I631: Foundation of Computational Geometry (9) — **November 7, 2011**

Exercises — **Yoshio Okamoto**

Warning: Your solutions have to be substantiated, namely, you have to provide proofs for the given answers.

Recital Exercise 9.1 Provide the definition of each of the following terms: V-polytope, H-polytope, polytope, dimension (of an affine subspace), dimension (of a polytope), cube, crosspolytope, affine independence, simplex, face, vertex (of a polytope), edge (of a polytope), ridge, facet, face lattice, combinatorial equivalence (of polytopes), simple polytope, simplicial polytope.

Recital Exercise 9.2 Prove that a face of a polytope is also a polytope.

Recital Exercise 9.3 Draw a Hasse diagram of the face lattice of a 3-dimensional cube.

Recital Exercise 9.4 Draw a Hasse diagram of the face lattice of a 3-dimensional crosspolytope.

Recital Exercise 9.5 Draw a Hasse diagram of the face lattice of a 4-dimensional simplex.

Complementary Exercise 9.6 Prove that if $P \subseteq \mathbb{R}^d$ is an affinely independent set of points, then $|P| \leq d+1$.

Complementary Exercise 9.7 Prove that the d-dimensional crosspolytope has at least $2d$ vertices.

Complementary Exercise 9.8 Prove that the d-dimensional cube has at least 2^d vertices.

Complementary Exercise 9.9 Prove the following. For two polytopes P and P', if P is a face of P' and P' is a face of P, then $P = P'$.

Supplementary Exercise 9.10 Let S be an affine subspace of $\mathbb{R}^d$ given by

$$\{x \in \mathbb{R}^d \mid Ax = b\}$$

for some natural number $k \leq d$, $A \in \mathbb{R}^{k \times d}$ and $b \in \mathbb{R}^k$. Prove or disprove: The dimension of S is equal to $d - \operatorname{rank}(A)$.

Supplementary Exercise 9.11 Consider the following polytope P:

$$P = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d] \subseteq \mathbb{R}^d,$$

where $a_i \leq b_i$ for all $i \in \{1, \ldots, d\}$. What is the dimension of P? (Notice the case when $a_i = b_i$ for some i.)

Supplementary Exercise 9.12 Prove or disprove: There exists a 3-dimensional polytope with six vertices, nine edges, and five facets.

Supplementary Exercise 9.13 Let $V = \{v_1, \ldots, v_n\} \subseteq \mathbb{R}^d$ be the vertex set of a polytope P. Prove that the convex hull of $V' \subseteq V$ is a face of P if and only if there exists a signed covector of V such that the components corresponding to the elements of V' are zero and the other components have the same sign ($+$ or $-$).

Supplementary Exercise 9.14 Draw a Hasse diagram of the face lattice of a regular dodecahedron.

Supplementary Exercise 9.15 Prove that every 2-dimensional polytope is simple and simplicial.

Supplementary Exercise 9.16 Prove or disprove: For every $d \geq 3$, a d-dimensional polytope is simple and simplicial if and only if it is a simplex.

Supplementary Exercise 9.17 Prove that every facet of any simplicial polytope is a simplex.

Supplementary Exercise 9.18 Prove or disprove: For every $n \geq 4$, there exists a 3-dimensional simple polytope with n vertices.