I631: Foundation of Computational Geometry (8) Order Types of Points

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Japan Advanced Institute of Science and Technology

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Schedule

Lectures: Mon 9:20–10:30, Wed 11:00–12:30
Office hours: Mon 13:30–15:00

8	Order types of points	Nov 2 (Wed) 11:00–12:30
9	Polytopes 1	Nov 7 (Mon) 9:20–10:30
•	(Office hour)	Nov 7 (Mon) 13:30–15:00
10	Polytopes 2	Nov 9 (Wed) 11:00–12:30
11	Hyperplane arrangements 1	Nov 14 (Mon) 9:20–10:30
12	Hyperplane arrangements 2	Nov 14 (Mon) 13:30–15:00
13	Envelopes and Levels 1	Nov 16 (Wed) 11:00–12:30
•	(Canceled)	Nov 21 (Mon) 9:20–10:30
14	Envelopes and Levels 2	Nov 28 (Mon) 9:20–10:30
•	(Office hour)	Nov 28 (Mon) 13:30–15:00
15	Exam	Nov 30 (Wed) 11:00–12:30

Exercises

Each exercise set consists of three types:

- Recital Exercises:
  Repeating the contents of lectures
- Complementary Exercises:
  Filling the gaps in the contents of lectures
- Supplementary Exercises:
  Enhancing the understanding of lectures

The exam will be based on the exercises,
so the easiest way to prepare for the exam is to work on them

Office Hours

- Discuss over some exercises in Office Hours
- Students should come to the lecture room
- In advance, students should solve at least one complementary or supplementary exercise, and summarize the solution as a report
- Students should submit the report at Office Hours

Remark: Reports will be graded

Schedule

- Nov 7 (Mon) 13:30–15:00
  - Exercises from Lectures 8–9
- Nov 28 (Mon) 13:30–15:00
  - Exercises from Lectures 10–14

Exam

- Nov 30 (Wed) 11:00–12:30
- Six problems: three from Prof. Asano, three from me
- Solve two problems from Prof. Asano and two from me

Problem types (from me)

- Identical to Complementary or Supplementary Exercises

Computational Geometry in High Dimension

What's "high" dimension?

The dimension is arbitrary (and typically more than three)

Why high dimension?

Many problems are intrinsically high-dimensional

- Data analysis
- Optimization
- Robotics
- ...

Why high dimension?: Data analysis

Dimension = The number of attributes for data

ID	Sepal length (cm)	Sepal width (cm)	Petal length (cm)	Petal width (cm)	Class
1	5.1	3.5	1.4	0.2	Iris-setosa
2	4.9	3.0	1.4	0.2	Iris-setosa
3	4.7	3.2	1.3	0.2	Iris-setosa
4	4.6	3.1	1.5	0.2	Iris-setosa
5	5.0	3.6	1.4	0.2	Iris-setosa
6	5.4	3.9	1.7	0.4	Iris-setosa
7	4.6	3.4	1.4	0.3	Iris-setosa
8	5.0	3.4	1.5	0.2	Iris-setosa
9	4.4	2.9	1.4	0.2	Iris-setosa
⋮	⋮	⋮	⋮	⋮	⋮

(Fisher's Iris Data '36)

Why high dimension?: Optimization

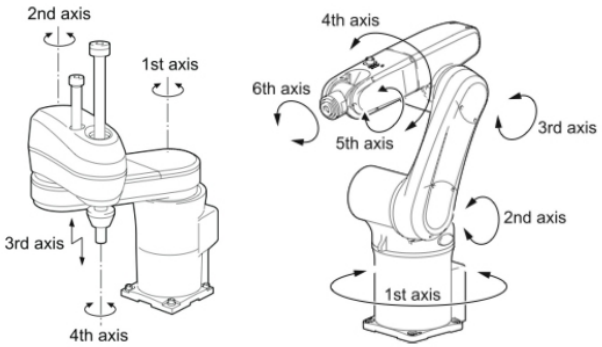
Dimension = The number of decision variables

Status	Name	Sets	C	Rows	Cols	NZs	In
●	30_70_45_095_100	P	MBP	12526	10976	46640	
●	30n20b8	B	IP	576	18380	109706	734
●	50v-10	C	MIP	233	2013	2745	18
●	a1c1s1	C	MBP	3312	3648	10178	
●	acc-tight4	PR	BP	3285	1620	17073	
●	acc-tight5	BPR	BP	3052	1339	16134	
●	acc-tight6	PR	BP	3047	1335	16108	
●	aflow40b	B	MBP	1442	2728	6783	
●	air04	B	BP	823	8904	72965	
●	app1-2	B	MBP	53467	26871	199175	

http://miplib.zib.de/miplib2010.php

Why high dimension?: Robotics

Dimension = The degrees of freedom



http://www.processonline.com.au/articles/36410-Packaging-automation-trends-using-small-assembly-robots-in-upstream-packaging-processes

A general strategy for computational geometry

Characteristic of problems in computational geometry

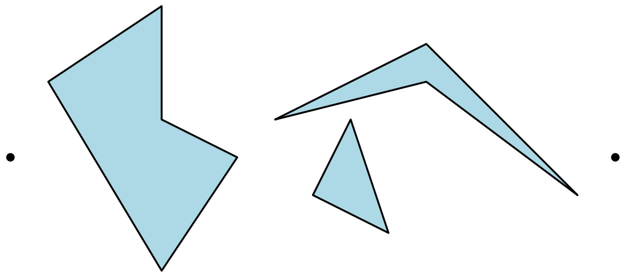
A search space is **continuous**

General strategy: Combinatorialization

Reduce the problem to a **discrete problem**

Example 1: A shortest path problem

Given two points on the plane with polygonal obstacles,
find a shortest path connecting the two points

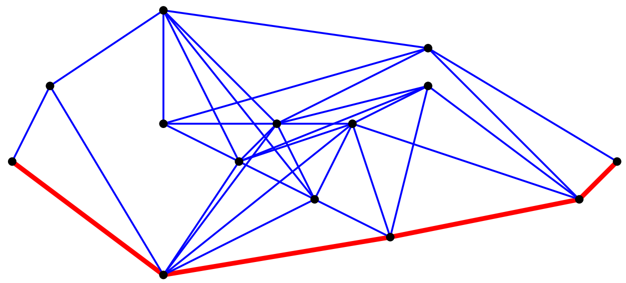


Crucial observation

A shortest path makes turns only at corners of obstacles

Example 1: A shortest path problem: Combinatorialization

Given two points on the plane with polygonal obstacles,
find a shortest path connecting the two points

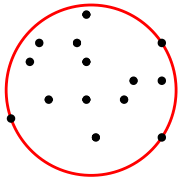


Approach

Build a "visibility graph" and run a graph algorithm

Example 2: Smallest enclosing disk problem

Given a finite set P of points on the plane,
find a smallest disk that encloses all of them

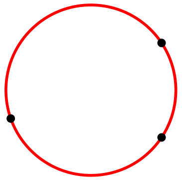


Crucial observation

$\exists$ three points p, q, r such that
the smallest encl. disk of P = the smallest encl. disk of $\{p, q, r\}$

Example 2: Smallest enclosing disk problem: Combinatorialization

Given a finite set P of points on the plane,
find a smallest disk that encloses all of them



Approach

Going through all triples of points,
and find the smallest enclosing disks of each of them

Focus on the second half of the course

Main topic

- How to describe high-dimensional objects in terms of combinatorics
- How to extract the essential combinatorial information
- What is essential?

Ambient space

$d \geq 1$ a natural number

Ambient space

We work on the space $\mathbb{R}^d$ (with Euclidean metric)

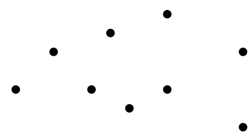
- $d = 1$: $\mathbb{R}^1$ is a line
- $d = 2$: $\mathbb{R}^2$ is a plane
- $d \geq 3$: We don't have a particular name for $\mathbb{R}^d$

In the sequel,
$d \geq 1$ always represents the dimension of the ambient space

Points

Definition: Point

A **point** is an element of $\mathbb{R}^d$



Hyperplanes

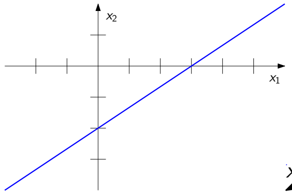
Definition: Hyperplane

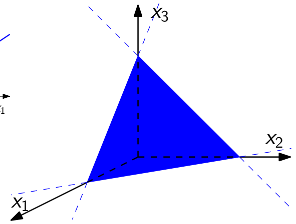
A **hyperplane** is a subset of $\mathbb{R}^d$ that can be represented as

$$\{x \in \mathbb{R}^d \mid a \cdot x = b\}$$

for some $a \in \mathbb{R}^d \setminus \{0\}$ and $b \in \mathbb{R}$

When $d = 2$: $\{(x_1, x_2) \in \mathbb{R}^2 \mid 2x_1 - 3x_2 = 6\}$When $d = 3$: $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 2x_1 + x_2 + x_3 = 2\}$



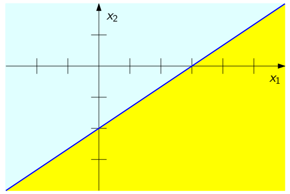


Hyperplane: A fact

A fact

Any hyperplane in $\mathbb{R}^d$ partitions $\mathbb{R}^d$ into three regions:

- $\{x \in \mathbb{R}^d \mid a \cdot x > b\}$ (open halfspace)
- $\{x \in \mathbb{R}^d \mid a \cdot x = b\}$ (hyperplane)
- $\{x \in \mathbb{R}^d \mid a \cdot x < b\}$ (open halfspace)



This can be proved via the intermediate value theorem

Affine subspaces

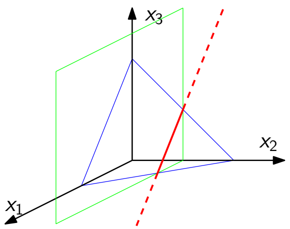
Definition: Affine subspace

An **affine subspace** of $\mathbb{R}^d$ is a subset of $\mathbb{R}^d$ that can be represented as

$$\{x \in \mathbb{R}^d \mid Ax = b\}$$

for some natural number $k \leq d$, $A \in \mathbb{R}^{k \times d}$ and $b \in \mathbb{R}^k$

When $d = 3$: $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 2x_1 + x_2 + x_3 = 2, x_2 = 1\}$



Points, lines, planes are affine subspaces in $\mathbb{R}^d$ ($d \geq 2$)

Closed halfspaces

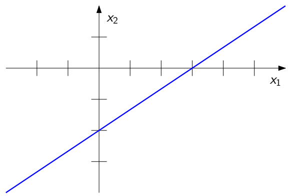
Definition: Closed halfspace

A closed **halfspace** is a subset of $\mathbb{R}^d$ that can be represented as

$$\{x \in \mathbb{R}^d \mid a \cdot x \leq b\}$$

for some $a \in \mathbb{R}^d \setminus \{0\}$ and $b \in \mathbb{R}$

When $d = 2$: $\{(x_1, x_2) \in \mathbb{R}^2 \mid 2x_1 - 3x_2 \leq 6\}$



Open halfspaces

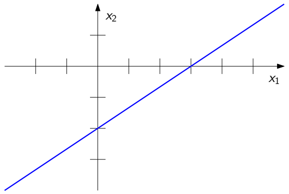
Definition: Open halfspace

An open **halfspace** is a subset of $\mathbb{R}^d$ that can be represented as

$$\{x \in \mathbb{R}^d \mid a \cdot x < b\}$$

for some $a \in \mathbb{R}^d \setminus \{0\}$ and $b \in \mathbb{R}$

When $d = 2$: $\{(x_1, x_2) \in \mathbb{R}^2 \mid 2x_1 - 3x_2 < 6\}$

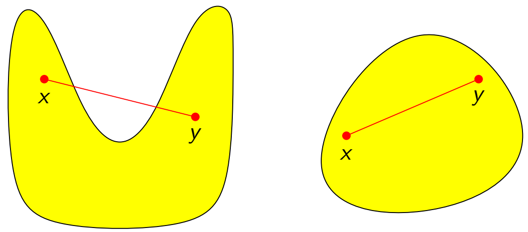


Convex sets

Definition: Convex set

A set $S \subseteq \mathbb{R}^d$ is **convex** if it satisfies the following condition

$$x, y \in S \quad \Rightarrow \quad \forall \lambda \in [0,1]: \ \lambda x + (1-\lambda) y \in S$$



Properties of convex sets

Properties of convex sets

1. A closed halfspace and an open halfspace are convex (Exercise)
2. The intersection of two convex sets is convex

Proof of (2): Let S, T be convex, and will prove $S \cap T$ is convex

- To prove: $x, y \in S \cap T \Rightarrow \forall \lambda \in [0,1]$: $\lambda x + (1-\lambda) y \in S \cap T$
- Fix $\lambda \in [0,1]$ arbitrarily. Then
  - $\lambda x + (1-\lambda) y \in S$ (since $x, y \in S \cap T \subseteq S$)
  - $\lambda x + (1-\lambda) y \in T$ (since $x, y \in S \cap T \subseteq T$)
  - $\therefore \lambda x + (1-\lambda) y \in S \cap T$ $\square$

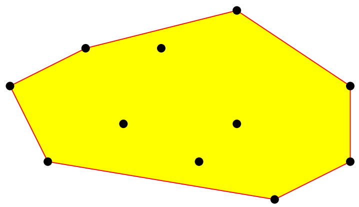
Convex hull

$X \subseteq \mathbb{R}^d$ a set

Definition: Convex hull

The **convex hull** of X is the intersection of all convex sets containing X:

$$\mathrm{conv}(X) = \bigcap_{S \text{ convex}, X \subseteq S} S$$



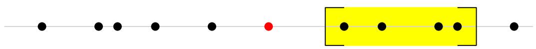
Goal of this section

- Look at geometric phenomena around finite point sets (without proofs)
- Look at some open problems in discrete geometry

Median of a 1-dimensional point set

- $P \subseteq \mathbb{R}$ a finite point set on the line
- $I \subseteq \mathbb{R}$ an interval
- I doesn't contain the median of $P \Rightarrow |I \cap P| \leq n/2$

$n = 11$



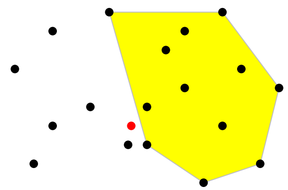
How can the "median" be generalized when $d \geq 2$?

Centerpoint theorem

Centerpoint theorem (Rado '47)

$\forall\ d \geq 1$ a natural number
$\forall\ n \geq 0$ a natural number, $P \subseteq \mathbb{R}^d$ (with $|P| = n$)
$\exists\ x \in \mathbb{R}^d$
$\forall$ convex set $S \subseteq \mathbb{R}^d$:

$$S \cap \{x\} = \emptyset \quad \Rightarrow \quad |S \cap P| \leq \frac{d}{d+1} n$$



$d = 2$,
$n = 18$,
$\frac{d}{d+1} n = 12$

Weak ε-net theorem

If a convex set needs to be smaller, then it should avoid more points

Weak ε-net theorem (Alon, Bárány, Füredi, Kleitman '92)

$\forall\ d \geq 1$ a natural number, $\varepsilon > 0$ a real number
$\exists\ f(d, \varepsilon) > 0$
$\forall\ n \geq 0$ a natural number, $P \subseteq \mathbb{R}^d$ (with $|P| = n$)
$\exists\ X \subseteq \mathbb{R}^d$ (with $|X| = f(d, \varepsilon)$)
$\forall$ convex set $S \subseteq \mathbb{R}^d$:

$$S \cap X = \emptyset \quad \Rightarrow \quad |S \cap P| \leq \varepsilon n$$

Open problem

Determine $f(2, \varepsilon)$

- Best upper bound: $\mathrm{O}(\frac{1}{\varepsilon^2})$ (Alon, Bárány, Füredi, Kleitman '92)
- Best lower bound: $\Omega(\frac{1}{\varepsilon} \log \frac{1}{\varepsilon})$ (Bukh, Matoušek, Nivasch '11)

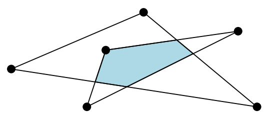
Radon's lemma

Radon's lemma (Radon '21)

$\forall\ d \geq 1$ a natural number
$\forall\ n \geq d + 2$ a natural number, $P \subseteq \mathbb{R}^d$ (with $|P| = n$)
$\exists$ a bipartition P_1, P_2 of P ($P = P_1 \cup P_2$ and $P_1 \cap P_2 = \emptyset$):

$$\mathrm{conv}(P_1) \cap \mathrm{conv}(P_2) \neq \emptyset$$

$d = 2,\ n = 6$



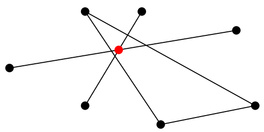
Tverberg's theorem

Tverberg's theorem (Tverberg '66)

$\forall\ d \geq 1, r \geq 2$ natural numbers
$\forall\ n \geq (d + 1)(r - 1) + 1$ a natural number, $P \subseteq \mathbb{R}^d$ (with $|P| = n$)
$\exists$ an r-partition $P_1, \ldots, P_r$ of P:

$$\mathrm{conv}(P_1) \cap \cdots \cap \mathrm{conv}(P_r) \neq \emptyset$$

$d = 2,\ r = 3,\ n = 7$



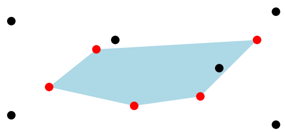
Erdős–Szekeres theorem

Erdős–Szekeres theorem (Erdős, Szekeres '35)

$\forall\ k \geq 1$ a natural number
$\exists\ n \geq 1$ a natural number
$\forall\ P \subseteq \mathbb{R}^2$ with $|P| = n$, no three points collinear
$\exists\ X \subseteq P$ with $|X| = k$:

$$x \in X \quad \Rightarrow \quad x \notin \mathrm{conv}(X \setminus \{x\})$$

$k = 5,\ n = 11$



Erdős–Szekeres theorem: Open problem

Question

How small can n be?

Let

$$n(k) = \text{ smallest } n \text{ for which Erdős–Szekeres theorem is true}$$

Open problem

Determine $n(k)$

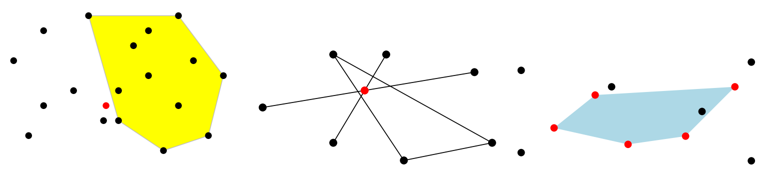
- Best upper bound: $n(k) \leq \binom{2k-5}{k-2} + 1$ (Tóth, Valtr '04)
- Best lower bound: $n(k) \geq 2^{k-2} + 1$ (Erdős, Szekeres '35)

Goal of this section

- Understand an idea to extract combinatorial information of a finite point set
- Especially, the order type of a point set

Common features of the theorems we saw

- Answers don't change by rotation and scaling
- Answers only depend on "relative positions" of the points

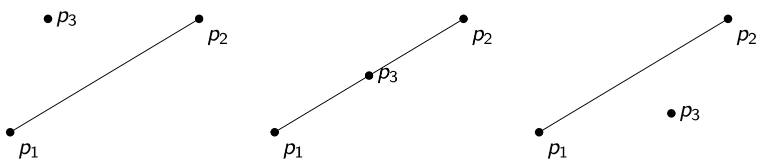


Question

- What is a relative position?
- Can we formalize what a relative position means?

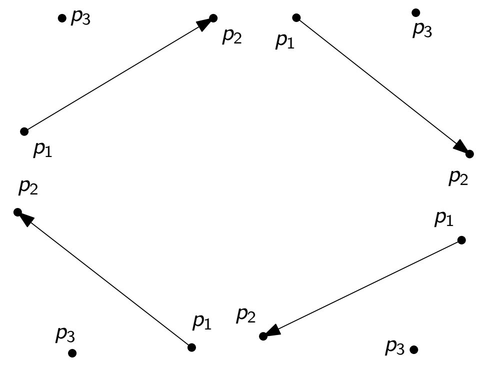
Three points on the plane

- 3 points $p_1 = (p_{11}, p_{12})$, $p_2 = (p_{21}, p_{22})$, $p_3 = (p_{31}, p_{32})$ on the plane $\mathbb{R}^2$
- One of the following three (and exactly one of them) occurs
  - p_3 lies "above" the line spanned by p_1 and p_2
  - p_3 lies "on" the the line spanned by p_1 and p_2
  - p_3 lies "below" the line spanned by p_1 and p_2



The line spanned by p_1 and p_2...

Remember: We seek for a property that is invariant of rotation



Thus, we consider a "directed line"

Three points on the plane: Determination by a determinant

Let

$$\Delta(p_1, p_2, p_3) = \begin{vmatrix} 1 & p_{11} & p_{12} \\ 1 & p_{21} & p_{22} \\ 1 & p_{31} & p_{32} \end{vmatrix}$$

Then

$$p_3 \text{ lies } \left\{ \begin{array}{c} \text{above} \\ \text{on} \\ \text{below} \end{array} \right\} \text{ the directed line spanned by } p_1 \text{ and } p_2$$

$$\Updownarrow$$

$$\Delta(p_1, p_2, p_3) \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} 0$$

Namely, we're only interested in the sign of $\Delta(p_1, p_2, p_3)$!

Sign

Sign

The **sign** of a real number $x \in \mathbb{R}$ is a symbol $\mathrm{sgn}(x) \in \{+, -, 0\}$ defined as

$$\mathrm{sgn}(x) = \left\{ \begin{array}{c} + \\ 0 \\ - \end{array} \right\} \Leftrightarrow x \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} 0$$

- $\mathrm{sgn}(3.4) = +$
- $\mathrm{sgn}(-4) = -$
- $\mathrm{sgn}(0) = 0$

Order type of a planar point set

$P = \{p_1, \dots, p_n\} \subseteq \mathbb{R}^2$ a finite point set

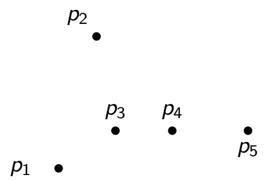
Order type (when $d = 2$) (Goodman, Pollack '83)

The **order type** of P is a map $\chi\colon \{1, \dots, n\}^3 \to \{+, -, 0\}$ such that

$$\chi(i_1, i_2, i_3) = \operatorname{sgn} \Delta(p_{i_1}, p_{i_2}, p_{i_3})$$

The order type of P is also called the **chirotope** of P

Example



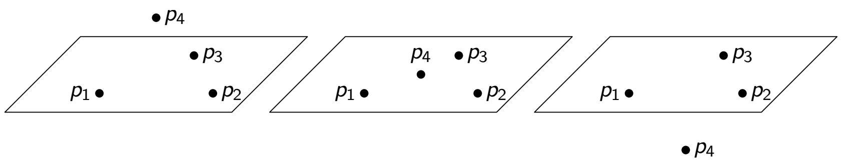
$\chi(1,2,3) = -,\quad \chi(1,2,4) = -,\quad \chi(1,2,5) = -,$
$\chi(1,3,4) = -,\quad \chi(1,3,5) = -,\quad \chi(1,4,5) = -,$
$\chi(2,3,4) = +,\quad \chi(2,3,5) = +,\quad \chi(2,4,5) = +,$
$\chi(3,4,5) = 0$

Note

The values not appearing here can be easily derived,
e.g., $\chi(i_1, i_2, i_3) = -\chi(i_1, i_3, i_2)$

Four points in $\mathbb{R}^3$

- Four points $p_1 = (p_{11}, p_{12}, p_{13})$, $p_2 = (p_{21}, p_{22}, p_{23})$, $p_3 = (p_{31}, p_{32}, p_{33})$, $p_4 = (p_{41}, p_{42}, p_{43})$ in $\mathbb{R}^3$
- One of the following three (and exactly one of them) occurs
  - p_4 lies "above" the plane spanned by p_1, p_2, p_3
  - p_4 lies "on" the plane spanned by p_1, p_2, p_3
  - p_4 lies "below" the plane spanned by p_1, p_2, p_3

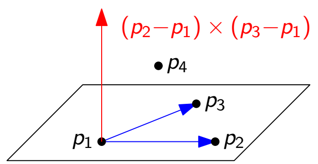


The plane spanned by p_1, p_2, p_3

With an ordering p_1, p_2, p_3,
we can canonically find a normal vector of the plane by

$$(p_2 - p_1) \times (p_3 - p_1)$$

This is invariant under rotation, scaling, ...



A care should be taken if p_1, p_2, p_3 are collinear

Four points in $\mathbb{R}^3$: Determination by a determinant

Let

$$\Delta(p_1, p_2, p_3, p_4) = \begin{vmatrix} 1 & p_{11} & p_{12} & p_{13} \\ 1 & p_{21} & p_{22} & p_{23} \\ 1 & p_{31} & p_{32} & p_{33} \\ 1 & p_{41} & p_{42} & p_{43} \end{vmatrix}$$

Then

$$p_4 \text{ lies } \left\{ \begin{array}{c} \text{above} \\ \text{on} \\ \text{below} \end{array} \right\} \text{ the plane spanned by } p_1, p_2, p_3$$

$$\Updownarrow$$

$$\Delta(p_1, p_2, p_3, p_4) \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} 0$$

(Exercise)

Order type of a point set in $\mathbb{R}^3$

$P = \{p_1, \ldots, p_n\} \subseteq \mathbb{R}^3$ a finite point set

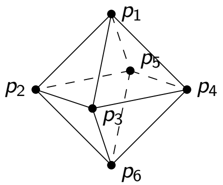
Order type (when $d = 3$) (Goodman, Pollack '83)

The **order type** of P is a map $\chi \colon \{1, \ldots, n\}^4 \to \{+, -, 0\}$ such that

$$\chi(i_1, i_2, i_3, i_4) = \operatorname{sgn} \Delta(p_{i_1}, p_{i_2}, p_{i_3}, p_{i_4})$$

The order type of P is also called the **chirotope** of P

Example: A regular octahedron



$$\begin{array}{lll} \chi(1,2,3,4) = -, & \chi(1,2,3,5) = -, & \chi(1,2,3,6) = -, \\ \chi(1,2,4,5) = -, & \chi(1,2,4,6) = 0, & \chi(1,2,5,6) = +, \\ \chi(1,3,4,5) = -, & \chi(1,3,4,6) = -, & \chi(1,3,5,6) = 0, \\ \chi(1,4,5,6) = -, & \chi(2,3,4,5) = 0, & \chi(2,3,4,6) = -, \\ \chi(2,3,5,6) = -, & \chi(2,4,5,6) = -, & \chi(3,4,5,6) = - \end{array}$$

Note

The values not appearing here can be easily derived,
e.g., $\chi(i_1, i_2, i_3, i_4) = -\chi(i_1, i_2, i_4.i_3)$

$d + 1$ points in $\mathbb{R}^d$

- $d + 1$ points $p_1, p_2, \ldots, p_{d+1}$ in $\mathbb{R}^d$
- We employ a similar approach to the case $d = 3$, but we don't have a cross product when $d > 3$
- We may employ Exterior Algebra for our purpose (and we do!), but a formal treatment is far beyond the scope of this lecture

$d+1$ points in $\mathbb{R}^d$: Determination by a determinant

Let

$$\Delta(p_1, p_2, \ldots, p_{d+1}) = \begin{vmatrix} 1 & p_{11} & p_{12} & \cdots & p_{1d} \\ 1 & p_{21} & p_{22} & \cdots & p_{2d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & p_{d+1,1} & p_{d+1,2} & \cdots & p_{d+1,d} \end{vmatrix}$$

Order type of a point set in $\mathbb{R}^d$

$P = \{p_1, \ldots, p_n\} \subseteq \mathbb{R}^d$ a finite point set

Order type (Goodman, Pollack '83)

The **order type** of P is a map $\chi \colon \{1, \ldots, n\}^{d+1} \to \{+, -, 0\}$ s.t.

$$\chi(i_1, i_2, \ldots, i_{d+1}) = \operatorname{sgn} \Delta(p_{i_1}, p_{i_2}, \ldots, p_{i_{d+1}})$$

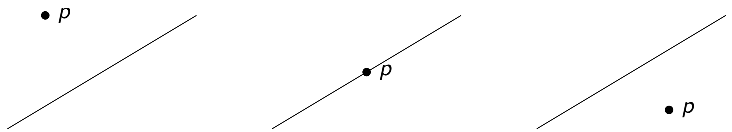
The order type of P is also called the **chirotope** of P

Goal of this section

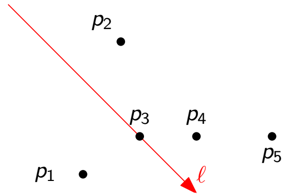
- Understand another (but essentially identical) idea to extract combinatorial information of a finite point set
- Especially, the signed covectors and the signed cocircuits of a point set

A line and a point in $\mathbb{R}^2$

- A line $\{x \in \mathbb{R}^2 \mid a_1x_1 + a_2x_2 = b\}$ and a point $p \in \mathbb{R}^2$
- One of the following three (and exactly one of them) occurs
  - $a_1p_1 + a_2p_2 > b$ (p lies above the line)
  - $a_1p_1 + a_2p_2 = b$ (p lies on the line)
  - $a_1p_1 + a_2p_2 < b$ (p lies below the line)



Separation of a point set by lines in $\mathbb{R}^2$



- p_2, p_4, p_5 lie above ℓ
- p_3 lies on ℓ
- p_1 lies below ℓ

$\rightsquigarrow$ a sign vector $(-, +, 0, +, +)$

Signed covectors of a point set in $\mathbb{R}^2$

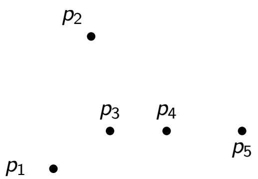
$P = \{p_1, p_2, \ldots, p_n\} \subseteq \mathbb{R}^2$ a point set

Signed covectors (when $d = 2$)

The **signed covectors** of P are the vectors in $\{+, -, 0\}^n$ defined as

$$\begin{aligned}\mathcal{V}^*(P) &= \{(\mathrm{sgn}(a \cdot p_1 - b), \ldots, \mathrm{sgn}(a \cdot p_n - b)) \mid a \in \mathbb{R}^2, b \in \mathbb{R}\} \\ &\subseteq \{+, -, 0\}^n\end{aligned}$$

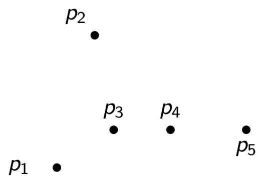
Example



$$\begin{aligned}\mathcal{V}^*(P) = \{&(\pm,\pm,\pm,\pm,\pm), (\mp,\pm,\pm,\pm,\pm), (\pm,\mp,\pm,\pm,\pm), (\pm,\pm,\pm,\pm,\mp), \\ &(\mp,\mp,\pm,\pm,\pm), (\mp,\pm,\mp,\pm,\pm), (\mp,\pm,\pm,\pm,\mp), (\pm,\mp,\mp,\pm,\pm), \\ &(\pm,\mp,\pm,\pm,\mp), (\pm,\pm,\pm,\mp,\mp), (0,\pm,\pm,\pm,\pm), (0,\mp,\pm,\pm,\pm), \\ &(0,\mp,\mp,\pm,\pm), (0,\pm,\pm,\pm,\mp), (\pm,0,\pm,\pm,\pm), (\mp,0,\pm,\pm,\pm), \\ &(\mp,0,\mp,\pm,\pm), (\pm,0,\pm,\pm,\mp), (\mp,\pm,0,\pm,\pm), (\pm,\mp,0,\pm,\pm), \\ &(\mp,\mp,0,\pm,\pm), (\pm,\pm,\pm,0,\mp), (\mp,\pm,\mp,0,\pm), (\pm,\pm,\pm,\pm,0), \\ &(\mp,\pm,\pm,\pm,0), (\pm,\mp,\pm,\pm,0), (0,0,\pm,\pm,\pm), (0,\pm,0,\mp,\mp), \\ &(0,\pm,\pm,0,\mp), (0,\pm,\pm,\pm,0), (\mp,0,0,\pm,\pm), (\mp,0,\mp,0,\pm), \\ &(\mp,0,\mp,\mp,0), (\mp,\pm,0,0,0), (0,0,0,0,0)\}\end{aligned}$$

A bit of thought

The signed covectors are redundant



For example, ...

- $(+,0,+,-,-),(+,0,0,-,-),(+,0,+,0,-) \in \mathcal{V}^*(P)$
- But, "$(+,0,0,-,-),(+,0,+,0,-) \in \mathcal{V}^*(P)$" tells you "$(+,0,+,-,-) \in \mathcal{V}^*(P)$"
- So $(+,0,+,-,-)$ is redundant

Can we get rid of such redundancy? $\rightsquigarrow$ Signed cocircuits $\mathcal{C}^*(P)$

Example

$$\begin{aligned}
\mathcal{V}^*(P) \quad = \quad & \{(\pm,\pm,\pm,\pm,\pm),(\mp,\pm,\pm,\pm,\pm),(\pm,\mp,\pm,\pm,\pm),(\pm,\pm,\pm,\pm,\mp),\\
& (\mp,\mp,\pm,\pm,\pm),(\mp,\pm,\mp,\pm,\pm),(\mp,\pm,\pm,\pm,\mp),(\pm,\mp,\mp,\pm,\pm),\\
& (\pm,\mp,\pm,\pm,\mp),(\pm,\pm,\pm,\mp,\mp),(0,\pm,\pm,\pm,\pm),(0,\mp,\pm,\pm,\pm),\\
& (0,\mp,\mp,\pm,\pm),(0,\pm,\pm,\pm,\mp),(\pm,0,\pm,\pm,\pm),(\mp,0,\pm,\pm,\pm),\\
& (\mp,0,\mp,\pm,\pm),(\pm,0,\pm,\pm,\mp),(\mp,\pm,0,\pm,\pm),(\pm,\mp,0,\pm,\pm),\\
& (\mp,\mp,0,\pm,\pm),(\pm,\pm,\pm,0,\mp),(\mp,\pm,\mp,0,\pm),(\pm,\pm,\pm,\pm,0),\\
& (\mp,\pm,\pm,\pm,0),(\pm,\mp,\pm,\pm,0),(0,0,\pm,\pm,\pm),(0,\pm,0,\mp,\mp),\\
& (0,\pm,\pm,0,\mp),(0,\pm,\pm,\pm,0),(\mp,0,0,\pm,\pm),(\mp,0,\mp,0,\pm),\\
& (\mp,0,\mp,\mp,0),(\mp,\pm,0,0,0),(0,0,0,0,0)\}\\
\mathcal{C}^*(P) \quad = \quad & \{(0,0,\pm,\pm,\pm),(0,\pm,0,\mp,\mp),(0,\pm,\pm,0,\mp),(0,\pm,\pm,\pm,0),\\
& (\mp,0,0,\pm,\pm),(\mp,0,\mp,0,\pm),(\mp,0,\mp,\mp,0),(\mp,\pm,0,0,0)\}
\end{aligned}$$

A partial order on the set of signs

A partial order on the set of signs

We define a partial order $\leq$ on the set $\{+,-,0\}$ of signs as

- $0 \leq +$ and $0 \leq -$
- ($+$ and $-$ are incomparable)

The order $\leq$ is extended to the set $\{+,-,0\}^n$ of sign vectors as

$$x \leq y \quad \Leftrightarrow \quad x_i \leq y_i \text{ for all } i \in \{1,\dots,n\}$$

For example,

- $(+,-,+,0,-) \leq (+,-,+,+,-)$
- $(+,-,0,0,0) \leq (+,-,+,+,-)$
- $(0,-,-,0,+) \not\leq (+,-,+,+,-)$

Signed cocircuits

$P = \{p_1, p_2, \dots, p_n\} \subseteq \mathbb{R}^2$ a point set

Signed cocircuits (when $d = 2$)

The **signed cocircuits** of P are the minimal elements in $\mathcal{V}^*(P) \setminus \{0\}$;
The set of signed cocircuits of P is denoted by $\mathcal{C}^*(P)$

$$\begin{aligned}
\mathcal{V}^*(P) \quad = \quad & \{(\pm,\pm,\pm,\pm,\pm),(\mp,\pm,\pm,\pm,\pm),(\pm,\mp,\pm,\pm,\pm),(\pm,\pm,\pm,\pm,\mp),\\
& (\mp,\mp,\pm,\pm,\pm),(\mp,\pm,\mp,\pm,\pm),(\mp,\pm,\pm,\pm,\mp),(\pm,\mp,\mp,\pm,\pm),\\
& (\pm,\mp,\pm,\pm,\mp),(\pm,\pm,\pm,\mp,\mp),(0,\pm,\pm,\pm,\pm),(0,\mp,\pm,\pm,\pm),\\
& (0,\mp,\mp,\pm,\pm),(0,\pm,\pm,\pm,\mp),(\pm,0,\pm,\pm,\pm),(\mp,0,\pm,\pm,\pm),\\
& (\mp,0,\mp,\pm,\pm),(\pm,0,\pm,\pm,\mp),(\mp,\pm,0,\pm,\pm),(\pm,\mp,0,\pm,\pm),\\
& (\mp,\mp,0,\pm,\pm),(\pm,\pm,\pm,0,\mp),(\mp,\pm,\mp,0,\pm),(\pm,\pm,\pm,\pm,0),\\
& (\mp,\pm,\pm,\pm,0),(\pm,\mp,\pm,\pm,0),(0,0,\pm,\pm,\pm),(0,\pm,0,\mp,\mp),\\
& (0,\pm,\pm,0,\mp),(0,\pm,\pm,\pm,0),(\mp,0,0,\pm,\pm),(\mp,0,\mp,0,\pm),\\
& (\mp,0,\mp,\mp,0),(\mp,\pm,0,0,0),(0,0,0,0,0)\}
\end{aligned}$$

Signed covectors and signed cocircuits in $\mathbb{R}^d$

When $d \geq 3$

The definitions are naturally extended

$P = \{p_1, p_2, \ldots, p_n\} \subseteq \mathbb{R}^d$ a point set

Signed covectors

The **signed covectors** of P are the vectors in $\{+, -, 0\}^n$ defined as

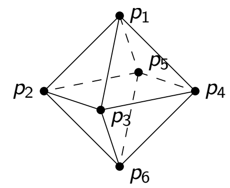
$$\mathcal{V}^*(P) = \{(\mathrm{sgn}(a \cdot p_1 - b), \ldots, \mathrm{sgn}(a \cdot p_n - b)) \mid a \in \mathbb{R}^d, b \in \mathbb{R}\}$$

Signed cocircuits

The **signed cocircuits** of P are the minimal elements in $\mathcal{V}^*(P) \setminus \{0\}$;
The set of signed cocircuits of P is denoted by $\mathcal{C}^*(P)$

(Bland, Las Vergnas '78; Folkman, Lawrence '78)

Example: A regular octahedron



$$\begin{aligned}\mathcal{C}^*(P) \quad = \quad & \{(0,0,0,\pm,\pm,\pm), (0,0,\pm,\pm,0,\pm), (0,\pm,0,0,\pm,\pm), (0,\pm,\pm,0,0,\pm),\\ & (\pm,0,0,\pm,\pm,0), (\pm,0,\pm,\pm,0,0), (\pm,\pm,0,0,\pm,0), (\pm,\pm,\pm,0,0,0),\\ & (0,0,\pm,0,\mp,0), (0,\pm,0,\mp,0,0), (\pm,0,0,0,0,\mp)\}\end{aligned}$$

Summary

Three ways of extracting combinatorics of point sets

- Order type
- Signed covectors
- Signed cocircuits

Remarks

- It's known (but we don't prove in the lecture) that these three objects carry the same information
  - We can transform one to another, without passing through coordinates of the points
- Such combinatorial descriptions motivate us to study "oriented matroids"
  - Interface between combinatorics, topology, and geometry

Further reading

- Ziegler: *Lectures on Polytopes*
  - Lecture 6
- Matoušek: *Lectures on Discrete Geometry*
  - Chapters 1, 3, 8, 10
- Edelsbrunner: *Algorithms in Combinatorial Geometry*
  - Chapter 1
- Björner, Las Vergnas, Sturmfels, White, Ziegler: *Oriented Matroids*
- Bokowski: *Computational Oriented Matroids*