

# Topics on Computing and Mathematical Sciences I Graph Theory (13) Summary and Review

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## Schedule

### Basic Topics

- 1 Definition of Graphs; Paths and Cycles
- 2 Cycles; Extremality
- 3 Trees; Matchings I
- 4 Matchings II
- 5 Connectivity
- 6 Coloring I
- 7 Coloring II
- 8 Planarity

### Advanced Topics

- 9 Extremal Graph Theory I
- 10 (Random Graphs)
- 11 Extremal Graph Theory II
- 12 Ramsey Theory

## Today's contents

- 1 Encountered Topics
  - Good characterization
  - Weak duality and strong duality
  - Extremal problems
  - Ramsey theory
  - Modern graph theory
- 2 Encountered Proof Techniques
  - Induction
  - Double counting
  - Extremality
  - Pigeonhole principle
  - Regularity method
  - Graph-theoretic method
- 3 Further study and a computational view

## Good characterizations

A **good characterization** provides an efficiently verifiable certificate

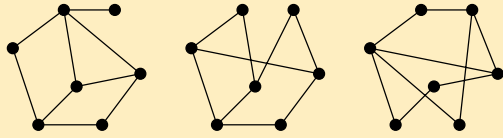
### Example of good characterizations

- König (for bipartite graphs)
- Euler's theorem (for an Euler circuit)
- Hall's theorem (for a perfect matching in a bipartite graph)
- Tutte's theorem (for a perfect matching in a graph)

## How can we tell the graph is bipartite?

### Question

Among these graphs, which is bipartite?



**Theorem 2.1** (A characterization of bipartite graphs, König '36)

A graph  $G = (V, E)$  is bipartite  $\iff G$  contains no odd cycle

## Euler circuits and Eulerian graphs

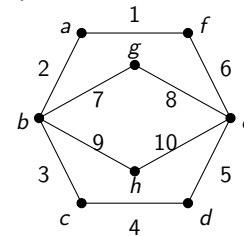
### Definition (Euler circuit)

An **Euler circuit** in  $G$  is a circuit in  $G$  in which every edge of  $G$  appears exactly once

### Definition (Eulerian graph)

$G$  is **Eulerian** if it contains an Euler circuit

Example:



$a, b, g, e, h, b, c, d, e, f, a$

## A characterization of Eulerian graphs

**Theorem 2.3** (A characterization of Eulerian graphs, Euler 1736)

A graph  $G$  is Eulerian  $\iff G$  has the following properties

- ① All edges of  $G$  lie in the same component of  $G$
- ② The degree of each vertex of  $G$  is even

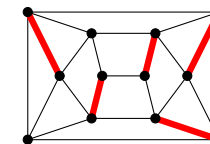
## Matchings

$G = (V, E)$  a graph

### Definition (Matching)

An edge subset  $M \subseteq E$  is a **matching** of  $G$  if no two edges of  $M$  share a common vertex

Example

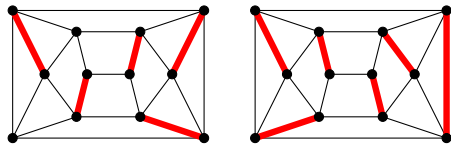


### Saturation

$G = (V, E)$  a graph;  $M \subseteq E$  a matching of  $G$

#### Definition (Saturation)

A vertex  $v \in V$  is  **$M$ -saturated** if  $v$  is incident to some edge of  $M$ ;  
 Otherwise,  $v$  is  **$M$ -unsaturated**;  
 We say  **$M$  saturates  $X$**  ( $X \subseteq V$ ) if every vertex  $v \in X$  is  $M$ -saturated



#### Definition (Perfect matching)

$M$  is **perfect** if it saturates  $V$

### A characterization of graphs with a perfect matching: Tutte's theorem

#### Theorem 4.5 (Tutte '47)

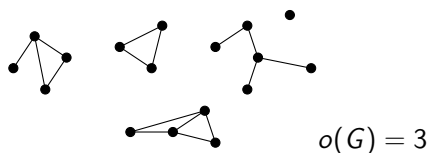
$G$  has a perfect matching  $\Leftrightarrow \forall S \subseteq V(G): o(G-S) \leq |S|$

#### Definition (Odd component)

An **odd component** of a graph  $G$  is a connected component with odd number of vertices

#### Notation

$o(G)$  = the number of odd components of  $G$

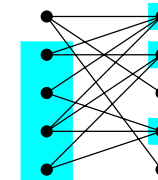


### Matchings in bipartite graphs: Hall's theorem

First, a certificate for the non-existence of a perfect matching

#### Theorem 4.1 (Hall's theorem '35)

$G = (V, E)$  a bipartite graph with partite sets  $X, Y$   
 $G$  has a matching saturating  $X \Leftrightarrow |N(S)| \geq |S|$  for all  $S \subseteq X$



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## Weak duality

Weak duality can certify the optimality of a substructure

### Examples of weak duality

- $\alpha'(G) \leq \beta(G)$
- $\lambda_G(u, v) \leq \kappa_G(u, v)$
- $\lambda'_G(u, v) \leq \kappa'_G(u, v)$
- $\omega(G) \leq \chi(G)$
- $\Delta(G) \leq \chi'(G)$

Usually, easy to prove via double counting

## Strong duality

Strong duality ensures the existence of a certificate

### Examples of strong duality

- $\alpha'(G) = \beta(G)$  bipartite (König '31, Egerváry '31)
- $\alpha'(G) = \min_{S \subseteq V} \{(n + |S| - o(G-S))/2\}$  (Berge '58)
- $\lambda_G(u, v) = \kappa_G(u, v)$  (Menger '27)
- $\lambda'_G(u, v) = \kappa'_G(u, v)$  (Menger '27)
- $\chi(G) = \omega(G)$  for interval graphs
- $\Delta(G) = \chi'(G)$  bipartite (König '16)

Usually, not easy to prove but a good characterization helps

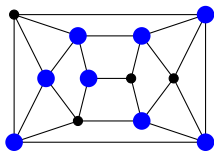
## Vertex covers

$G = (V, E)$  a graph

### Definition (Vertex cover)

A vertex subset  $C \subseteq V$  is a vertex cover of  $G$  if every edge is incident to a vertex of  $C$

Example



## Weak duality: matchings and vertex covers

### Definition (Matching number and covering number)

$\alpha'(G)$  = the size of a maximum matching of  $G$

$\beta(G)$  = the size of a minimum vertex cover of  $G$

### Corollary 3.12 (Weak duality)

For any graph  $G$ ,  $\alpha'(G) \leq \beta(G)$

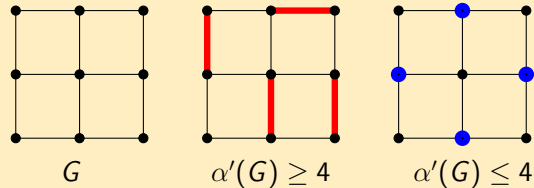
### How to certify the optimality of a matching

You find a matching of size  $k$ , then you **prove**  $\alpha'(G) \geq k$

You find a vertex cover of size  $k$ , then you **prove**  $\alpha'(G) \leq \beta(G) \leq k$

Then, you may conclude that  $\alpha'(G) = k$

#### Example



### König-Egerváry theorem: Strong duality

Theorem 4.3 (König-Egerváry Theorem, König '31, Egerváry '31)

For any **bipartite** graph  $G$ ,  $\alpha'(G) = \beta(G)$

#### Remark

For every bipartite graph  $G$ , we had two theorems

- $\alpha'(G) \leq \beta(G)$  (weak duality)
  - If we find a matching  $M$  and a vertex cover  $C$  of the same size ( $|M| = |C|$ ), then this *proves*  $M$  is a maximum matching and  $C$  is a minimum vertex cover
- $\alpha'(G) = \beta(G)$  (strong duality, *min-max theorem*)
  - We can always find such  $M$  and  $C$

### The Berge-Tutte formula

What about a min-max theorem for general graphs?

Exercise 4.4 (The Berge-Tutte formula; Berge '58)

For any graph  $G$ , it holds

$$\alpha'(G) = \min \left\{ \frac{1}{2}(n(G) + |S| - o(G-S)) \mid S \subseteq V(G) \right\}$$

One direction should be easy (corresponding to weak duality)

### Vertex cut

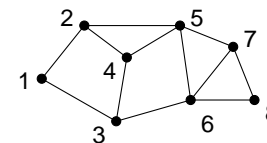
$G = (V, E)$  a graph;  $u, v \in V$  two distinct vertices

Definition (Vertex cut)

$S \subseteq V \setminus \{u, v\}$  is a  **$u, v$ -vertex cut** (or a  **$u, v$ -separating set**) of  $G$  if  $G-S$  has no  $u, v$ -walk (or  $u, v$ -path)

In this case, we also say  $S$  *separates*  $u$  and  $v$

Example



- $\{3, 5\}$  is a 1, 8-vertex cut
- $\{3, 5\}$  is not a 1, 4-vertex cut
- $\{2, 3\}$  is a 1, 4-vertex cut
- $\{2, 4, 6\}$  is a 3, 5-vertex cut

Local vertex connectivity

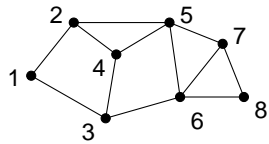
$G = (V, E)$  a graph;  $u, v \in V$  two distinct *non-adjacent* vertices

Definition (Local vertex connectivity)

The **local vertex connectivity** of  $G$  between  $u, v$  is the minimum size of a  $u, v$ -vertex cut of  $G$ ;

Denoted by  $\kappa_G(u, v)$  (or  $\kappa(u, v)$  when  $G$  is clear from the context)

Example



$$\begin{aligned} \kappa(\{1, 8\}) &= 2 \\ \kappa(\{1, 4\}) &= 2 \\ \kappa(\{3, 5\}) &= 3 \end{aligned}$$

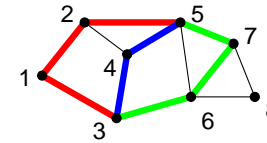
Internally disjoint paths

$G = (V, E)$  a graph;  $u, v \in V$  two distinct vertices

Definition (Internally disjoint paths)

Two  $u, v$ -paths  $P, Q$  are **internally disjoint** if  $V(P) \cap V(Q) = \{u, v\}$ , namely  $P$  and  $Q$  do not share any vertex other than  $u$  and  $v$ ;

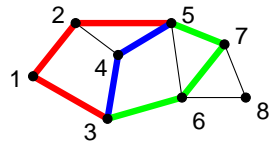
Some  $u, v$ -paths  $P_1, \dots, P_k$  are **pairwise internally disjoint** if for any  $i, j, i \neq j, P_i$  and  $P_j$  are internally disjoint



Weak duality for  $\kappa$  and  $\lambda$

Definition ( $\lambda_G(u, v)$ )

$\lambda_G(u, v) = \max\{k \mid \exists k \text{ pairwise internally disjoint } u, v\text{-paths in } G\}$



$$\lambda_G(3, 5) = 3$$

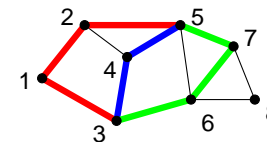
Lemma 5.4 (Weak duality)

$G = (V, E)$  a graph;  $u, v \in V$  two distinct non-adjacent vertices  
 $\implies \lambda_G(u, v) \leq \kappa_G(u, v)$

Strong duality for vertex connectivity

Theorem 5.5 (Menger's theorem; Menger '27)

$G = (V, E)$  a graph;  $u, v \in V$  two distinct non-adjacent vertices  
 $\implies \lambda_G(u, v) = \kappa_G(u, v)$



$$\begin{aligned} \lambda_G(3, 5) &= 3 \\ \kappa_G(3, 5) &= 3 \end{aligned}$$

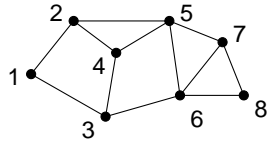
Disconnecting set

$G = (V, E)$  a graph;  $u, v \in V$  two distinct vertices

Definition (Disconnecting set)

$D \subseteq E$  is a  **$u, v$ -disconnecting set** of  $G$  if  $G - D$  has no  $u, v$ -walk (or  $u, v$ -path)

Example



$\{\{1, 2\}, \{1, 3\}\}$  is a 1, 8-disconnecting set  
 $\{\{1, 2\}, \{3, 4\}, \{3, 6\}\}$  is a 3, 5-disconnecting set

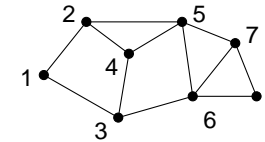
Local edge connectivity

$G = (V, E)$  a graph;  $u, v \in V$  two distinct vertices (not necessarily non-adjacent)

Definition (Local edge connectivity)

The **local edge connectivity** of  $G$  between  $u, v$  is the minimum size of a  $u, v$ -disconnecting set of  $G$ ;  
 Denoted by  $\kappa'_G(u, v)$  (or  $\kappa(u, v)$  when  $G$  is clear from the context)

Example



$$\begin{aligned} \kappa'_G(\{1, 8\}) &= 2 \\ \kappa'_G(\{1, 4\}) &= 2 \\ \kappa'_G(\{3, 5\}) &= 3 \end{aligned}$$

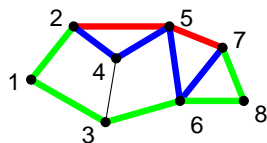
Edge-disjoint paths

$G = (V, E)$  a graph;  $u, v \in V$  two distinct vertices

Definition (Edge-disjoint path)

Two  $u, v$ -paths  $P$  and  $Q$  are **edge-disjoint** if  $E(P) \cap E(Q) = \emptyset$ , namely  $P$  and  $Q$  do not share any edge;

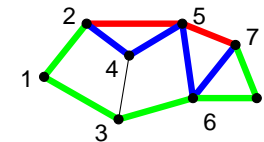
Some  $u, v$ -paths  $P_1, \dots, P_k$  are **pairwise edge-disjoint** if for any  $i, j, i \neq j, P_i$  and  $P_j$  are edge disjoint



Weak duality: Edge version

Definition ( $\lambda'_G(u, v)$ )

$\lambda'_G(u, v) = \max\{k \mid \exists k \text{ pairwise edge disjoint } u, v\text{-paths in } G\}$



$$\lambda'_G(2, 7) = 3$$

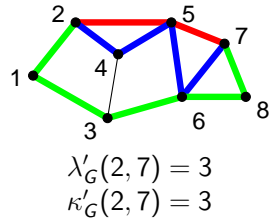
Lemma 5.6 (Weak duality)

$G = (V, E)$  a graph;  $u, v \in V$  two distinct vertices  
 $\implies \lambda'_G(u, v) \leq \kappa'_G(u, v)$

### Menger's theorem: Edge version

Theorem 5.7 (Menger's theorem; Menger '27)

$G = (V, E)$  a graph;  $u, v \in V$  two distinct vertices  
 $\implies \lambda'_G(u, v) = \kappa'_G(u, v)$



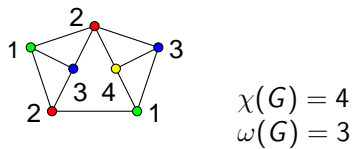
### Easy lower bound

Definition (clique, clique number (recap))

A set  $S \subseteq V$  is a **clique** if every pair of vertices of  $S$  are adjacent;  
 $\omega(G)$  = the size of a largest clique of  $G$

Proposition 6.1 (Easy lower bound for the chromatic number)

$\chi(G) \geq \omega(G)$  for every graph  $G$



### Chromatic numbers

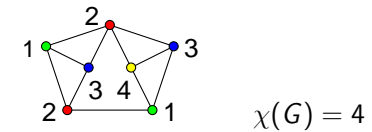
$G = (V, E)$  a graph

Definition (Chromatic number)

The **chromatic number** of  $G$  is the min  $k$  for which  $G$  is  $k$ -colorable

Notation

$\chi(G)$  = the chromatic number of  $G$

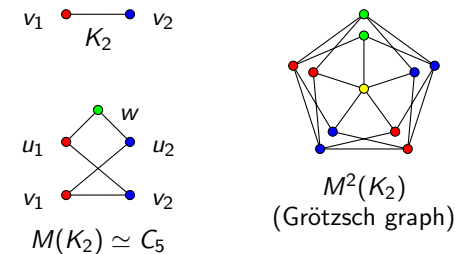


### Iterative application of Mycielski's construction

Definition (Mycielski's construction)

From  $G$ , **Mycielski's construction** produces a graph  $M(G)$  containing  $G$ , as follows:

- Let  $V = \{v_1, \dots, v_n\}$
- $V(M(G)) = V \cup \{u_1, \dots, u_n, w\}$
- $E(M(G)) = E \cup \{\{u_i, v_j\} \mid v_j \in N_G(v_i)\} \cup \{w\}$





Properties of Mycielski's construction

Theorem 7.1 (Mycielski '55)

- ①  $G \not\cong K_3 \Rightarrow M(G) \not\cong K_3$
- ②  $\chi(G) = k \Rightarrow \chi(M(G)) = k+1$

Corollary 7.2

- ①  $\forall k \geq 2 \exists$  a graph  $G: \omega(G) = 2$  and  $\chi(G) = k$
- ②  $\forall k \geq \ell \geq 2 \exists$  a graph  $G: \omega(G) = \ell$  and  $\chi(G) = k$

Namely, the bound  $\chi(G) \geq \omega(G)$  can be arbitrarily bad

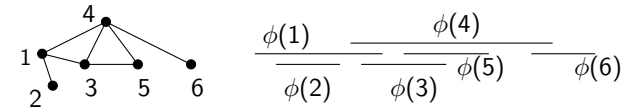
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Interval graphs

Definition (Interval graph)

$G$  is an **interval graph** if  $\exists$  a set  $\mathcal{I}$  of (closed) intervals and a bijection  $\phi: V(G) \rightarrow \mathcal{I}$  s.t.  $u, v$  adjacent iff  $\phi(u) \cap \phi(v) \neq \emptyset$



Theorem 6.7 (Chromatic number of an interval graph)

$G$  an interval graph  $\Rightarrow \chi(G) = \omega(G)$

What's an extremal problem?: Informal definition

Informal definition: Extremal problem

Determination of the max (or min) value of a parameter over some class of objects ( $n$ -vertex graphs in this course)

For example...

objective parameter class	maximization minimum degree non-Hamiltonian graphs
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## Dirac's theorem for Hamiltonicity

Theorem 2.6 (Min-degree condition for Hamiltonicity, Dirac '52)

$$n(G) \geq 3, \delta(G) \geq \lceil n(G)/2 \rceil \implies G \text{ Hamiltonian}$$

## Observation

For every  $n \geq 3$ , there exists an  $n$ -vertex non-Hamiltonian graph  $G$  with  $\delta(G) = \lceil n/2 \rceil - 1$

Therefore

$$\max\{\delta(G) \mid G \text{ } n\text{-vertex and non-Hamiltonian}\} = \lceil n/2 \rceil - 1$$

## A typical extremal problem

## Question

Given a natural number  $n$  and a graph  $H$ ,  
What is the maximum number of edges in an  $n$ -vertex graph that doesn't contain  $H$ ?

## Notation

$$\text{ex}(n, H) = \max\{e(G) \mid n(G) = n, G \not\supseteq H\}$$

Extremality for having no  $H$ 

Theorem 2.7 (Extremality for having no  $K_3$ , Mantel 1907)

$$\text{ex}(n, K_3) = \lfloor n^2/4 \rfloor$$

Theorem 9.2 (Extremality for having no  $K_r$ ; Turán '41)

$$\text{ex}(n, K_r) = e(T_{n,r}) \quad \text{for all } r \geq 3$$

Theorem 9.3 (Erdős, Simonovits '66; Erdős-Simonovits-Stone thm)

$$\text{ex}(n, H) = \left(1 - \frac{1}{\chi(H) - 1}\right) \binom{n}{2} + o(n^2) \quad \text{for all } H$$

Exercise 9.3 (Extremality for having no  $K_{s,t}$ ; Kővári, Sós, Turán '54)

$$\text{ex}(n, K_{s,t}) = O(n^{2-1/\min\{s,t\}})$$

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## What is Ramsey theory

## Quotation from D. West '01

"Ramsey theory" refers to the study of partitions of large structures. Typical results state that a special substructure must occur in some class of the partition. Motzkin described this by saying that "Complete disorder is impossible." The objects we consider are merely sets and numbers, ...

## Definition (Ramsey number)

The **Ramsey number**  $R(k, \ell)$  is the minimum  $r$  for which every 2-coloring (say with **red** and **blue**) of the edges of  $K_r$  contains a **red**  $K_k$  or a **blue**  $K_\ell$

There are several variants: Graph Ramsey numbers, multi-color Ramsey numbers...

## Upper bound for Ramsey numbers

## Theorem 12.4 (Recursion for Ramsey numbers; Ramsey '30)

For  $k, \ell > 1$ , it holds  $R(k, \ell) \leq R(k, \ell-1) + R(k-1, \ell)$

## Exercise 12.1 (Upper bound for Ramsey numbers)

For  $k, \ell \geq 1$ , it holds  $R(k, \ell) \leq \binom{k+\ell-2}{k-1}$

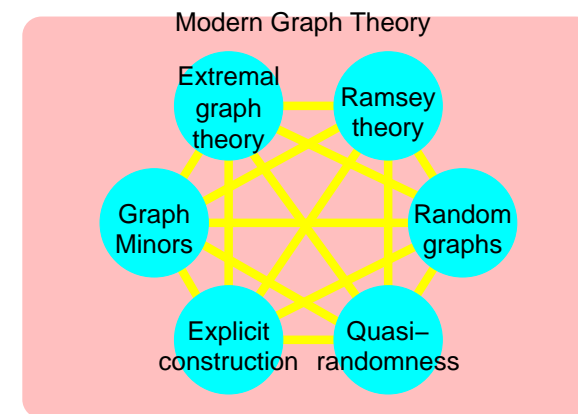
Some small Ramsey numbers

- $R(3, 3) = 6$  (Prop 12.2)
- $R(3, 4) \leq 9$  (Exer 12.2.1)
- $R(4, 4) \leq 18$  (Exer 12.2.2)

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## Modern theory of graphs



## What we didn't touch upon

and how they are related to what we saw

- Graph minors  $\leftrightarrow$  Planarity
  - Planarity can be characterized by having no  $K_5$ -minor or  $K_{3,3}$ -minor (Wagner's theorem). Graph minor theorem by Robertson and Seymour says every minor-closed property can be characterized by finitely many forbidden minors.
- Quasi-randomness  $\leftrightarrow$  Szemerédi's regularity
  - Regular pairs look like random bipartite graphs, but they are not random but concrete graphs. A quasi-random graph is a concrete graph with some properties similar to random graphs.
- Explicit construction  $\leftrightarrow$  Exercise 9.5
  - In Exer 9.5, we find that a construction of graphs via finite fields gives a tight lower bound for  $ex(n, K_{2,2})$ . It is often the case that good constructions rely heavily on algebra, number theory, and so on.

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## Principle of induction

Assuming the statement holds for the smaller value...

Appeared frequently, for example,

- A  $u, v$ -walk contains a  $u, v$ -path (Prop 1.6)
- A characterization of trees (Thm 3.5)
- Menger's theorem on connectivity (Thm 5.5)
- Kőnig's theorem on the chromatic index of a bipartite graph (Thm 7.4)
- Euler's formula for planar graphs (Thm 8.1)
- Five-color theorem (Thm 8.7)
- Turán's theorem (Thm 9.2)
- Erdős-Stone theorem (Lem 9.5)

and in some of the exercises

## Double counting

Counting one thing in two different ways

Appeared many times

- Handshaking lemma (Thm 1.1)
- Mantel's theorem (Thm 2.7)
- Weak duality (Prop 3.11, Lem 5.4, Lem 5.6)
- Every regular bipartite graph has a perfect matching (Thm 4.2)
- Every 3-reg graph w/o cut-edge has a perfect matching (Thm 4.5)
- Number of edges in a planar graph (Prop 8.2)

and in some of the exercises

## Extremality

When choosing an object, pick an extremal one and use this property

Encountered many times

- A characterization of Eulerian graphs (Thm 2.3)
- Dirac's theorem for Hamiltonicity (Thm 2.6)
- Every tree has at least two leaves ( $n \geq 2$ ) (Lem 3.3)
- Tutte's theorem for perfect matchings (Thm 4.4)

and in some of the exercises

## Pigeonhole principle

When many objects are put in few boxes, at least one box has many

- Dirac's theorem for Hamiltonicity (Thm 2.6)
- List chromatic numbers can be arbitrarily larger than chromatic numbers (Page 38, Lect 7)
- Erdős-Stone theorem (Lem 9.5)
- Ramsey theorem (Thm 12.4)

and in some of the exercises

## Regularity method

If you have a dense graph, then regularize and hope the regularity graph has a nice property

- Quantitative ver of Mantel's theorem (Prop 11.4)
- Quantitative ver of Turán's theorem (Exer 11.2)
- Erdős-Stone theorem (Thm 9.4)
- Ramsey number of bounded-degree graphs (Thm 12.6)

and in some of the exercises

## Today's contents

- 1 Encountered Topics
  - Good characterization
  - Weak duality and strong duality
  - Extremal problems
  - Ramsey theory
  - Modern graph theory
- 2 Encountered Proof Techniques
  - Induction
  - Double counting
  - Extremality
  - Pigeonhole principle
  - Regularity method
  - Graph-theoretic method
- 3 Further study and a computational view

## Graph-theoretic method

For a seemingly non-graph-theoretic problem, derive a graph structure and apply results for graphs

Sometimes appeared in Exercises

- System of distinct representatives (Exer 4.1)
- Number of point-pairs far from each other in an equilateral triangle (Exer 9.2)
- Number of unit-distance pairs on the plane and in the 3-dim space (Exer 9.4)
- Roth's thm for 3-term arithmetic progressions (Exer 11.6)
- Schur's theorem (next slide)

## Schur's theorem about integers

## Quiz

Can we partition  $\{1, 2, \dots, 13\}$  into three parts so that no part has (not necessarily distinct) three numbers  $x, y, z$  with  $x + y = z$ ?

## Theorem 13.1 (Schur's theorem; Schur '16)

$\forall n \in \mathbb{N} \exists r \in \mathbb{N}$

$\forall$  a partition of  $\{1, \dots, r\}$  into  $n$  parts  $\exists$  a part containing (not necessarily distinct) three numbers  $x, y, z$  s.t.  $x + y = z$

## Proof idea of Schur's theorem

Given  $n$ ; Let  $r = R(\underbrace{3, 3, \dots, 3}_{n \text{ times}})$  the multi-color Ramsey number

- Let  $X_1, X_2, \dots, X_n$  be a given partition of  $\{1, \dots, r\}$
- Consider a complete graph  $G = (V, E)$  where  $V = \{1, \dots, r\}$
- Color the edges with  $n$  colors as follows:  
 $\{a, b\}$  is colored by  $i$  iff  $a - b \in X_i$  ( $a > b$ )
- $\therefore \exists$  a  $K_3$  of color  $i$  for some  $i \in \{1, \dots, n\}$
- Let  $K_3$  be formed by  $a, b, c \in \{1, \dots, r\}$  (wlog  $a > b > c$ )
- $a - b, a - c, b - c \in X_i$  □

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## Relation to computational view

- Mathematical characterizations can be classified from the computational perspective (good characterization)
- Efficiently-solvable optimization problems rely on strong duality
- Hard problems lack in good characterizations and strong duality
- Large structures can be approximated by a random-looking object (regularity method), so we may regard them as a random object as long as approximation is concerned
- Large structures contain a still large well-organized substructure (Ramsey theory), so finding such a substructure might help for your computation

## Hypergraphs

## Caution

- For some applications, the graph-theoretic method is not powerful enough
- Hypergraph-theoretic method would be an alternative

## Definition (Hypergraph)

A **hypergraph** is a pair  $(V, E)$  of a finite set  $V$  and a family  $E \subseteq 2^V$  of subsets of  $V$ ;  
Elements of  $E$  are called **hyperedges**

There are concepts in hypergraph theory corresponding to graphs: degrees, paths, cycles, trees, connectivity, coloring, regularity, Ramsey theorem, ...

## The final “piece of paper”

- **Write something!**
  - whatever you got in the course of lectures
  - whatever you felt during the lectures
  - whatever you think about this special education program
  - whatever you saw this morning
  - whatever you like to share with me
  - ...
- **Please hand in before you leave**

Thanks a lot for your patience!