Topics on Computing and Mathematical Sciences I Graph Theory (12) Ramsey Theory

Yoshio Okamoto

Tokyo Institute of Technology

July 9, 2008

"Last updated: Tue Jul 08 22:26 JST 2008"

Today's contents

1. What is Ramsey theory?
2. Ramsey numbers
3. Graph Ramsey numbers
4. Ramsey numbers of bounded-degree graphs
5. Multi-color Ramsey numbers
6. Algorithmic issue

What is Ramsey theory

Quotation from D. West '01

> *"Ramsey theory" referes to the study of partitions of large structures. Typical results state that a special substructure must occur in some class of the partition. Motzkin described this by saying that "Complete disorder is impossible." The objects we consider are merely sets and numbers, ...*

Ramsey theory somewhat generalizes pigeonhole principle

"1-dimensional" Ramsey theory	pigeonhole principle
"2-dimensional" Ramsey theory	graph Ramsey theory
"higher-dimensional" Ramsey theory	hypergraph Ramsey theory

Pigeonhole principle

$n, r, s \in \mathbb{N}$ natural numbers; $V = \{1, \ldots, n\}$ a set

Proposition 12.1 (Pigeonhole principle, PHP)

Assume $n \geq rs - r + 1$;
If we color each element of V with one of the colors in $\{1, \ldots, r\}$,
then, there are at least s elements that have the same color

Example: $r = 3$, $s = 4$, $n = 10$

Proof of the pigeonhole principle

Proof idea.

Suppose not

- For each color, there are at most $s-1$ elements with that color
- In total, there are at most $r(s-1) = rs-r$ elements in V □

Example: $r = 3$, $s = 4$, $n = 10$

Note

The inequality "$n \geq rs-r+1$" is best possible

Example: $r = 3$, $s = 4$, $n = 9$

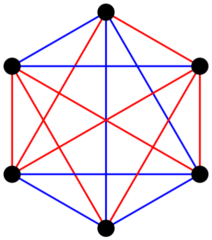
A flavor of Ramsey theory

"One-dimensional" to "two-dimensional"

Now, we have a graph; Instead of coloring vertices, we color edges

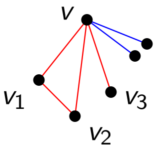
Proposition 12.2

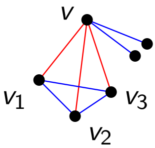
If we color the edge set of K_6 with two colors (say red and blue), then at least one color class contains a copy of K_3



Proof idea

- Look at one vertex v and its five incident edges
- At least three of those have the same color (say red) (by PHP)
- Look at the three vertices v_1, v_2, v_3 incident to those edges but no v itself
- Case 1: Some two of v_1, v_2, v_3 are joined by a red edge
  - This edge together with the red edges incident to v creates a K_3
- Case 2: Any two of v_1, v_2, v_3 are joined by a blue edge
  - These three vertices form a K_3 □





Today's contents

1. What is Ramsey theory?
2. Ramsey numbers
3. Graph Ramsey numbers
4. Ramsey numbers of bounded-degree graphs
5. Multi-color Ramsey numbers
6. Algorithmic issue

Ramsey numbers

Question

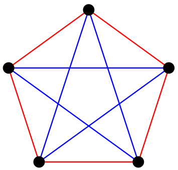
What about finding K_4, K_5, ...?

$k, \ell \in \mathbb{N}$ natural numbers

Definition (Ramsey number)

The Ramsey number $R(k, \ell)$ is the minimum r for which every 2-coloring (say with red and blue) of the edges of K_r contains K_k of red edges or K_ℓ of blue edges

Proposition 11.2 shows $R(3, 3) \leq 6$, and in fact $R(3, 3) = 6$



Basic properties of Ramsey numbers

Another equilavent definition of Ramsey numbers

The Ramsey number $R(k,\ell)$ is the minimum r for which every r-vertex graph G contains K_k or its complement $\overline{G}$ contains K_ℓ

Observation 12.3 (Basic properties of Ramsey numbers)

$k, \ell \in \mathbb{N}$ natural numbers

1. $R(k,\ell) = R(\ell,k)$
2. $R(k,\ell) \leq R(k+1,\ell)$

Proof idea.

1. Switch the role of red and blue □
2. Let $r = R(k+1,\ell)$; Then, every 2-edge-coloring of K_r contains red K_{k+1} or blue K_ℓ, ∴ it contains red K_k or blue K_ℓ as well □

Upper bound for Ramsey numbers

Theorem 12.4 (Recursion for Ramsey numbers; Ramsey '30)

For $k, \ell > 1$, it holds $R(k,\ell) \leq R(k,\ell-1) + R(k-1,\ell)$

Rediscovered by Erdős, Szekeres '35, Greenwood, Gleason '55

Proof idea.

- Let $n = R(k,\ell-1) + R(k-1,\ell)$
- Consider an arbitrary 2-edge-coloring (red/blue) of K_n
- Choose any vertex v and let
  $R = \{u \mid \{u,v\} \text{ red}\}$ and $B = \{u \mid \{u,v\} \text{ blue}\}$
- $|R| \geq R(k-1,\ell)$ or $|B| \geq R(k,\ell-1)$ $\quad (\because |R|+|B| = n-1)$

Upper bound for Ramsey numbers (continued)

Proof idea (cont'd).

- Case 1: $|R| \geq R(k-1, \ell)$
  - $K_n[R]$ contains a red K_{k-1} or a blue K_ℓ
  - A red K_{k-1} with v form a red K_k
- Case 2: $|B| \geq R(k, \ell-1) \rightsquigarrow$ similar $\square$

Upper bound for Ramsey numbers: Conclusion

Upper bound for Ramsey numbers (Exercise)

For $k, \ell \geq 1$, it holds $R(k,\ell) \leq \binom{k+\ell-2}{k-1}$

Table of $\binom{k+\ell-2}{k-1}$

$k \backslash \ell$	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7
3	1	3	6	10	15	21	28
4	1	4	10	20	35	56	84
5	1	5	15	35	70	126	210
6	1	6	21	56	126	252	462
7	1	7	28	84	210	462	924

Table of small Ramsey numbers

Table of $R(k, \ell)$

$k \backslash \ell$	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7
3	1	3	6	9	14	18	23
4	1	4	9	18	25	35–41	49–61
5	1	5	14	25	43–49	58–87	80–143
6	1	6	18	35–41	58–87	102–165	113–298
7	1	7	23	49–61	80–143	113–298	205–540

(taken from Radziszowski '06)

Open problem

Fill the gaps in the table

Diagonal Ramsey numbers

- When $k = \ell$, we have

$$R(k,k) \leq \binom{2k-2}{k-1} = \mathrm{O}(4^k/\sqrt{k})$$

- $\exists$ a lower bound (due to Erdős '47 by a probabilistic method)

$$R(k,k) = \Omega(k2^{k/2})$$

- They give

$$\sqrt{2} \leq \liminf_{k\to\infty} R(k,k)^{1/k} \leq \limsup_{k\to\infty} R(k,k)^{1/k} \leq 4$$

Open problems (Important!)

- Fill the gap
- Does $\lim R(k,k)^{1/k}$ exist? (Erdős '47)

Off-diagonal Ramsey numbers: Asympototics

Not much is known

For $R(k, 3)$

- $R(k, 3) \leq \mathrm{O}(k^2 / \log k)$ (Ajtai, Komlós, Szemerédi '80)
- $R(k, 3) \geq \Omega(k^2 / \log k)$ (Kim '95; Fulkerson Prize Winner)

For $R(k, 4)$

- $R(k, 4) \leq \mathrm{O}(k^3 / \log^2 k)$ (Ajtai, Komlós, Szemerédi '80)
- $R(k, 3) \geq \Omega(k^{2.5} / \log^{2.5} k)$ (Spencer '77)

Open problem

Determine $R(k, 4)$ asymptotically

Today's contents

1. What is Ramsey theory?
2. Ramsey numbers
3. Graph Ramsey numbers
4. Ramsey numbers of bounded-degree graphs
5. Multi-color Ramsey numbers
6. Algorithmic issue

What about other graphs?

So far...

We've tried to find a red K_k or a blue K_ℓ in a 2-edge-coloring of K_r

A possible generalization

We may try to find a red G or a blue H in a 2-edge-coloring of K_r (G, H arbitrary)

Graph Ramsey numbers

G, H graphs

Definition (Graph Ramsey number)

The graph Ramsey number $R(G, H)$ is the minimum r for which every 2-coloring (say with red and blue) of the edges of K_r contains a red G or a blue H

Remark

- $R(K_k, K_\ell) = R(k, \ell)$
- $G_1 \subseteq G_2,\ H_1 \subseteq H_2 \Longrightarrow R(G_1, H_1) \leq R(G_2, H_2)$
- $n(G) = k,\ n(H) = \ell \Longrightarrow R(G, H) \leq R(k, \ell)$

Examples

- $R(P_4, C_4) = 5$ (Exercise)
- $R(C_4, C_4) = 6$ (Exercise)
- $R(P_k, P_\ell) = k - 1 + \lfloor \ell/2 \rfloor$ if $k \geq \ell$ (Erdős '47)
- $R(K_{1,k}, K_{1,\ell}) = \begin{cases} k + \ell - 1 & \text{if } k, \ell \text{ even,} \\ k + \ell & \text{otherwise} \end{cases}$ (Burr, Roberts '73)
- $R(T, K_\ell) = (k-1)(\ell-1)+1$ if T a k-vtx tree (Next slide)
- $R(T, K_{1,\ell}) = k+\ell-1$ if T a k-vtx tree and $k-1|\ell-1$ (Burr '74; Exercise)
- $R(G, G) \leq \mathrm{O}(k)$ if G a k-vtx graph w/ $\Delta(G)$ const (Later)
- $R(G, G) \leq \mathrm{O}(k)$ if G a k-vtx planar graph (Chen, Schelp '93)
- $R(G, G) \leq \mathrm{O}(k)$ if G a k-vtx graph w/o K_r-subdiv for fixed r (Rödl, Thomas '97)

One example

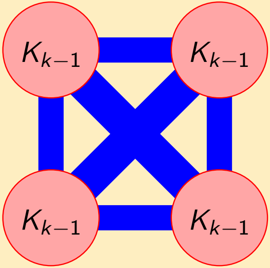
Theorem 12.5 (Chvátal '77)

T is any k-vertex tree $\Longrightarrow R(T, K_\ell) = (k-1)(\ell-1)+1$

Proof idea.

$R(T, K_\ell) \geq (k-1)(\ell-1)+1$

- Consider the following 2-edge-coloring of $K_{(k-1)(\ell-1)}$



One example (continued)

Proof idea (cont'd).

$R(T, K_\ell) \leq (k-1)(\ell-1)+1$

- We show every 2-edge-coloring of $K_{(k-1)(\ell-1)+1}$ contains a red T or a blue K_ℓ
- Fix a 2-edge-coloring and suppose it contains no blue K_ℓ (o.w. we're done)
- Look at the red graph $G \subseteq K_{(k-1)(\ell-1)+1}$
- $\alpha(G) \leq \ell-1$ $(\because K_\ell \not\subseteq \overline{G})$
- $\chi(G) \geq n(G)/\alpha(G) > k-1$ (Prop 6.2)
- G contains a subgraph H s.t. $\delta(H) \geq k-1$ (Contrapositive of Prop 6.4)
- $\therefore T \subseteq H \subseteq G$ (cf. Hint of Exer 3.4)

□

Today's contents

1. What is Ramsey theory?
2. Ramsey numbers
3. Graph Ramsey numbers
4. Ramsey numbers of bounded-degree graphs
5. Multi-color Ramsey numbers
6. Algorithmic issue

Ramsey numbers of bounded-degree graphs

Theorem 12.6 (Chvátal, Rödl, Szemerédi, Trotter '83)

$\forall\ \Delta \geq 1\ \exists\ c\ \forall\ H$ a graph w/ $\Delta(H) \leq \Delta$: $R(H,H) \leq cn(H)$

In words:

The Ramsey number of a bounded-degree graph is linear

An incomplete list of bounded-degree graphs

- paths $(\Delta \leq 2)$
- cycles $(\Delta = 2)$
- k-regular graphs $(\Delta = k)$
- binary trees $(\Delta \leq 3)$
- ...

Proof outline of Theorem 12.6

Proof outline:

Use Szemerédi's regularity lemma, the embedding lemma, Turán's theorem, Ramsey theorem

- Let G be any graph w/ $n(G) \geq cn(H)$
- Show: $H \subseteq G$ or $H \subseteq \overline{G}$
- A regularity graph R of G dense $\Rightarrow K_r \subseteq R$ for some large r (Turán)
- Color each edge of R w/ red if the corresp. pair is dense and w/ blue if the corresp. pair is sparse
- The red graph R_r or the blue R_b graph contains K_k where k is still large (Ramsey)
- $K_k \subseteq R_r \Rightarrow H \subseteq R_r(s) \Rightarrow H \subseteq G$ (Emb Lem)
- $K_k \subseteq R_b \Rightarrow H \subseteq R_b(s) \Rightarrow H \subseteq \overline{G}$ (Emb Lem)

Proof idea (1/5)

Given $\Delta \geq 1$, first we fix c as follows

- Apply the embedding lemma for $d = 1/2$ and this Δ
  $\rightsquigarrow$ Obtain $\gamma_0 > 0$ (we may assume $\gamma_0 < 1$)
- Let $n_0 = R(\Delta+1, \Delta+1)$
- Let $\gamma < \gamma_0$ sufficiently large s.t.

$$2\gamma < \frac{1}{n_0 - 1} - \frac{1}{n_0}$$

- Apply Szemerédi's regularity lemma for $\varepsilon = \gamma$ and n_0
  $\rightsquigarrow$ Obtain N_0
- Let $c = \dfrac{N_0}{\gamma_0(1-\gamma)}$ (Note: c depends only on Δ)

Proof idea (2/5)

Given a graph H with $\Delta(H) \leq \Delta$

- Let $s = n(H)$ (for a later use)
- Let G be a graph with $n(G) = n \geq cs$
- WANT: $H \subseteq G$ or $H \subseteq \overline{G}$

- $n \geq cs \geq c = \frac{N_0}{\gamma_0(1-\gamma)} \geq N_0 \geq n_0$
- $\therefore$ $\exists$ a γ-reg partition $\{V_0, V_1, \ldots, V_k\}$ of G w/ $n_0 \leq k \leq N_0$ and $|V_1| = \cdots = |V_k| = \ell$ (Szemerédi)
- Note $\ell = \dfrac{n - |V_0|}{k} \geq \dfrac{(1-\gamma)n}{N_0} \geq \dfrac{(1-\gamma)cs}{N_0} = \dfrac{s}{\gamma_0}$

Proof idea (3/5)

Let R be a regularity graph of $\{V_i\}$ w/ param's $\gamma, \ell, 0$

- $n(R) = k$
- On the other hand

$$\begin{aligned} e(R) &\geq \binom{k}{2} - \gamma k^2 = \frac{k^2 - k}{2} - \gamma k^2 = \frac{k^2}{2}\left(1 - \frac{1}{k} - 2\gamma\right) \\ &> \frac{k^2}{2}\left(1 - \frac{1}{n_0} - \left(\frac{1}{n_0 - 1} - \frac{1}{n_0}\right)\right) \\ &= \frac{k^2}{2}\left(1 - \frac{1}{n_0 - 1}\right) > \mathrm{ex}(k, K_{n_0}) \end{aligned}$$

- $\therefore K_{n_0} \subseteq R$ (Turán)

Proof idea (4/5)

Reminder: $n_0 = R(\Delta+1, \Delta+1)$

- Let R_r be a regularity graph of $\{V_i\}$ w/ param's $\gamma, \ell, 1/2$
- Note: $R_r \subseteq R$
- Let R_b be the following subgraph of R:

$$E(R_b) = \{\{V_i, V_j\} \in E(R) \mid d(V_i, V_j) \leq 1/2\}$$

- Note: $\{V_i\}$ is a γ-reg partition of $\overline{G}$ (Exer 11.3) and so, R_b is a regularity graph of $\{V_i\}$ in $\overline{G}$ w/ param's $\gamma, \ell, 1/2$
- R_r or R_b contains $K_{\Delta+1}$ ($\because$ $E(R) = E(R_r) \cup E(R_b)$ and Ramsey)

Proof idea (5/5)

Reminder: $s = n(H)$, $\gamma \leq \gamma_0$, $\ell \geq s/\gamma_0$

- $\therefore$ $R_r(s)$ or $R_b(s)$ contains $K_{\Delta+1}(s)$
- $\therefore$ $R_r(s)$ or $R_b(s)$ contains H $\quad$ ($\because$ $H \subseteq K_{\chi(H)}(s) \subseteq K_{\Delta+1}(s)$)
- $\therefore$ G or $\overline{G}$ contains H $\quad$ (Embedding lem)

$\square$

Conjectures

Conjecture (Burr, Erdős '75)

$R(G, G) = \mathrm{O}(n(G))$ if the average deg of G is bounded

Note: This is true if we replace "average" by "maximum" (as we saw in Thm 12.6)

Conjecture (Loebl)

$R(T, T) \leq 2k$ if T is a k-vtx tree

What about having many colors?

So far...

We've tried to find a red G or a blue H in a 2-edge-coloring of K_r (G, H arbitrary)

Possible generalization

We may try to find a red H_1 or a blue H_2 or a green H_3 in a 3-edge-coloring of K_r (H_1, H_2, H_3 arbitrary)

More generally...

We may try to find a H_1 of color 1 or a H_2 of color 2 or ... or a H_m of color m in an m-edge-coloring of K_r ($H_1, H_2, \ldots, H_m$ arbitrary)

Multi-color Ramsey numbers

$H_1, \ldots, H_m$ graphs

Definition (Multi-color graph Ramsey number)

The *m-color graph Ramsey number* $R(H_1, \ldots, H_m)$ is the minimum r for which every m-edge-coloring of K_r contains a copy of H_i of color i for some $i \in \{1, \ldots, m\}$

Definition (Multi-color Ramsey number)

The *m-color Ramsey number* $R(k_1, \ldots, k_m)$ is $R(K_{k_1}, \ldots, K_{k_m})$

Facts on multi-color graph Ramsey numbers

Known facts

- $R(k_1, \ldots, k_m) \le \dfrac{(k_1 + \cdots + k_m - m)!}{(k_1-1)! \cdots (k_m - 1)!}$ (Exercise)
- $R(3,3,3) = 17$ (Greenwood, Gleason '55)
- $51 \le R(3,3,3,3) \le 62$
  (LB: Chung '73; UB: Fettes, Kramer, Radziszowski '04)
- $128 \le R(4,4,4) \le 236$ (LB: Hill, Irving '82)
- Several results on paths and cycles
- $R(C_n, C_n, C_n) = 4n - 3$ for sufficiently large odd n
  (Kohayakawa, Simonovits, Skokan '05)

Conjecture (Bondy, Erdős)

$R(C_n, C_n, C_n) \le 4n - 3$ for $n \ge 4$

Today's contents

Deciding a Ramsey property

Problem RAMSEY

Input: A natural number n, two graphs H_1, H_2
Question: Does every 2-edge-coloring (by red/blue) of K_n contains a red H_1 or a blue H_2?

Fact

- Belongs to Π_2^p
- NP-hard (Burr '84)

Open problem: Is RAMSEY Π_2^p-complete?

Deciding a Ramsey property

Problem ARROWING

Input: three graphs G, H_1, H_2

Question: Does every 2-edge-coloring (by red/blue) of G contains a red H_1 or a blue H_2?

Fact

ARROWING is Π_2^p-complete (Schaefer '01)

Use of Ramsey theory in Algorithms design

Approximation algorithms

- Maximum independent set (Boppana, Halldórsson '92)
    - Repeatedly find a clique and an independent set of size guaranteed by Ramsey's theorem
    - Gave an efficient algorithm with approx ratio $\mathrm{O}(n/\log^2 n)$
- Minimum vertex cover (Monien, Speckenmeyer '85)
    - Considered the max number of vertices in a graph for which every 2-edge-coloring contains a long red odd cycle or a large blue complete subgraph
    - Gave an efficient algorithm with approximation ratio $2 - \frac{\log\log n}{2\log n}$

Fixed-parameter algorithms

- Finding a small subgraph with hereditary properties (Khot, Raman '02)

Other applications to theory of computation: surveyed by Rosta '04