Topics on Computing and Mathematical Sciences I Graph Theory (13) Summary and Review

Yoshio Okamoto

Tokyo Institute of Technology

July 16, 2008

"Last updated: Tue Jul 15 21:47 JST 2008"

Schedule

Basic Topics

1. Definition of Graphs; Paths and Cycles
2. Cycles; Extremality
3. Trees; Matchings I
4. Matchings II
5. Connectivity
6. Coloring I
7. Coloring II
8. Planarity

Advanced Topics

9. Extremal Graph Theory I
10. (Random Graphs)
11. Extremal Graph Theory II
12. Ramsey Theory

Today's contents

1. Encountered Topics
   - Good characterization
   - Weak duality and strong duality
   - Extremal problems
   - Ramsey theory
   - Modern graph theory
2. Encountered Proof Techniques
   - Induction
   - Double counting
   - Extremality
   - Pigeonhole principle
   - Regularity method
   - Graph-theoretic method
3. Further study and a computational view

Good characterizations

A good characterization provides an efficiently verifiable certificate

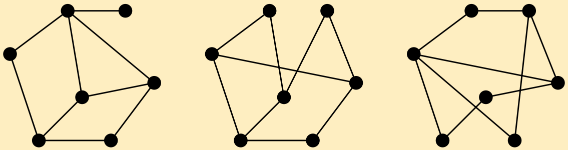
Example of good characterizations

- Kőnig (for bipartite graphs)
- Euler's theorem (for an Euler circuit)
- Hall's theorem (for a perfect matching in a bipartite graph)
- Tutte's theorem (for a perfect matching in a graph)

How can we tell the graph is bipartite?

Question

Among these graphs, which is bipartite?



Theorem 2.1 (A characterization of bipartite graphs, Kőnig '36)

A graph $G = (V, E)$ is bipartite $\iff$ G contains no odd cycle

Euler circuits and Eulerian graphs

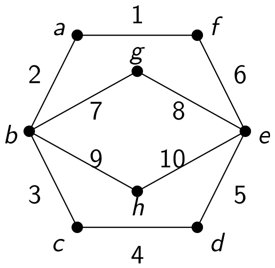
Definition (Euler circuit)

An Euler circuit in G is a circuit in G in which every edge of G appears exactly once

Definition (Eulerian graph)

G is Eulerian if it contains an Euler circuit

Example:



$a, b, g, e, h, b, c, d, e, f, a$

A characterization of Eulerian graphs

Theorem 2.3 (A characterization of Eulerian graphs, Euler 1736)

A graph G is Eulerian $\Leftrightarrow$ G has the following properties

1. All edges of G lie in the same component of G
2. The degree of each vertex of G is even

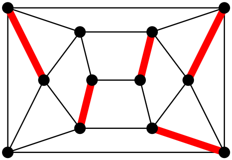
Matchings

$G = (V, E)$ a graph

Definition (Matching)

An edge subset $M \subseteq E$ is a matching of G if no two edges of M share a common vertex

Example

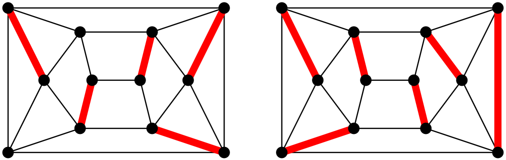


Saturation

$G = (V, E)$ a graph; $M \subseteq E$ a matching of G

Definition (Saturation)

A vertex $v \in V$ is M-saturated if v is incident to some edge of M;
Otherwise, v is M-unsaturated;
We say M saturates X ($X \subseteq V$) if every vertex $v \in X$ is M-saturated



Definition (Perfect matching)

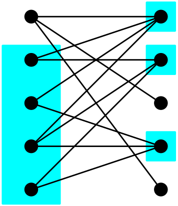
M is perfect if it saturates V

Matchings in bipartite graphs: Hall's theorem

First, a certificate for the non-existence of a perfect matching

Theorem 4.1 (Hall's theorem '35)

$G = (V, E)$ a bipartite graph with partite sets X, Y
G has a matching saturating $X \Leftrightarrow |N(S)| \geq |S|$ for all $S \subseteq X$



A characterization of graphs with a perfect matching: Tutte's theorem

Theorem 4.5 (Tutte '47)

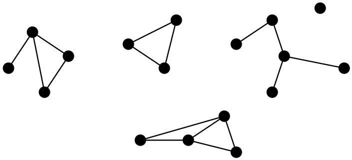
G has a perfect matching $\Leftrightarrow \forall\, S \subseteq V(G)$: $o(G - S) \leq |S|$

Definition (Odd component)

An odd component of a graph G is a connected component with odd number of vertices

Notation

$o(G)$ = the number of odd components of G



$o(G) = 3$

Today's contents

1. Encountered Topics
   - Good characterization
   - Weak duality and strong duality
   - Extremal problems
   - Ramsey theory
   - Modern graph theory
2. Encountered Proof Techniques
   - Induction
   - Double counting
   - Extremality
   - Pigeonhole principle
   - Regularity method
   - Graph-theoretic method
3. Further study and a computational view

Weak duality

Weak duality can certify the optimality of a substructure

Examples of weak duality

- $\alpha'(G) \leq \beta(G)$
- $\lambda_G(u, v) \leq \kappa_G(u, v)$
- $\lambda'_G(u, v) \leq \kappa'_G(u, v)$
- $\omega(G) \leq \chi(G)$
- $\Delta(G) \leq \chi'(G)$

Usually, easy to prove via double counting

Strong duality

Strong duality ensures the existence of a certificate

Examples of strong duality

- $\alpha'(G) = \beta(G)$ biparite (Kőnig '31, Egerváry '31)
- $\alpha'(G) = \min_{S \subseteq V}\{(n + |S| - o(G-S))/2\}$ (Berge '58)
- $\lambda_G(u, v) = \kappa_G(u, v)$ (Menger '27)
- $\lambda'_G(u, v) = \kappa'_G(u, v)$ (Menger '27)
- $\chi(G) = \omega(G)$ for interval graphs
- $\Delta(G) = \chi'(G)$ bipartite (Kőnig '16)

Usually, not easy to prove but a good characterization helps

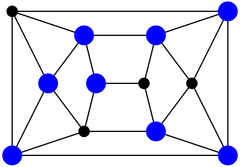
Vertex covers

$G = (V, E)$ a graph

Definition (Vertex cover)

A vertex subset $C \subseteq V$ is a vertex cover of G if every edge is incident to a vertex of C

Example



Weak duality: matchings and vertex covers

Definition (Matching number and covering number)

$\alpha'(G) =$ the size of a maximum matching of G
$\beta(G) =$ the size of a minimum vertex cover of G

Corollary 3.12 (Weak duality)

For any graph G, $\alpha'(G) \leq \beta(G)$

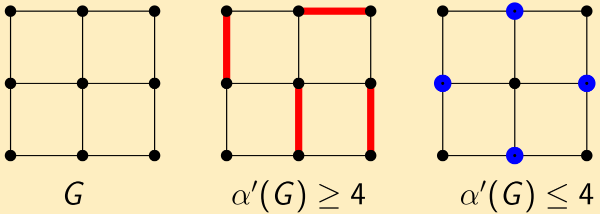
How to certify the optimality of a matching

You find a matching of size k, then you prove $\alpha'(G) \geq k$

You find a vertex cover of size k, then you prove $\alpha'(G) \leq \beta(G) \leq k$

Then, you may conclude that $\alpha'(G) = k$

Example



Kőnig-Egerváry theorem: Strong duality

Theorem 4.3 (Kőnig-Egerváry Theorem, Kőnig '31, Egerváry '31)

For any bipartite graph G, $\alpha'(G) = \beta(G)$

Remark

For every bipartite graph G, we had two theorems

- $\alpha'(G) \leq \beta(G)$ (weak duality)
  - If we find a matching M and a vertex cover C of the same size ($|M| = |C|$), then this *proves* M is a maximum matching and C is a minimum vertex cover
- $\alpha'(G) = \beta(G)$ (strong duality, *min-max theorem*)
  - We can always find such M and C

The Berge-Tutte formula

What about a min-max theorem for general graphs?

Exercise 4.4 (The Berge-Tutte formula; Berge '58)

For any graph G, it holds

$$\alpha'(G) = \min \left\{ \frac{1}{2}(n(G) + |S| - o(G-S)) \mid S \subseteq V(G) \right\}$$

One direction should be easy (corresponding to weak duality)

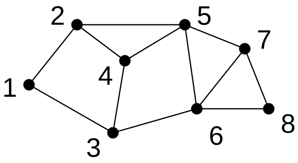
Vertex cut

$G = (V, E)$ a graph; $u, v \in V$ two distinct vertices

Definition (Vertex cut)

$S \subseteq V \setminus \{u, v\}$ is a u, v-vertex cut (or a u, v-separating set) of G if $G - S$ has no u, v-walk (or u, v-path)
In this case, we also say S *separates* u *and* v

Example



$\{3, 5\}$ is a $1, 8$-vertex cut
$\{3, 5\}$ is not a $1, 4$-vertex cut
$\{2, 3\}$ is a $1, 4$-vertex cut
$\{2, 4, 6\}$ is a $3, 5$-vertex cut

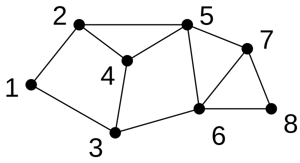
Local vertex connectivity

$G = (V, E)$ a graph; $u, v \in V$ two distinct *non-adjacent* vertices

Definition (Local vertex connectivity)

The local vertex connectivity of G between u, v is the minimum size of a u, v-vertex cut of G;
Denoted by $\kappa_G(u, v)$ (or $\kappa(u, v)$ when G is clear from the context)

Example



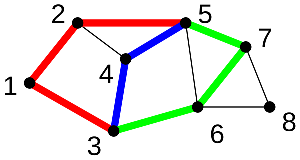
$$\kappa(\{1, 8\}) = 2$$
$$\kappa(\{1, 4\}) = 2$$
$$\kappa(\{3, 5\}) = 3$$

Internally disjoint paths

$G = (V, E)$ a graph; $u, v \in V$ two distinct vertices

Definition (Internally disjoint paths)

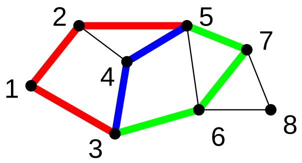
Two u, v-paths P, Q are internally disjoint if $V(P) \cap V(Q) = \{u, v\}$, namely P and Q do not share any vertex other than u and v;
Some u, v-paths $P_1, \ldots, P_k$ are pairwise internally disjoint if for any i, j, $i \neq j$, P_i and P_j are internally disjoint



Weak duality for κ and λ

Definition ($\lambda_G(u, v)$)

$\lambda_G(u, v) = \max\{k \mid \exists\ k$ pairwise internally disjoint u, v-paths in $G\}$



$\lambda_G(3, 5) = 3$

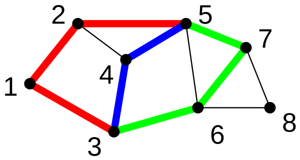
Lemma 5.4 (Weak duality)

$G = (V, E)$ a graph; $u, v \in V$ two distinct non-adjacent vertices
$\Longrightarrow \lambda_G(u, v) \leq \kappa_G(u, v)$

Strong duality for vertex connectivity

Theorem 5.5 (Menger's theorem; Menger '27)

$G = (V, E)$ a graph; $u, v \in V$ two distinct non-adjacent vertices
$\Longrightarrow \lambda_G(u, v) = \kappa_G(u, v)$



$$\lambda_G(3, 5) = 3$$
$$\kappa_G(3, 5) = 3$$

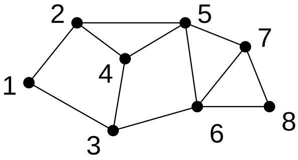
Disconnecting set

$G = (V, E)$ a graph; $u, v \in V$ two distinct vertices

Definition (Disconnecting set)

$D \subseteq E$ is a u, v-disconnecting set of G if $G - D$ has no u, v-walk (or u, v-path)

Example



$\{\{1, 2\}, \{1, 3\}\}$ is a $1, 8$-disconnecting set
$\{\{1, 2\}, \{3, 4\}, \{3, 6\}\}$ is a $3, 5$-disconnecting set

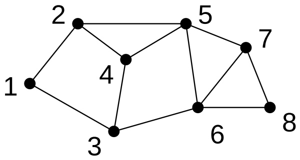
Local edge connectivity

$G = (V, E)$ a graph; $u, v \in V$ two distinct vertices (not necessarily non-adjacent)

Definition (Local edge connectivity)

The local edge connectivity of G between u, v is the minimum size of a u, v-disconnecting set of G;
Denoted by $\kappa'_G(u, v)$ (or $\kappa(u, v)$ when G is clear from the context)

Example



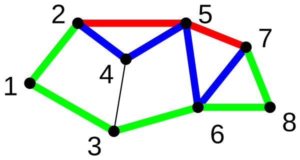
$\kappa'(\{1, 8\}) = 2$
$\kappa'(\{1, 4\}) = 2$
$\kappa'(\{3, 5\}) = 3$

Edge-disjoint paths

$G = (V, E)$ a graph; $u, v \in V$ two distinct vertices

Definition (Edge-disjoint path)

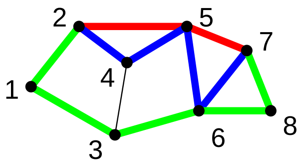
Two u, v-paths P and Q are edge-disjoint if $E(P) \cap E(Q) = \emptyset$, namely P and Q do not share any edge;
Some u, v-paths $P_1, \ldots, P_k$ are pairwise edge-disjoint if for any i, j, $i \neq j$, P_i and P_j are edge disjoint



Weak duality: Edge version

Definition ($\lambda'_G(u, v)$)

$\lambda'_G(u, v) = \max\{k \mid \exists\ k$ pairwise edge disjoint u, v-paths in $G\}$



$\lambda'_G(2, 7) = 3$

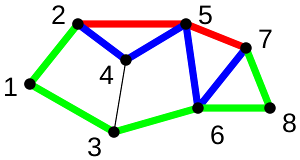
Lemma 5.6 (Weak duality)

$G = (V, E)$ a graph; $u, v \in V$ two distinct vertices
$\Longrightarrow \lambda'_G(u, v) \leq \kappa'_G(u, v)$

Menger's theorem: Edge version

Theorem 5.7 (Menger's theorem; Menger '27)

$G = (V, E)$ a graph; $u, v \in V$ two distinct vertices
$\Longrightarrow \lambda'_G(u, v) = \kappa'_G(u, v)$



$$\lambda'_G(2, 7) = 3$$
$$\kappa'_G(2, 7) = 3$$

Chromatic numbers

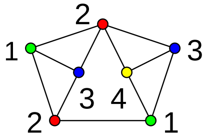
$G = (V, E)$ a graph

Definition (Chromatic number)

The chromatic number of G is the min k for which G is k-colorable

Notation

$\chi(G)$ = the chromatic number of G



$\chi(G) = 4$

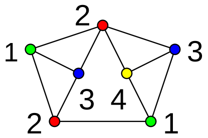
Easy lower bound

Definition (clique, clique number (recap))

A set $S \subseteq V$ is a clique if every pair of vertices of S are adjacent;
$\omega(G) =$ the size of a largest clique of G

Proposition 6.1 (Easy lower bound for the chromatic number)

$\chi(G) \geq \omega(G)$ for every graph G



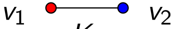
$\chi(G) = 4$
$\omega(G) = 3$

Iterative application of Mycielski's construction

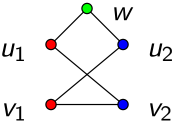
Definition (Mycielski's construction)

From G, Mycielski's construction produces a graph $M(G)$ containing G, as follows:

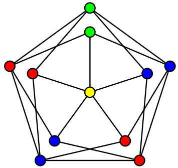
- Let $V = \{v_1, \ldots, v_n\}$
- $V(M(G)) = V \cup \{u_1, \ldots, u_n, w\}$
- $E(M(G)) = E \cup \{\{u_i, v\} \mid v \in N_G(v_i) \cup \{w\}\}$



K_2



$M(K_2) \simeq C_5$



$M^2(K_2)$
(Grötzsch graph)

Properties of Mycielski's construction

Theorem 7.1 (Mycielski '55)

1. $G \not\supseteq K_3 \Rightarrow M(G) \not\supseteq K_3$
2. $\chi(G) = k \Rightarrow \chi(M(G)) = k+1$

Corollary 7.2

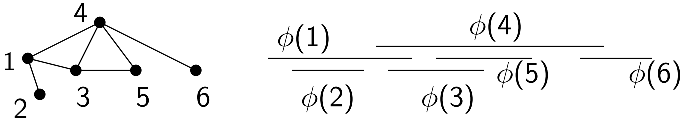
1. $\forall\ k \geq 2\ \exists$ a graph G: $\omega(G) = 2$ and $\chi(G) = k$
2. $\forall\ k \geq \ell \geq 2\ \exists$ a graph G: $\omega(G) = \ell$ and $\chi(G) = k$

Namely, the bound $\chi(G) \geq \omega(G)$ can be arbitrarily bad

Interval graphs

Definition (Interval graph)

G is an interval graph if $\exists$ a set $\mathcal{I}$ of (closed) intervals and a bijection $\phi\colon V(G) \to \mathcal{I}$ s.t. u, v adjacent iff $\phi(u) \cap \phi(v) \neq \emptyset$



Theorem 6.7 (Chromatic number of an interval graph)

G an interval graph $\Rightarrow \chi(G) = \omega(G)$

Today's contents

1. Encountered Topics
   - Good characterization
   - Weak duality and strong duality
   - Extremal problems
   - Ramsey theory
   - Modern graph theory
2. Encountered Proof Techniques
   - Induction
   - Double counting
   - Extremality
   - Pigeonhole principle
   - Regularity method
   - Graph-theoretic method
3. Further study and a computational view

What's an extremal problem?: Informal definition

Informal definition: Extremal problem

Determination of the max (or min) value of a parameter over some class of objects (n-vertex graphs in this course)

For example...

objective	maximization
parameter	minimum degree
class	non-Hamiltonian graphs

Dirac's theorem for Hamiltonicity

Theorem 2.6 (Min-degree condition for Hamiltonicity, Dirac '52)

$n(G) \geq 3$, $\delta(G) \geq \lceil n(G)/2 \rceil \Longrightarrow G$ Hamiltonian

Observation

For every $n \geq 3$, there exists an n-vertex non-Hamiltonian graph G with $\delta(G) = \lceil n/2 \rceil - 1$

Therefore

$$\max\{\delta(G) \mid G \text{ } n\text{-vertex and non-Hamiltonian}\} = \lceil n/2 \rceil - 1$$

A typical extremal problem

Question

Given a natrual number n and a graph H,
What is the maximum number of edges in an n-vertex graph that doesn't contain H?

Notation

$\mathrm{ex}(n, H) = \max\{e(G) \mid n(G) = n, G \not\supseteq H\}$

Extremality for having no H

Theorem 2.7 (Extremality for having no K_3, Mantel 1907)

$$\mathrm{ex}(n, K_3) = \lfloor n^2/4 \rfloor$$

Theorem 9.2 (Extremality for having no K_r; Turán '41)

$$\mathrm{ex}(n, K_r) = e(T_{n,r}) \qquad \text{for all } r \geq 3$$

Theorem 9.3 (Erdős, Simonovits '66; Erdős-Simonovits-Stone thm)

$$\mathrm{ex}(n, H) = \left(1 - \frac{1}{\chi(H) - 1}\right)\binom{n}{2} + \mathrm{o}(n^2) \qquad \text{for all } H$$

Exercise 9.3 (Extremality for having no $K_{s,t}$; Kővári, Sós, Turán '54)

$$\mathrm{ex}(n, K_{s,t}) = \mathrm{O}(n^{2-1/\min\{s,t\}})$$

Today's contents

1. Encountered Topics
   - Good characterization
   - Weak duality and strong duality
   - Extremal problems
   - Ramsey theory
   - Modern graph theory
2. Encountered Proof Techniques
   - Induction
   - Double counting
   - Extremality
   - Pigeonhole principle
   - Regularity method
   - Graph-theoretic method
3. Further study and a computational view

What is Ramsey theory

Quotation from D. West '01

"Ramsey theory" referes to the study of partitions of large structures. Typical results state that a special substructure must occur in some class of the partition. Motzkin described this by saying that "Complete disorder is impossible." The objects we consider are merely sets and numbers, ...

Definition (Ramsey number)

The Ramsey number $R(k, \ell)$ is the minimum r for which every 2-coloring (say with red and blue) of the edges of K_r contains a red K_k or a blue K_ℓ

There are several variants: Graph Ramsey numbers, multi-color Ramsey numbers...

Upper bound for Ramsey numbers

Theorem 12.4 (Recursion for Ramsey numbers; Ramsey '30)

For $k, \ell > 1$, it holds $R(k,\ell) \leq R(k,\ell-1) + R(k-1,\ell)$

Exercise 12.1 (Upper bound for Ramsey numbers)

For $k, \ell \geq 1$, it holds $R(k,\ell) \leq \binom{k+\ell-2}{k-1}$

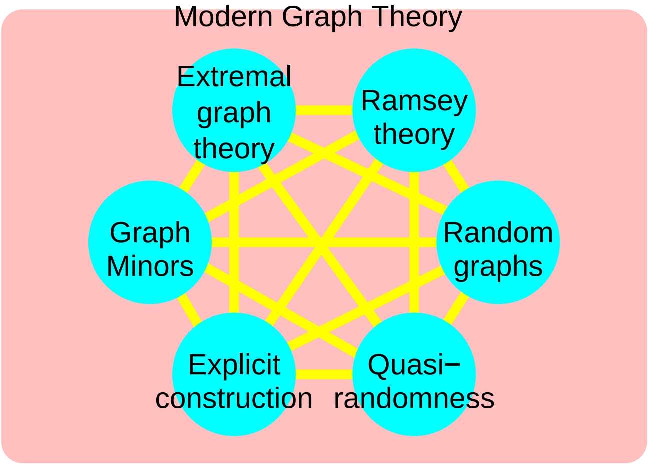
Some small Ramsey numbers

- $R(3,3) = 6$ (Prop 12.2)
- $R(3,4) \leq 9$ (Exer 12.2.1)
- $R(4,4) \leq 18$ (Exer 12.2.2)

Today's contents

1. Encountered Topics
   - Good characterization
   - Weak duality and strong duality
   - Extremal problems
   - Ramsey theory
   - Modern graph theory
2. Encountered Proof Techniques
   - Induction
   - Double counting
   - Extremality
   - Pigeonhole principle
   - Regularity method
   - Graph-theoretic method
3. Further study and a computational view

Modern theory of graphs



What we didn't touch upon

and how they are related to what we saw

- Graph minors ↔ Planarity
  - Planarity can be characterized by having no K_5-minor or $K_{3,3}$-minor (Wagner's theorem). Graph minor theorem by Robertson and Seymour says every minor-closed property can be characterized by finitely many forbidden minors.
- Quasi-randomness ↔ Szemerédi's regularity
  - Regular pairs look like random bipartite graphs, but they are not random but concrete graphs. A quasi-random graph is a concrete graph with some properties similar to random graphs.
- Explicit construction ↔ Exercise 9.5
  - In Exer 9.5, we find that a construction of graphs via finite fields gives a tight lower bound for $\mathrm{ex}(n, K_{2,2})$. It is often the case that good constructions rely heavily on algebra, number theory, and so on.

Today's contents

1. Encountered Topics
   - Good characterization
   - Weak duality and strong duality
   - Extremal problems
   - Ramsey theory
   - Modern graph theory
2. Encountered Proof Techniques
   - Induction
   - Double counting
   - Extremality
   - Pigeonhole principle
   - Regularity method
   - Graph-theoretic method
3. Further study and a computational view

Principle of induction

Assuming the statement holds for the smaller value...

Appeared frequently, for example,

- A u, v-walk contains a u, v-path (Prop 1.6)
- A characterization of trees (Thm 3.5)
- Menger's theorem on connectivity (Thm 5.5)
- Kőnig's theorem on the chromatic index of a bipartite graph (Thm 7.4)
- Euler's formula for planar graphs (Thm 8.1)
- Five-color theorem (Thm 8.7)
- Turán's theorem (Thm 9.2)
- Erdős-Stone theorem (Lem 9.5)

and in some of the exercises

Double counting

Counting one thing in two different ways

Appeared many times

- Handshaking lemma (Thm 1.1)
- Mantel's theorem (Thm 2.7)
- Weak duality (Prop 3.11, Lem 5.4, Lem 5.6)
- Every regular bipartite graph has a perfect matching (Thm 4.2)
- Every 3-reg graph w/o cut-edge has a perfect matching (Thm 4.5)
- Number of edges in a planar graph (Prop 8.2)

and in some of the exercises

Extremality

When choosing an object, pick an extremal one and use this property

Encountered many times

- A characterization of Eulerian graphs (Thm 2.3)
- Dirac's theorem for Hamiltonicity (Thm 2.6)
- Every tree has at least two leaves ($n \geq 2$) (Lem 3.3)
- Tutte's theorem for perfect matchings (Thm 4.4)

and in some of the exercises

Pigeonhole principle

When may objects are put in few boxes, at least one box has many

- Dirac's theorem for Hamiltonicity (Thm 2.6)
- List chromatic numbers can be arbitrarily larger than chromatic numbers (Page 38, Lect 7)
- Erdős-Stone theorem (Lem 9.5)
- Ramsey theorem (Thm 12.4)

and in some of the exercises

Regularity method

If you have a dense graph, then regularize and hope the regularity graph has a nice property

- Quantative ver of Mantel's theorem (Prop 11.4)
- Quantative ver of Turán's theorem (Exer 11.2)
- Erdős-Stone theorem (Thm 9.4)
- Ramsey number of bounded-degree graphs (Thm 12.6)

and in some of the exercises

Today's contents

1. Encountered Topics
   - Good characterization
   - Weak duality and strong duality
   - Extremal problems
   - Ramsey theory
   - Modern graph theory
2. Encountered Proof Techniques
   - Induction
   - Double counting
   - Extremality
   - Pigeonhole principle
   - Regularity method
   - Graph-theoretic method
3. Further study and a computational view

Graph-theoretic method

For a seemingly non-graph-theoretic problem, derive a graph structure and apply results for graphs

Sometimes appeared in Exercises

- System of distinct representatives (Exer 4.1)
- Number of point-pairs far from each other in an equilateral triangle (Exer 9.2)
- Number of unit-distance pairs on the plane and in the 3-dim space (Exer 9.4)
- Roth's thm for 3-term arithmetic progressions (Exer 11.6)
- Schur's theorem (next slide)

Schur's theorem about integers

Quiz

Can we partition $\{1, 2, \ldots, 13\}$ into three parts so that no part has (not necessarily distinct) three numbers x, y, z with $x + y = z$?

Schur's theorem about integers

Quiz

Can we partition $\{1, 2, \ldots, 13\}$ into three parts so that no part has (not necessarily distinct) three numbers x, y, z with $x + y = z$?

Answer

Yes: $\{1, 4, 10, 13\}$, $\{2, 3, 11, 12\}$, $\{5, 6, 7, 8, 9\}$

Schur's theorem about integers

Quiz

Can we partition $\{1, 2, \ldots, 13\}$ into three parts so that no part has (not necessarily distinct) three numbers x, y, z with $x + y = z$?

Answer

Yes: $\{1, 4, 10, 13\}$, $\{2, 3, 11, 12\}$, $\{5, 6, 7, 8, 9\}$

Theorem 13.1 (Schur's theorem; Schur '16)

$\forall\ n \in \mathbb{N}\ \exists\ r \in \mathbb{N}$
$\forall$ a partition of $\{1, \ldots, r\}$ into n parts $\exists$ a part containing (not necessarily distinct) three numbers x, y, z s.t. $x + y = z$

Proof idea of Schur's theorem

Given n; Let $r = R(\underbrace{3, 3, \ldots, 3}_{n \text{ times}})$ the multi-color Ramsey number

- Let $X_1, X_2, \ldots, X_n$ be a given partition of $\{1, \ldots, r\}$
- Consider a complete graph $G = (V, E)$ where $V = \{1, \ldots, r\}$
- Color the edges with n colors as follows:
  $\{a, b\}$ is colored by i iff $a - b \in X_i$ $(a > b)$
- $\therefore \exists$ a K_3 of color i for some $i \in \{1, \ldots, n\}$
- Let K_3 be formed by $a, b, c \in \{1, \ldots, r\}$ (wlog $a > b > c$)
- $a-b, a-c, b-c \in X_i$ $\square$

Today's contents

1. Encountered Topics
   - Good characterization
   - Weak duality and strong duality
   - Extremal problems
   - Ramsey theory
   - Modern graph theory
2. Encountered Proof Techniques
   - Induction
   - Double counting
   - Extremality
   - Pigeonhole principle
   - Regularity method
   - Graph-theoretic method
3. Further study and a computational view

Relation to computational view

- Mathematical characterizations can be classified from the computational perspective (good characterization)
- Efficiently-solvable optimization problems rely on strong duality
- Hard problems lack in good characterizations and strong duality

- Large structures can be approximated by a random-looking object (regularity method), so we may regard them as a random object as long as approximation is concerned
- Large structures contain a still large well-organized substructure (Ramsey theory), so finding such a substructure might help for your computation

Hypergraphs

Caution

- For some applications, the graph-theoretic method is not powerful enough
- Hypergraph-theoretic method would be an alternative

Definition (Hypergraph)

A hypergraph is a pair (V, E) of a finite set V and a family $E \subseteq 2^V$ of subsets of V;
Elements of E are called hyperedges

There are concepts in hypergraph theory corresponding to graphs: degrees, paths, cycles, trees, connectivity, coloring, regularity, Ramsey theorem, ...

The final “piece of paper”

- Write something!
  - whatever you got in the course of lectures
  - whatever you felt during the lectures
  - whatever you think about this special education program
  - whatever you saw this morning
  - whatever you like to share with me
  - ...
- Please hand in before you leave

Thanks a lot for your patience!