I631: Foundation of Computational Geometry (10) **November 9, 2011**
Exercises **Yoshio Okamoto**

Warning: Your solutions have to be substantiated, namely, you have to provide proofs for the given answers.

Recital Exercise 10.1 Provide the definition of each of the following terms: polar set, the f-vector (of a polytope), moment curve, cyclic polytope, Gale's evenness criterion

Recital Exercise 10.2 Prove that the polar of the set $S = [-1,1]^2 \subseteq \mathbb{R}^2$ is determined by

$$S^* = \{y \mid y_1 + y_2 \leq 1, y_1 - y_2 \leq 1, -y_1 + y_2 \leq 1, -y_1 - y_2 \leq 1\}.$$

Recital Exercise 10.3 Prove that the polar of any set in $\mathbb{R}^d$ is convex.

Recital Exercise 10.4 Prove that the intersection of the moment curve and any hyperplane in $\mathbb{R}^d$ consists of at most d points, and if it is exactly d, then the moment curve is not tangent to the hyperplane at the intersections.

Recital Exercise 10.5 Prove that Gale's evenness criterion characterizes the facets of cyclic polytopes.

Recital Exercise 10.6 Using Gale's evenness criterion, prove that the number of facets of a d-dimensional cyclic polytope with n vertices is at least $\Omega(n^{\lfloor d/2 \rfloor})$, where $d \geq 2$ is constant.

Recital Exercise 10.7 Prove that the number of facets of any d-dimensional simplicial polytope with n vertices is $O(n^{\lfloor d/2 \rfloor})$. (Hint: First observe that it is enough to prove that the number of vertices of any d-dimensional simple polytope with n facets is at most twice the number of $\lceil d/2 \rceil$-dimensional faces of the polytope.)

Complementary Exercise 10.8 Prove that $\left(\frac{a}{b}\right)^b \leq \binom{a}{b}$ for every natural numbers a, b with $0 \leq b \leq a$.

Complementary Exercise 10.9 Let P be a d-dimensional polytope with n vertices, $n \geq d+1$. Prove that $f_i(P) \leq \binom{n}{i+1}$ for all $i \in \{-1, 0, 1, \ldots, d\}$. Note: This inequality was used in the proof of the asymptotic upper bound theorem.

Complementary Exercise 10.10 Prove that every cyclic polytope is simplicial.

Supplementary Exercise 10.11 What is the polar of the set $S = [0,2]^2 \subseteq \mathbb{R}^2$?

Supplementary Exercise 10.12 What is the polar of the set $S = [-1,1] \times [1,3] \subseteq \mathbb{R}^2$?

Supplementary Exercise 10.13 Prove or disprove: For any set $S \subseteq \mathbb{R}^d$, it holds that $(S^*)^* = S$.

Supplementary Exercise 10.14 Provide the f-vector of the 3-dimensional cube.

Supplementary Exercise 10.15 Provide the f-vector of a d-dimensional simplex for all $d \geq 0$.

Supplementary Exercise 10.16 Prove that for every d-dimensional cyclic polytope with n vertices, where $d \geq 4$ and $n \geq d+1$, the convex hull of any two vertices forms an edge.

Supplementary Exercise 10.17 Prove that the number of facets of every d-dimensional cyclic polytope with n vertices is

$$\begin{cases} \binom{n - \lfloor d/2 \rfloor}{\lfloor d/2 \rfloor} + \binom{n - \lfloor d/2 \rfloor - 1}{\lfloor d/2 \rfloor - 1} & \text{if } d \text{ is even,} \\ 2\binom{n - \lfloor d/2 \rfloor - 1}{\lfloor d/2 \rfloor} & \text{if } d \text{ is odd,} \end{cases}$$

where $d \geq 2$ and $n \geq d+1$.