Topics on
Computing and Mathematical Sciences I
Graph Theory
(12) Ramsey Theory

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Today's contents

What is Ramsey theory

Quotation from D. West '01

"Ramsey theory" referes to the study of partitions of large structures. Typical results state that a special substructure must occur in some class of the partition. Motzkin described this by saying that "Complete disorder is impossible." The objects we consider are merely sets and numbers, ...

Ramsey theory somewhat generalizes pigeonhole principle

"1-dimensional" Ramsey theory	pigeonhole principle
"2-dimensional" Ramsey theory	graph Ramsey theory
"higher-dimensional" Ramsey theory	hypergraph Ramsey theory

Pigeonhole principle

$n, r, s \in \mathbb{N}$ natural numbers; $V = \{1, \ldots, n\}$ a set

Proposition 12.1 (Pigeonhole principle, PHP)

Assume $n \geq rs - r + 1$;
If we color each element of V with one of the colors in $\{1, \ldots, r\}$,
then, there are at least s elements that have the same color

Example: $r = 3$, $s = 4$, $n = 10$

Proof of the pigeonhole principle

Proof idea.

Suppose not

- For each color, there are at most $s-1$ elements with that color
- In total, there are at most $r(s-1) = rs-r$ elements in V □

Example: $r = 3$, $s = 4$, $n = 10$

Note

The inequality "$n \geq rs-r+1$" is best possible

Example: $r = 3$, $s = 4$, $n = 9$

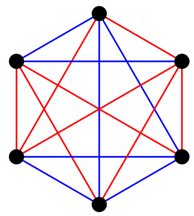
A flavor of Ramsey theory

"One-dimensional" to "two-dimensional"

Now, we have a graph; Instead of coloring vertices, we color edges

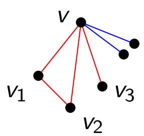
Proposition 12.2

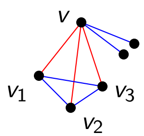
If we color the edge set of K_6 with two colors (say red and blue), then at least one color class contains a copy of K_3



Proof idea

- Look at one vertex v and its five incident edges
- At least three of those have the same color (say red) (by PHP)
- Look at the three vertices v_1, v_2, v_3 incident to those edges but no v itself
- Case 1: Some two of v_1, v_2, v_3 are joined by a red edge
  - This edge together with the red edges incident to v creates a K_3
- Case 2: Any two of v_1, v_2, v_3 are joined by a blue edge
  - These three vertices form a K_3 □





Today's contents

Ramsey numbers

Question

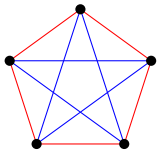
What about finding K_4, K_5, ...?

$k, \ell \in \mathbb{N}$ natural numbers

Definition (Ramsey number)

The Ramsey number $R(k, \ell)$ is the minimum r for which every 2-coloring (say with red and blue) of the edges of K_r contains K_k of red edges or K_ℓ of blue edges

Proposition 11.2 shows $R(3, 3) \leq 6$, and in fact $R(3, 3) = 6$



Basic properties of Ramsey numbers

Another equilavent definition of Ramsey numbers

The Ramsey number $R(k, \ell)$ is the minimum r for which every r-vertex graph G contains K_k or its complement $\overline{G}$ contains K_ℓ

Observation 12.3 (Basic properties of Ramsey numbers)

$k, \ell \in \mathbb{N}$ natural numbers

1. $R(k, \ell) = R(\ell, k)$
2. $R(k, \ell) \leq R(k+1, \ell)$

Proof idea.

1. Switch the role of red and blue □
2. Let $r = R(k+1, \ell)$; Then, every 2-edge-coloring of K_r contains red K_{k+1} or blue K_ℓ, $\therefore$ it contains red K_k or blue K_ℓ as well □

Upper bound for Ramsey numbers

Theorem 12.4 (Recursion for Ramsey numbers; Ramsey '30)

For $k, \ell > 1$, it holds $R(k, \ell) \leq R(k, \ell-1) + R(k-1, \ell)$

Rediscovered by Erdős, Szekeres '35, Greenwood, Gleason '55

Proof idea.

- Let $n = R(k, \ell-1) + R(k-1, \ell)$
- Consider an arbitrary 2-edge-coloring (red/blue) of K_n
- Choose any vertex v and let $R = \{u \mid \{u, v\} \text{ red}\}$ and $B = \{u \mid \{u, v\} \text{ blue}\}$
- $|R| \geq R(k-1, \ell)$ or $|B| \geq R(k, \ell-1)$ $\quad (\because |R|+|B| = n-1)$

Upper bound for Ramsey numbers (continued)

Proof idea (cont'd).

- Case 1: $|R| \geq R(k-1, \ell)$
  - $K_n[R]$ contains a red K_{k-1} or a blue K_ℓ
  - A red K_{k-1} with v form a red K_k
- Case 2: $|B| \geq R(k, \ell-1) \rightsquigarrow$ similar □

Upper bound for Ramsey numbers: Conclusion

Upper bound for Ramsey numbers (Exercise)

For $k, \ell \geq 1$, it holds $R(k,\ell) \leq \binom{k+\ell-2}{k-1}$

Table of $\binom{k+\ell-2}{k-1}$

$k \backslash \ell$	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7
3	1	3	6	10	15	21	28
4	1	4	10	20	35	56	84
5	1	5	15	35	70	126	210
6	1	6	21	56	126	252	462
7	1	7	28	84	210	462	924

Table of small Ramsey numbers

Table of $R(k,\ell)$

$k \backslash \ell$	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7
3	1	3	6	9	14	18	23
4	1	4	9	18	25	35–41	49–61
5	1	5	14	25	43–49	58–87	80–143
6	1	6	18	35–41	58–87	102–165	113–298
7	1	7	23	49–61	80–143	113–298	205–540

(taken from Radziszowski '06)

Open problem

Fill the gaps in the table

Diagonal Ramsey numbers

- When $k = \ell$, we have

$$R(k,k) \leq \binom{2k-2}{k-1} = \mathrm{O}(4^k/\sqrt{k})$$

- $\exists$ a lower bound (due to Erdős '47 by a probabilistic method)

$$R(k,k) = \Omega(k2^{k/2})$$

- They give

$$\sqrt{2} \leq \liminf_{k\to\infty} R(k,k)^{1/k} \leq \limsup_{k\to\infty} R(k,k)^{1/k} \leq 4$$

Open problems (Important!)

- Fill the gap
- Does $\lim R(k,k)^{1/k}$ exist? (Erdős '47)

Off-diagonal Ramsey numbers: Asympototics

Not much is known

For $R(k,3)$

- $R(k,3) \leq \mathrm{O}(k^2/\log k)$ (Ajtai, Komlós, Szemerédi '80)
- $R(k,3) \geq \Omega(k^2/\log k)$ (Kim '95; Fulkerson Prize Winner)

For $R(k,4)$

- $R(k,4) \leq \mathrm{O}(k^3/\log^2 k)$ (Ajtai, Komlós, Szemerédi '80)
- $R(k,3) \geq \Omega(k^{2.5}/\log^{2.5} k)$ (Spencer '77)

Open problem

Determine $R(k,4)$ asymptotically

Today's contents

1. What is Ramsey theory?
2. Ramsey numbers
3. Graph Ramsey numbers
4. Ramsey numbers of bounded-degree graphs
5. Multi-color Ramsey numbers
6. Algorithmic issue

What about other graphs?

So far...

We've tried to find a red K_k or a blue K_ℓ in a 2-edge-coloring of K_r

A possible generalization

We may try to find a red G or a blue H in a 2-edge-coloring of K_r (G, H arbitrary)

Graph Ramsey numbers

G, H graphs

Definition (Graph Ramsey number)

The graph Ramsey number $R(G, H)$ is the minimum r for which every 2-coloring (say with red and blue) of the edges of K_r contains a red G or a blue H

Remark

- $R(K_k, K_\ell) = R(k, \ell)$
- $G_1 \subseteq G_2,\ H_1 \subseteq H_2 \implies R(G_1, H_1) \le R(G_2, H_2)$
- $n(G) = k,\ n(H) = \ell \implies R(G, H) \le R(k, \ell)$

Examples

- $R(P_4, C_4) = 5$ (Exercise)
- $R(C_4, C_4) = 6$ (Exercise)
- $R(P_k, P_\ell) = k - 1 + \lfloor \ell/2 \rfloor$ if $k \ge \ell$ (Erdős '47)
- $R(K_{1,k}, K_{1,\ell}) = \begin{cases} k + \ell - 1 & \text{if } k, \ell \text{ even,} \\ k + \ell & \text{otherwise} \end{cases}$ (Burr, Roberts '73)
- $R(T, K_\ell) = (k-1)(\ell-1)+1$ if T a k-vtx tree (Next slide)
- $R(T, K_{1,\ell}) = k+\ell-1$ if T a k-vtx tree and $k-1 | \ell-1$ (Burr '74; Exercise)
- $R(G, G) \le \mathrm{O}(k)$ if G a k-vtx graph w/ $\Delta(G)$ const (Later)
- $R(G, G) \le \mathrm{O}(k)$ if G a k-vtx planar graph (Chen, Schelp '93)
- $R(G, G) \le \mathrm{O}(k)$ if G a k-vtx graph w/o K_r-subdiv for fixed r (Rödl, Thomas '97)

One example

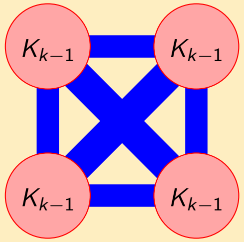
Theorem 12.5 (Chvátal '77)

T is any k-vertex tree $\Longrightarrow R(T, K_\ell) = (k-1)(\ell-1)+1$

Proof idea.

$R(T, K_\ell) \geq (k-1)(\ell-1)+1$

- Consider the following 2-edge-coloring of $K_{(k-1)(\ell-1)}$



One example (continued)

Proof idea (cont'd).

$R(T, K_\ell) \leq (k-1)(\ell-1)+1$

- We show every 2-edge-coloring of $K_{(k-1)(\ell-1)+1}$ contains a red T or a blue K_ℓ
- Fix a 2-edge-coloring and suppose it contains no blue K_ℓ (o.w. we're done)
- Look at the red graph $G \subseteq K_{(k-1)(\ell-1)+1}$
- $\alpha(G) \leq \ell-1$ $(\because K_\ell \not\subseteq \overline{G})$
- $\chi(G) \geq n(G)/\alpha(G) > k-1$ (Prop 6.2)
- G contains a subgraph H s.t. $\delta(H) \geq k-1$ (Contrapositive of Prop 6.4)
- $\therefore T \subseteq H \subseteq G$ (cf. Hint of Exer 3.4)

□

Today's contents

Ramsey numbers of bounded-degree graphs

Theorem 12.6 (Chvátal, Rödl, Szemerédi, Trotter '83)

$\forall\ \Delta \geq 1\ \exists\ c\ \forall\ H$ a graph w/ $\Delta(H) \leq \Delta$: $R(H, H) \leq cn(H)$

In words:

The Ramsey number of a bounded-degree graph is linear

An incomplete list of bounded-degree graphs

- paths $(\Delta \leq 2)$
- cycles $(\Delta = 2)$
- k-regular graphs $(\Delta = k)$
- binary trees $(\Delta \leq 3)$
- ...

Proof outline of Theorem 12.6

Proof outline:
Use Szemerédi's regularity lemma, the embedding lemma, Turán's theorem, Ramsey theorem

- Let G be any graph w/ $n(G) \geq cn(H)$
- Show: $H \subseteq G$ or $H \subseteq \overline{G}$
- A regularity graph R of G dense $\Rightarrow K_r \subseteq R$ for some large r (Turán)
- Color each edge of R w/ red if the corresp. pair is dense and w/ blue if the corresp. pair is sparse
- The red graph R_r or the blue R_b graph contains K_k where k is still large (Ramsey)
- $K_k \subseteq R_r \Rightarrow H \subseteq R_r(s) \Rightarrow H \subseteq G$ (Emb Lem)
- $K_k \subseteq R_b \Rightarrow H \subseteq R_b(s) \Rightarrow H \subseteq \overline{G}$ (Emb Lem)

Proof idea (1/5)

Given $\Delta \geq 1$, first we fix c as follows

- Apply the embedding lemma for $d = 1/2$ and this Δ
  $\rightsquigarrow$ Obtain $\gamma_0 > 0$ (we may assume $\gamma_0 < 1$)
- Let $n_0 = R(\Delta{+}1, \Delta{+}1)$
- Let $\gamma < \gamma_0$ sufficiently large s.t.

$$2\gamma < \frac{1}{n_0 - 1} - \frac{1}{n_0}$$

- Apply Szemerédi's regularity lemma for $\varepsilon = \gamma$ and n_0
  $\rightsquigarrow$ Obtain N_0
- Let $c = \dfrac{N_0}{\gamma_0(1-\gamma)}$ (Note: c depends only on Δ)

Proof idea (2/5)

Given a graph H with $\Delta(H) \leq \Delta$

- Let $s = n(H)$ (for a later use)
- Let G be a graph with $n(G) = n \geq cs$
- WANT: $H \subseteq G$ or $H \subseteq \overline{G}$
- $n \geq cs \geq c = \frac{N_0}{\gamma_0(1-\gamma)} \geq N_0 \geq n_0$
- $\therefore \exists$ a γ-reg partition $\{V_0, V_1, \ldots, V_k\}$ of G w/ $n_0 \leq k \leq N_0$ and $|V_1| = \cdots = |V_k| = \ell$ (Szemerédi)
- Note $\ell = \dfrac{n - |V_0|}{k} \geq \dfrac{(1-\gamma)n}{N_0} \geq \dfrac{(1-\gamma)cs}{N_0} = \dfrac{s}{\gamma_0}$

Proof idea (3/5)

Let R be a regularity graph of $\{V_i\}$ w/ param's $\gamma, \ell, 0$

- $n(R) = k$
- On the other hand

$$\begin{aligned} e(R) &\geq \binom{k}{2} - \gamma k^2 = \frac{k^2 - k}{2} - \gamma k^2 = \frac{k^2}{2}\left(1 - \frac{1}{k} - 2\gamma\right) \\ &> \frac{k^2}{2}\left(1 - \frac{1}{n_0} - \left(\frac{1}{n_0 - 1} - \frac{1}{n_0}\right)\right) \\ &= \frac{k^2}{2}\left(1 - \frac{1}{n_0 - 1}\right) > \mathrm{ex}(k, K_{n_0}) \end{aligned}$$

- $\therefore K_{n_0} \subseteq R$ (Turán)

Proof idea (4/5)

Reminder: $n_0 = R(\Delta+1, \Delta+1)$

- Let R_r be a regularity graph of $\{V_i\}$ w/ param's $\gamma, \ell, 1/2$
- Note: $R_r \subseteq R$
- Let R_b be the following subgraph of R:

$$E(R_b) = \{\{V_i, V_j\} \in E(R) \mid d(V_i, V_j) \leq 1/2\}$$

- Note: $\{V_i\}$ is a γ-reg partition of $\overline{G}$ (Exer 11.3) and so, R_b is a regularity graph of $\{V_i\}$ in $\overline{G}$ w/ param's $\gamma, \ell, 1/2$
- R_r or R_b contains $K_{\Delta+1}$ ($\because E(R) = E(R_r) \cup E(R_b)$ and Ramsey)

Proof idea (5/5)

Reminder: $s = n(H)$, $\gamma \leq \gamma_0$, $\ell \geq s/\gamma_0$

- $\therefore R_r(s)$ or $R_b(s)$ contains $K_{\Delta+1}(s)$
- $\therefore R_r(s)$ or $R_b(s)$ contains H $\quad (\because H \subseteq K_{\chi(H)}(s) \subseteq K_{\Delta+1}(s))$
- $\therefore G$ or $\overline{G}$ contains H (Embedding lem)

□

Conjectures

Conjecture (Burr, Erdős '75)

$R(G, G) = \mathrm{O}(n(G))$ if the average deg of G is bounded

Note: This is true if we replace "average" by "maximum" (as we saw in Thm 12.6)

Conjecture (Loebl)

$R(T, T) \leq 2k$ if T is a k-vtx tree

What about having many colors?

So far...

We've tried to find a red G or a blue H in a 2-edge-coloring of K_r (G, H arbitrary)

Possible generalization

We may try to find a red H_1 or a blue H_2 or a green H_3 in a 3-edge-coloring of K_r (H_1, H_2, H_3 arbitrary)

More generally...

We may try to find a H_1 of color 1 or a H_2 of color 2 or ... or a H_m of color m in an m-edge-coloring of K_r ($H_1, H_2, \ldots, H_m$ arbitrary)

Multi-color Ramsey numbers

$H_1, \dots, H_m$ graphs

Definition (Multi-color graph Ramsey number)

The *m-color graph Ramsey number* $R(H_1, \dots, H_m)$ is the minimum r for which every m-edge-coloring of K_r contains a copy of H_i of color i for some $i \in \{1, \dots, m\}$

Definition (Multi-color Ramsey number)

The *m-color Ramsey number* $R(k_1, \dots, k_m)$ is $R(K_{k_1}, \dots, K_{k_m})$

Facts on multi-color graph Ramsey numbers

Known facts

- $R(k_1, \dots, k_m) \le \dfrac{(k_1 + \cdots + k_m - m)!}{(k_1-1)! \cdots (k_m - 1)!}$ (Exercise)
- $R(3,3,3) = 17$ (Greenwood, Gleason '55)
- $51 \le R(3,3,3,3) \le 62$
  (LB: Chung '73; UB: Fettes, Kramer, Radziszowski '04)
- $128 \le R(4,4,4) \le 236$ (LB: Hill, Irving '82)
- Several results on paths and cycles
- $R(C_n, C_n, C_n) = 4n - 3$ for sufficiently large odd n
  (Kohayakawa, Simonovits, Skokan '05)

Conjecture (Bondy, Erdős)

$R(C_n, C_n, C_n) \le 4n - 3$ for $n \ge 4$

Today's contents

Deciding a Ramsey property

Problem RAMSEY

Input: A natural number n, two graphs H_1, H_2
Question: Does every 2-edge-coloring (by red/blue) of K_n contains a red H_1 or a blue H_2?

Fact

- Belongs to Π_2^p
- NP-hard (Burr '84)

Open problem: Is RAMSEY Π_2^p-complete?

Deciding a Ramsey property

Problem ARROWING

Input: three graphs G, H_1, H_2

Question: Does every 2-edge-coloring (by red/blue) of G contains a red H_1 or a blue H_2?

Fact

ARROWING is Π_2^p-complete (Schaefer '01)

Use of Ramsey theory in Algorithms design

Approximation algorithms

- Maximum independent set (Boppana, Halldórsson '92)
  - Repeatedly find a clique and an independent set of size guaranteed by Ramsey's theorem
  - Gave an efficient algorithm with approx ratio $\mathrm{O}(n/\log^2 n)$
- Minimum vertex cover (Monien, Speckenmeyer '85)
  - Considered the max number of vertices in a graph for which every 2-edge-coloring contains a long red odd cycle or a large blue complete subgraph
  - Gave an efficient algorithm with approximation ratio $2 - \frac{\log\log n}{2\log n}$

Fixed-parameter algorithms

- Finding a small subgraph with hereditary properties (Khot, Raman '02)

Other applications to theory of computation: surveyed by Rosta '04