightarrow$ P^* bounded





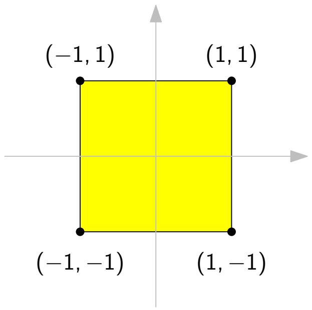
A V-representation of P gives an H-representation of P^*

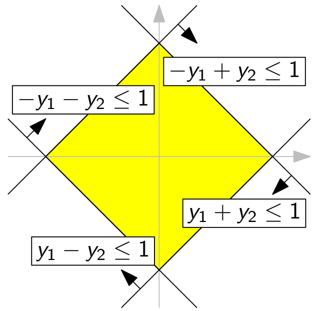
Fact

$P \subseteq \mathbb{R}^d$ a d-dimensional polytope, $0 \in P$ in its interior,

$P = \mathrm{conv}(V)$ for some finite point set $V = \{v_1, \ldots, v_n\} \subseteq \mathbb{R}^d$

$$\Rightarrow \qquad P^* = \{x \mid v_i \cdot x \leq 1 \quad \forall\, i \in \{1, \ldots, n\}\}$$





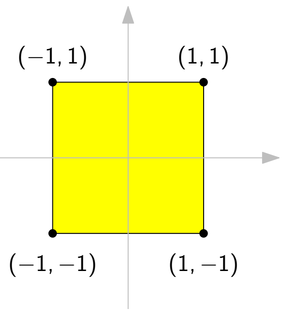
An H-representation of P gives a V-representation of P^*

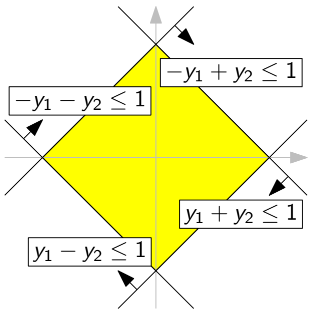
Fact

$P \subseteq \mathbb{R}^d$ a d-dimensional polytope, $0 \in P$ in its interior,

$P = \{x \mid a_i \cdot x \leq 1 \quad \forall\, i \in \{1, \ldots, n\}\}$ for some $a_1, \ldots, a_n \in \mathbb{R}^d$

$$\Rightarrow \qquad P^* = \mathrm{conv}(\{a_1, \ldots, a_n\})$$



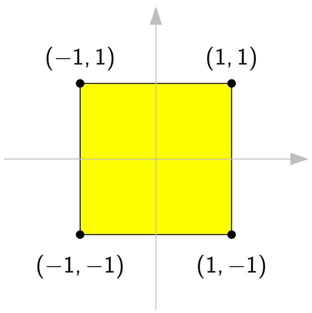


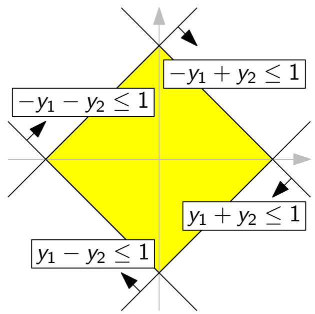
The polar of the polar of a polytope gives the polytope back

Corollary of the two facts above

$P \subseteq \mathbb{R}^d$ a d-dimensional polytope, $0 \in P$ in its interior

$$\Rightarrow \qquad (P^*)^* = P$$



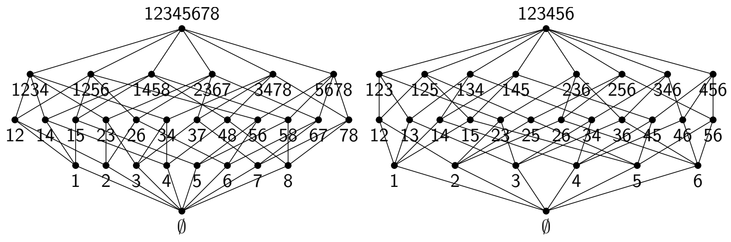


The polar polytope has the reverse face lattice

Fact

$P \subseteq \mathbb{R}^d$ a d-dimensional polytope, $0 \in P$ in its interior

$\Rightarrow \begin{cases} \text{The face lattice of } P^* \text{ is isomorphic to} \\ \text{the reverse of the face lattice of } P \end{cases}$



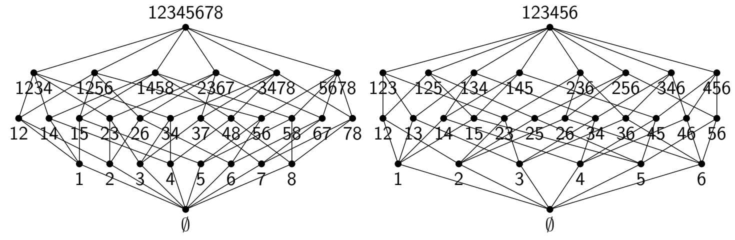
The face lattice of a 3-dim cube P The face lattice of a 3-dim crosspolytope P^*

Simple polytopes and simplicial polytopes

Corollary

$P \subseteq \mathbb{R}^d$ a d-dimensional polytope, $0 \in \mathbb{R}^d$ in its interior

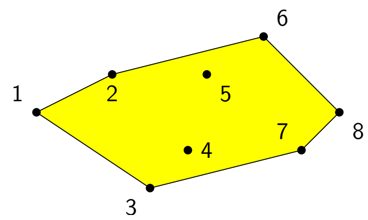
- P simple $\Rightarrow$ P^* simplicial
- P simplicial $\Rightarrow$ P^* simple



The face lattice of a 3-dim cube P The face lattice of a 3-dim crosspolytope P^*

Convex hull computation in the plane

- Input: A set of points in $\mathbb{R}^2$
- Output: The vertices of its convex hull in the clockwise order



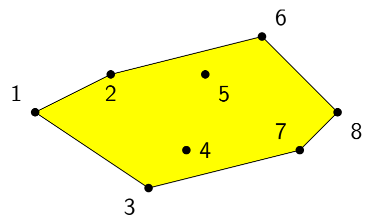
Output: 1–2–6–8–7–3

Basic facts for convex hulls in the plane

Facts

V a set of n points in $\mathbb{R}^2$

- $\mathrm{conv}(V)$ has at most n vertices
- $\mathrm{conv}(V)$ has at most n facets (or edges)



Convex hull computation in $\mathbb{R}^d$: From V to H

Problem: Convex hull computation (from V to H)

- Input: A V-representation of a polytope P
- Output: An H-representation of P

Typically

- Input: the set of vertices of P
- Output: the set of facets of P

Convex hull computation in $\mathbb{R}^d$: From H to V

Problem: Convex hull computation (from H to V)

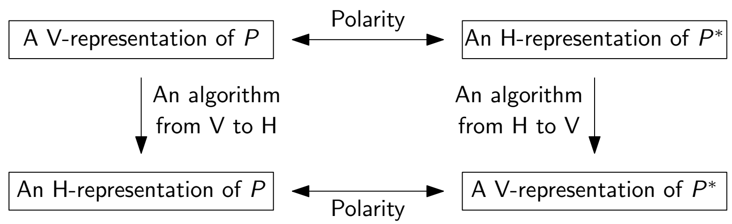
- Input: An H-representation of a polytope P
- Output: A V-representation of P

Typically

- Input: the set of facets of P
- Output: the set of vertices of P

A use of polarity

We have two kinds of convex hull computation problems,
but if you can solve one problem, you can solve the other...



Intrinsic difficulty is determined by the number of faces

Consider the convex hull computation from V to H

- Let n be the number of given points
- **How many facets can P have?**
  - This determines the time complexity that *every* algorithm for the convex hull computation problem from V to H needs to spend
- Trivial upper bound: $\binom{n}{d}$
  - Each facet contains at least d points from the input
- **What is the correct order of magnitude?**

The f-vector of a polytope

P a d-dimensional polytope

The f-vector of a polytope

The **f-vector** of P is a vector $f(P) = (f_{-1}(P), f_0(P), f_1(P), \dots, f_d(P))$ such that

$$f_i(P) = \text{ the number of } i\text{-dimensional faces of } P$$

for all $i \in \{1, 0, 1, \dots, d\}$

Remark

- $f_{-1}(P) = 1$
- $f_0(P) =$ the number of vertices of P
- $f_1(P) =$ the number of edges of P
- $f_{d-1}(P) =$ the number of facets of P
- $f_d(P) = 1$

Question

- P a d-dimensional polytope
- $f_0(P) = n$

Question

How large can $f_{d-1}(P)$ be?

We had a trivial upper bound $f_{d-1}(P) \leq \binom{n}{d} = O(n^d)$ if d const

The rest of this lecture

d is constant

- We look at polytopes with many facets
  - Cyclic polytopes
  - They have $\Omega(n^{\lfloor d/2 \rfloor})$ facets
- We look at the so-called "Upper Bound Theorem"
  - Saying "Cyclic polytopes have the largest number of facets"
  - We prove the asymptotics: Every polytope has $O(n^{\lfloor d/2 \rfloor})$ facets

1. Polarity
2. Convex hull computation: What does it mean?
3. **Cyclic polytopes: Polytopes with many faces**
4. The upper bound theorem

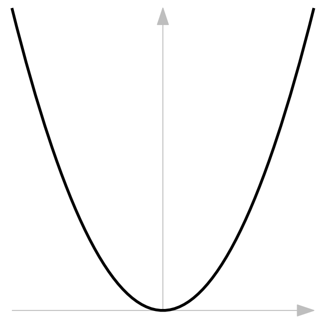
The moment curve

The moment curve

The **moment curve** is a curve in $\mathbb{R}^d$, $d \geq 2$, defined as

$$\{\gamma(t) = (t, t^2, \ldots, t^d) \in \mathbb{R}^d \mid t \in \mathbb{R}\}$$

When $d = 2$: $x_1 = t, x_2 = t^2$, so the curve is determined by $x_2 = x_1^2$



Every hyperplane intersects the moment curve with $\leq d$ pts

$d \geq 2$ a natural number

Observation

The intersection of the moment curve and every hyperplane in $\mathbb{R}^d$ consists of at most d points;
If it is exactly d, then the moment curve is not tangent to the hyperplane at the intersections

Proof of the 1st part: Let $a \cdot x = b$ defines a hyperplane in $\mathbb{R}^d$

- $\gamma(t)$ lies on the hyperplane $\Leftrightarrow a \cdot \gamma(t) = b$
- Then $a_1 t + a_2 t^2 + \cdots + a_d t^d = b$
- This is a degree-d polynomial in t
- Thus, it has at most d real solutions
- Each real solution corresponds to a point in the intersection □

Every hyperplane intersects the moment curve with $\leq d$ pts (cont'd)

$d \geq 2$ a natural number

Observation

The intersection of the moment curve and every hyperplane in $\mathbb{R}^d$ consists of at most d points;
If it is exactly d, then the moment curve is not tangent to the hyperplane at the intersections

Proof of the 2nd part: Let $a \cdot x = b$ defines a hyperplane in $\mathbb{R}^d$

- The polynomial has d distinct sol'ns $\Rightarrow$ they are simple roots
- Thus, not tangent at the corresponding intersections □

Corollary (Exercise)

Every cyclic polytope is simplicial

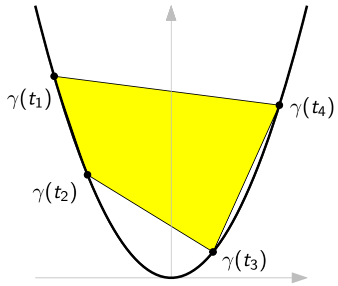
Cyclic polytopes

$d \geq 2$ a natural number, $n \geq d+1$ a natural number

Cyclic polytope

A **cyclic polytope** is the conv hull of n points on the moment curve: Let $t_1 < t_2 < \cdots < t_n$, and a cyclic polytope is defined as

$$\mathrm{conv}(\{\gamma(t_i) \mid i \in \{1,\dots,n\}\}) \subseteq \mathbb{R}^d$$



Gale's evenness criterion

$P = \mathrm{conv}(\{\gamma(t_1), \gamma(t_2), \dots, \gamma(t_n)\}) \subseteq \mathbb{R}^d$ a cyclic polytope

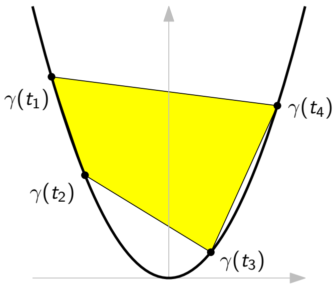
Theorem (Gale '63)

$F = \mathrm{conv}(\{\gamma(t_{i_1}), \gamma(t_{i_2}), \dots, \gamma(t_{i_d})\})$

F a facet of $P \Leftrightarrow$ $\#$ indices in $i_1, \dots, i_d$ between j & k is even $\forall\, j,k \notin \{i_1,\dots,i_d\}$

The theorem characterizes the facets of a cyclic polytope

Gale's evenness criterion: An example



	1	2	3	4
12	*	*		
23		*	*	
34			*	*
14	*			*

Gale's evenness criterion: Another example

- Consider a 4-dimensional cyclic polytope with 8 vertices
- The following table shows the list of its facets

	1	2	3	4	5	6	7	8
1678	*					*	*	*
1568	*				*	*		*
1458	*			*	*			*
1348	*		*	*				*
1238	*	*	*					*
1234	*	*	*	*				
1245	*	*		*	*			
1256	*	*			*	*		
1267	*	*				*	*	
1278	*	*					*	*

	1	2	3	4	5	6	7	8
2345		*	*	*	*			
2356		*	*		*	*		
2367		*	*			*	*	
2378		*	*				*	*
3456			*	*	*	*		
3467			*	*		*	*	
3478			*	*			*	*
4567				*	*	*	*	
4578				*	*		*	*
5678					*	*	*	*

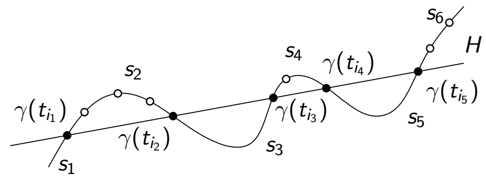
Consequences of Gale's evenness criterion

- Completely determines the facet of a cyclic polytope
- The number of facets can easily be calculated
- Implies that all d-dim cyclic polytopes with n vertices are combinatorially equivalent (having isomorphic face lattices)

Proof of Gale's evenness criterion

H the hyperplane containing F

- $\gamma(t_i) \in H$ for all $i \in \{i_1, \dots, i_d\}$
- H partitions the moment curve into $d+1$ pieces (why $d+1$?)
- F a facet $\Leftrightarrow$ $\gamma(t_i)$ lies on the same side of H $\forall\, i \notin \{i_1, \dots, i_d\}$
  $\Leftrightarrow$ $\gamma(t_i)$ lies only on the even-numbered pieces
  or only on the odd-numbered pieces
  $\Leftrightarrow$ Gale's evenness criterion is satisfied □



Asymptotic lower bound on the number of facets of a cyclic polytope

Observation

$C(d,n)$ a d-dimensional cyclic polytope with n vertices
$d \geq 2$ constant, $n \geq d+1$

$$\Rightarrow \quad f_{d-1}(C(d,n)) = \Omega(n^{\lfloor d/2 \rfloor})$$

The proof strategy

- Describe a way to choose d indices such that
  - they satisfy Gale's evenness criterion
  - we can choose sufficiently many ($\Omega(n^{\lfloor d/2 \rfloor})$)
- Idea: Pair adjacent indices

Asymptotic lower bound: d even

Indices $\{1, \dots, n\}$

- Construction
  - Make pairs $(1,2), (3,4), \dots, (i, i+1), \dots, (n-1, n)$ (or $(1,2), (3,4), \dots, (i, i+1), \dots, (n-2, n-1)$ if n odd)
  - Choose $d/2$ pairs
  - (This satisfies Gale's evenness criterion)
- The number of choices
  - Choosing $d/2$ pairs among $\lfloor n/2 \rfloor$ pairs
  - The number $= \binom{\lfloor n/2 \rfloor}{d/2} \geq \left(\frac{\lfloor n/2 \rfloor}{d/2}\right)^{d/2} = \Omega(n^{d/2})$

1	2	3	4	5	6	7	8	9	10	11
		*	*					*	*	

Asymptotic lower bound: d odd

Indices $\{1, \dots, n\}$

- Construction
  - Make pairs $(1,2), (3,4), \dots, (i, i+1), \dots, (n-2, n-1)$ (or $(1,2), (3,4), \dots, (i, i+1), \dots, (n-3, n-2)$ if n even)
  - Choose the index n and $(d-1)/2$ pairs
  - (This satisfies Gale's evenness criterion)
- The number of choices
  - Choosing $(d-1)/2$ pairs among $\lfloor (n-1)/2 \rfloor$ pairs
  - The number $= \Omega(n^{d/2})$ (with a similar calculation) □

1	2	3	4	5	6	7	8	9	10	11
		*	*			*	*			*

The upper bound theorem

Fact (McMullen '70)

P a d-dimensional polytope with n vertices
$C(d, n)$ a d-dimensional cyclic polytope with n vertices

$$\Rightarrow f_{d-1}(P) \leq f_{d-1}(C(d, n))$$

Namely, cyclic polytopes maximize the number of facets among all polytopes with the same number of vertices in the same dimension

The asymptotic upper bound theorem

The upper bound theorem is difficult to prove,
so we prove its asymptotic version, which is easier to prove

Theorem

P a d-dimensional *simplicial* polytope with n vertices
$d \geq 2$ constant, $n \geq d+1$

$$\Rightarrow f_{d-1}(P) = O(n^{\lfloor d/2 \rfloor})$$

The asymptotic upper bound theorem: Proof (1)

Proof strategy (Seidel '95)

- Consider P^*, which is a simple polytope

To prove

$f_0(P^*) \leq 2f_{\lceil d/2 \rceil}(P^*)$ for any d-dim simple polytope P^*

- If done, then we get $f_{d-1}(P) \leq 2f_{\lfloor d/2 \rfloor - 1}(P)$
- Easy: $f_{\lfloor d/2 \rfloor - 1}(P) \leq \binom{n}{\lfloor d/2 \rfloor}$
- Therefore,

$$f_{d-1}(P) \leq 2\binom{n}{\lfloor d/2 \rfloor} = O(n^{\lfloor d/2 \rfloor})$$

The asymptotic upper bound theorem: Proof (2)

To prove

$f_0(P^*) \leq 2f_{\lceil d/2 \rceil}(P^*)$ for any d-dim simple polytope P^*

WLOG: All vertices of P^* have distinct x_d-coordinates (by tiny rotation)

- We double-count the pairs (v, F) of
  - vertices v of P^* and
  - $\lceil d/2 \rceil$-dim faces F of P^* such that
  - v is the highest (or the lowest) vertex of F
- Each $\lceil d/2 \rceil$-dim face F has exactly one highest vertex and exactly one lowest vertex

The asymptotic upper bound theorem: Proof (3)

To prove

$f_0(P^*) \leq 2f_{\lceil d/2 \rceil}(P^*)$ for any d-dim simple polytope P^*

- Each vertex v is incident to d edges ($\because$ P^* simple)
- At least $\lceil d/2 \rceil$ edges going up from v, or at least $\lceil d/2 \rceil$ edges going down from v ($\because$ pigeonhole principle)
- The former case: These $\lceil d/2 \rceil$ edges determine a $\lceil d/2 \rceil$-dim face of P^* that has v as the lowest pt (c.f. Exer 9.17)
- The latter case: Similar

Therefore,

$$f_0(P^*) \leq \#\text{ pairs to count} = 2f_{\lceil d/2 \rceil}(P^*)$$

□

Summary

Polarity

- Definition
- A use of polarity: Convex hull computation

The number of facets

- The number of facets of a d-dim polytope with n vertices $= O(n^{\lfloor d/2 \rfloor})$ (Upper bound theorem)
- Cyclic polytopes show this bound is tight

This shows an intrinsic difficulty of the convex hull computation

Further reading

- Matoušek: *Lectures on Discrete Geometry*
  - Chapter 5
- Ziegler: *Lectures on Polytopes*
  - Lectures 0, 1, 2, 8
- Edelsbrunner: *Algorithms in Combinatorial Geometry*
  - Chapters 1, 8