I631: Foundation of Computational Geometry (13) **November 16, 2011**
Exercises **Yoshio Okamoto**

Warning: Your solutions have to be substantiated, namely, you have to provide proofs for the given answers.

Recital Exercise 13.1 Provide the definition of each of the following terms: Voronoi diagram, order-k Voronoi diagram, farthest-point Voronoi diagram, level (of a point in a hyperplane arrangement), k-level (of a hyperplane arrangement), upper envelope (of a hyperplane arrangement), lower envelope (of a hyperplane arrangement)

Recital Exercise 13.2 Let $d \geq 1$ and $U = \{x \in \mathbb{R}^{d+1} \mid x_{d+1} = x_1^2 + x_2^2 + \cdots x_d^2\}$ be the unit paraboloid in $\mathbb{R}^{d+1}$. For a point $p \in \mathbb{R}^d$, define $u(p) = (p, p_1^2 + p_2^2 + \cdots p_d^2) \in \mathbb{R}^{d+1}$ and

$$H(p) = \left\{ x \in \mathbb{R}^{d+1} \mid x_{d+1} = \sum_{i=1}^{d} (2p_i x_i - p_i^2) \right\}$$

Prove that for every two points $p, q \in \mathbb{R}^2$, the vertical distance from $u(q)$ to $H(p)$ is equal to $d(p,q)^2$.

Recital Exercise 13.3 Let $d \geq 1, n \geq 2$ and $P \subseteq \mathbb{R}^d$ be a set of n points. Prove that the Voronoi diagram of P is the projection of the upper envelope of the hyperplane arrangement $\{H(p) \mid p \in P\}$ on the hyperplane $\{x \in \mathbb{R}^{d+1} \mid x_{d+1} = 0\}$, where $H(p) = \left\{ x \in \mathbb{R}^{d+1} \mid x_{d+1} = \sum_{i=1}^{d} (2p_i x_i - p_i^2) \right\}$ for every $p \in P$. Similarly, prove that the farthest-point Voronoi diagram of P is the projection of the lower envelope of the hyperplane arrangement $\{H(p) \mid p \in P\}$ on the hyperplane $\{x \in \mathbb{R}^{d+1} \mid x_{d+1} = 0\}$.

Complementary Exercise 13.4 Let $d \geq 1$ and $U = \{x \in \mathbb{R}^{d+1} \mid x_{d+1} = x_1^2 + x_2^2 + \cdots x_d^2\}$ be the unit paraboloid in $\mathbb{R}^{d+1}$. For a point $p \in \mathbb{R}^d$, define $u(p) = (p, p_1^2 + p_2^2 + \cdots p_d^2) \in \mathbb{R}^{d+1}$. Verify that for every $p \in \mathbb{R}^d$, the hyperplane

$$H(p) = \left\{ x \in \mathbb{R}^{d+1} \mid x_{d+1} = \sum_{i=1}^{d} (2p_i x_i - p_i^2) \right\}$$

is tangent to U at the point $u(p)$.

Complementary Exercise 13.5 Let $S \subseteq \mathbb{R}^d$ be a convex set and $A \in \mathbb{R}^{k \times d}$ be a real matrix. Prove that the set $S' = \{Ax \mid x \in S\} \subseteq \mathbb{R}^k$ is convex.

Supplementary Exercise 13.6 Consider a 1-dimensional point set $P = \{1, \ldots, n\} \subseteq \mathbb{R}$. Determine the exact numbers of Voronoi cells and Voronoi vertices of the order-k Voronoi diagram of P for every pair (n, k) of natural numbers with $n \geq k + 1$.

Supplementary Exercise 13.7 Prove or disprove: For every natural number $n \geq 5$ and every set P of n points in $\mathbb{R}^2$, the Voronoi diagram of P has at least one bounded Voronoi cell.

Supplementary Exercise 13.8 Prove or disprove: For every natural number $n \geq 3$ and every set P of n points in $\mathbb{R}^2$, the farthest-point Voronoi diagram of P has at least three Voronoi cells.

Supplementary Exercise 13.9 Prove or disprove: In the 1-level of the arrangement of at least three lines in $\mathbb{R}^2$, in which no line is vertical or horizontal, no three consecutive edges have increasing values of slopes, or decreasing values of slopes.