I631: Foundation of Computational Geometry (11) | **November 14, 2011**
Exercises | **Yoshio Okamoto**

Warning: Your solutions have to be substantiated, namely, you have to provide proofs for the given answers.

Recital Exercise 11.1 Provide the definition of each of the following terms: hyperplane arrangement, face (of a hyperplane arrangement), cell (of a hyperplane arrangement), simple hyperplane arrangement, dual (of a hyperplane that does not contain the origin), dual (of a point that is not the origin), signed covectors (of a hyperplane arrangement), signed cocircuits (of a hyperplane arrangement)

Recital Exercise 11.2 Prove that for any point $p \in \mathbb{R}^d \setminus \{0\}$ and any hyperplane $H = \{x \in \mathbb{R}^d \mid a \cdot x = 1\}$ where $a \in \mathbb{R}^d \setminus \{0\}$, $p \in H$ if and only if $\mathcal{D}(p) \ni \mathcal{D}(H)$.

Complementary Exercise 11.3 Consider the following recurrence on $n \geq 0$ and $d \geq 1$:

$$\Phi_d(n) = \begin{cases} 1 & \text{if } n = 0, \\ 2 & \text{if } n = 1 \text{ and } d = 1, \\ \Phi_d(n-1) + \Phi_{d-1}(n-1) & \text{otherwise.} \end{cases}$$

Prove that this recurrence has a unique solution $\Phi_d(n) = \displaystyle\sum_{i=0}^{d} \binom{n}{i}$.

Complementary Exercise 11.4 For a hyperplane $H = \{x \in \mathbb{R}^d \mid a \cdot x = 1\}$, let $H^- = \{x \in \mathbb{R}^d \mid a \cdot x \leq 1\}$.

For a point $p \in \mathbb{R}^d \setminus \{0\}$ and a hyperplane $H = \{x \in \mathbb{R}^d \mid a \cdot x = 1\}$ with $a \in \mathbb{R}^d \setminus \{0\}$, prove that $p \in H^-$ if and only if $\mathcal{D}(p)^- \ni \mathcal{D}(H)$.

Supplementary Exercise 11.5 For all natural numbers $d \geq 1$ and $n \geq 0$, determine the number of vertices of a simple arrangement of n hyperplanes in $\mathbb{R}^d$.

Supplementary Exercise 11.6 For all natural numbers $d \geq 1$ and $n \geq 0$, determine the number of edges of a simple arrangement of n hyperplanes in $\mathbb{R}^d$.

Supplementary Exercise 11.7 Prove that for all natural numbers $d \geq 1$ (constant) and $n \geq 0$, the number of unbounded cells in a simple arrangement of n hyperplanes in $\mathbb{R}^d$ is $O(n^{d-1})$.

Supplementary Exercise 11.8 Let $d \geq 1$ be a natural number, and let $H_{i,j} = \{x \in \mathbb{R}^d \mid x_i = x_j\}$ a hyperplane for all i, j where $1 \leq i < j \leq d$. Determine the number of cells in the hyperplane arrangement $\{H_{i,j} \mid 1 \leq i < j \leq d\}$.

Supplementary Exercise 11.9 Prove or disprove: For every natural number $d \geq 1$ and every $a \in \mathbb{R}^d \setminus \{0\}$, $\mathcal{D}(H)$ is a unique element of the set $\bigcap_{z \in H} \mathcal{D}(z)$, where $H = \{x \in \mathbb{R}^d \mid a \cdot x = 1\}$ is a hyperplane.