I631: Foundation of Computational Geometry (8)	November 2, 2011
Exercises	Yoshio Okamoto

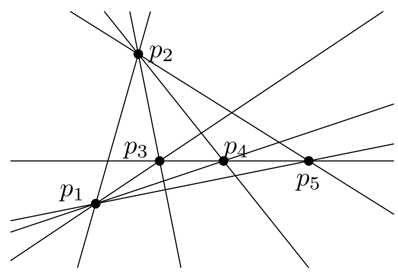
Warning: Your solutions have to be substantiated, namely, you have to provide proofs for the given answers.

Recital Exercise 8.1 Provide the definition of each of the following terms: hyperplane, affine subspace, closed halfspace, open halfspace, convex set, convex hull, order type (of a finite point set), signed covector (of a finite point set), signed cocircuit (of a finite point set).

Recital Exercise 8.2 Prove that the intersection of two convex sets in $\mathbb{R}^d$ is also convex.

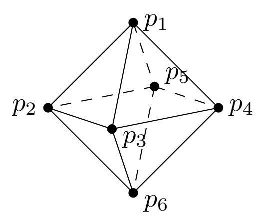
Recital Exercise 8.3 Prove that the convex hull of any set is convex.

Recital Exercise 8.4 Provide the order type, the signed covectors, and the signed cocircuits of the following set of points on the plane.



Some lines are added to help us understand the picture.

Recital Exercise 8.5 Provide the order type of the vertex set of a regular octahedron shown below.



Complementary Exercise 8.6 Prove that any closed halfspace and any open halfspace in $\mathbb{R}^d$ are convex (for any natural number $d \geq 1$).

Complementary Exercise 8.7 Prove that the convex hull of any set uniquely exists.

Complementary Exercise 8.8 For infinitely many natural numbers n, find a set P of n points such that there is no point $x \in \mathbb{R}^2$ such that every convex set $S \subseteq \mathbb{R}^2$ with $S \cap \{x\} = \emptyset$ satisfies $|S \cap P| < 2n/3$.

Complementary Exercise 8.9 Let p_1, p_2, p_3, p_4 be four points in $\mathbb{R}^3$. Prove that $\Delta(p_1, p_2, p_3, p_4) > 0$ if and only if p_4 lies in the open halfspace $\{x \in \mathbb{R}^3 \mid a \cdot x > b\}$, where $a = (p_2 - p_1) \times (p_3 - p_1) \in \mathbb{R}^3$ and $b = a \cdot p_1 \in \mathbb{R}$.

Complementary Exercise 8.10 Provide the signed cocircuits of the vertex set of the regular octahedron above.

Supplementary Exercise 8.11 Prove that the following set is convex for any natural number $d \geq 1$:

$$\left\{ x \in \mathbb{R}^d \mid \sum_{i=1}^{d} x_i^2 \leq 1 \right\}.$$

Supplementary Exercise 8.12 For two sets $X, Y \subseteq \mathbb{R}^d$, their *Minkowski sum* is defined as

$$\{x + y \in \mathbb{R}^d \mid x \in X \text{ and } y \in Y\}.$$

Prove that the Minkowski sum of two convex sets is convex.

Supplementary Exercise 8.13 Prove the centerpoint theorem, assuming that Tverberg's theorem is true.

Supplementary Exercise 8.14 Prove or disprove the following. Let $P = \{p_1, p_2, p_3, p_4, p_5\}$ be a set of five points in $\mathbb{R}^3$. Let $\chi\colon \{1, 2, 3, 4, 5\}^4 \to \{+, -, 0\}$ be the order type of P. If $\chi(1, 2, 3, 4) = \chi(2, 3, 4, 5) = \chi(3, 4, 5, 1) = \chi(4, 5, 1, 2) = +$, then $\chi(5, 1, 2, 3) = +$ too.

Supplementary Exercise 8.15 Prove or disprove the following. Let $P = \{p_1, p_2, p_3, p_4, p_5, p_6\}$ be a set of six points in $\mathbb{R}^3$, and let $\chi\colon \{1, 2, 3, 4, 5, 6\}^4 \to \{+, -, 0\}$ be the order type of P. Furthermore, let $\chi'\colon \{1, 2, 3, 4, 5, 6\}^4 \to \{+, -, 0\}$ be the order type of the vertex set of a regular octahedron. If $\chi(i_1, i_2, i_3, i_4) = \chi'(i_1, i_2, i_3, i_4)$ for all indices $1 \leq i_1 < i_2 < i_3 < i_4 \leq 6$, then P is the vertex set of a regular octahedron.

Supplementary Exercise 8.16 Consider the following set $P = \{p_1, p_2, p_3, p_4, p_5\}$ of points in $\mathbb{R}^3$: $p_1 = (0, 0, 0)$, $p_2 = (2, 0, 0)$, $p_3 = (0, 2, 0)$, $p_4 = (0, 0, 1)$, $p_5 = (1, 1, 0)$. Provide the order type of P. (You may restrict to the cases when the indices i_1, i_2, i_3, i_4 satisfy $i_1 < i_2 < i_3 < i_4$). Furthermore, provide the signed cocircuits of P.

Supplementary Exercise 8.17 Consider the following set P of points in $\mathbb{R}^3$:

$$P = \{(x_1, x_2, x_3) \mid x_1, x_2, x_3 \in \{0, 1\}\}.$$

(Namely, P is the vertex set of a 3-dimensional cube.) Provide the signed cocircuits of P.