I631: Foundation of Computational Geometry (14) — November 28, 2011

Exercises — **Yoshio Okamoto**

Warning: Your solutions have to be substantiated, namely, you have to provide proofs for the given answers.

In the problems below, $e_k(n)$ represents the maximum number of vertices of the k-level in a simple arrangement of n lines, and $e(n) = e_{\lfloor n/2 \rfloor}(n)$.

Recital Exercise 14.1 Prove that $e(n) \geq 2n - 2$ for all natural numbers $n \geq 2$.

Recital Exercise 14.2 Prove that $e(n) = O(n^{3/2})$ by following the steps below.

1. Fix an arbitrary simple arrangement $\mathcal{A}$ of n lines in the plane. Let $V_i(\mathcal{A})$ be the set of vertices of the i-level of $\mathcal{A}$ (let $V_i(\mathcal{A}) = \emptyset$ when $i < 0$ or $i \geq n$). Let $B_i = (V_{k-i}(\mathcal{A}) \setminus V_{k-i-1}(\mathcal{A})) \cup (V_{k+i}(\mathcal{A}) \setminus V_{k+i+1}(\mathcal{A}))$ and $I_i = (V_{k-i+1} \cup V_{k-i+2} \cup \cdots \cup V_{k+i-2} \cup V_{k+i-1}) \setminus (V_{k-i} \cup V_{k+i})$.
   Prove that $|I_i| \leq 2i \cdot |B_i| + 2i^2$
2. Prove that $B_i = I_{i+1} \setminus I_i$.
3. Prove that $|I_i| = O(n^{3/2} i^{1/2})$. You may use the complementary exercise below.
4. Prove that $e_k(n) = |I_2|$ for all k.
5. Conclude that $e(n) = O(n^{3/2})$.

Complementary Exercise 14.3 Prove that

$$\prod_{j=i}^{m} \frac{2j}{2j+1} \leq \sqrt{\frac{i}{m}}$$

for any integers i, m such that $1 \leq i \leq m$.

Supplementary Exercise 14.4 Prove that $e_{k+1}(n+2) \geq e_k(n) + 4$ for all natural numbers $n \geq 2$ and $k \in \{0, \ldots, \lfloor n/2 \rfloor - 1\}$.

Supplementary Exercise 14.5 Prove that $e_k(n) \leq e(2n+2)$ for all natural numbers $n \geq 2$ and $k \in \{0, \ldots, \lfloor n/2 \rfloor\}$.