Topics on
Computing and Mathematical Sciences I
Graph Theory
(11) Extremal Graph Theory II

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July 2, 2008

"Last updated: Tue Jul 15 21:40 JST 2008"

Today's contents

Erdős-Simonovits-Stone theorem

A complete answer for the Turán-type problems

Theorem 9.3 (Erdős, Simonovits '66; Erdős-Simonovits-Stone thm)

$\forall$ graph H

$$\lim_{n\to\infty} \frac{\mathrm{ex}(n,H)}{\binom{n}{2}} = 1 - \frac{1}{\chi(H)-1};$$

In other words,

$$\mathrm{ex}(n,H) = \left(1 - \frac{1}{\chi(H)-1}\right)\binom{n}{2} + \mathrm{o}(n^2)$$

Proof outline of Erdős-Simonovits-Stone theorem

Proof outline I (Lecture 9)

1. Deduce the Erdős-Simonovits-Stone thm from the Erdős-Stone thm
2. Deduce the Erdős-Stone thm from its weaker version
3. Prove the weaker version of Erdős-Stone thm

Proof outline II (today)

1. Deduce the Erdős-Simonovits-Stone thm from the Erdős-Stone thm
2. Use Szemerédi's regularity lemma to prove the embedding lemma
3. Use the embedding lemma to prove Erdős-Stone thm

Erdős-Stone theorem

Theorem 9.4 (Erdős, Stone '46)

$\forall\, s \geq 1, r \geq 2$ natural numbers

$$\lim_{n\to\infty} \frac{\mathrm{ex}(n, T_{rs,r})}{\binom{n}{2}} = 1 - \frac{1}{r-1}$$

We have seen the Erdős-Stone theorem 9.4 implies the Erdős-Stone-Simonovits theorem 9.3

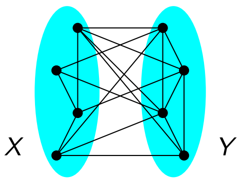
Today's contents

Density

Notation

$G = (V, E)$ a graph; $X, Y \subseteq V$ disjoint

- $e(X, Y) = |\{\{u, v\} \in E \mid u \in X, v \in Y\}|$
- $d(X, Y) = \dfrac{e(X, Y)}{|X||Y|}$ (called the density of (X, Y))



$e(X, Y) = 10$

$d(X, Y) = \dfrac{10}{4 \times 4} = \dfrac{5}{8}$

ε-Regular pairs

$G = (V, E)$ a graph; $A, B \subseteq V$ disjoint; $\varepsilon > 0$

Definition (ε-Regular pair)

(A, B) is ε-regular if for all $X \subseteq A$ and $Y \subseteq B$,

$$|X| \geq \varepsilon|A|,\ |Y| \geq \varepsilon|B| \Rightarrow |d(X, Y) - d(A, B)| \leq \varepsilon$$

Intuition

Large subsets X, Y of A, B (resp.) look similar to A, B (as far as the density is concerned)

Regular pairs look like random bipartite graphs (1)

Lemma 11.1

$G = (V, E)$ bipartite w/ its partite sets A, B, $|A| = |B| = n$;
(A, B) ε-regular w/ its density $d \Rightarrow$

$$|\{v \in A \mid (d-\varepsilon)n \le d(v) \le (d+\varepsilon)n\}| \ge (1-2\varepsilon)n.$$

Note

Construct a bipartite random graph $G = (A \cup B, E)$ by joining two vertices $v \in A, u \in B$ w/ prob. d/n^2 independently at random. Then

- $\mathbb{E}[d(A, B)] = d$
- $\mathbb{E}[d(v)] = dn$

Proof idea of Lemma 11.1

Suppose not

- $A_1 = \{v \in A \mid d(v) < (d-\varepsilon)n\}$
  - $\therefore d(A_1, B) < d-\varepsilon$
- $A_2 = \{v \in A \mid d(v) > (d+\varepsilon)n\}$ (Note: $A_1 \cap A_2 = \emptyset$)
  - $\therefore d(A_2, B) > d+\varepsilon$
- Then, $|A_1 \cup A_2| \ge 2\varepsilon n$
- $\therefore |A_1| \ge \varepsilon n$ or $|A_2| \ge \varepsilon n$
- $|A_1| \ge \varepsilon n \Rightarrow |d(A_1, B)-d(A, B)| \le \varepsilon$ (by ε-regularity of (A, B))
- $|A_2| \ge \varepsilon n \Rightarrow |d(A_2, B)-d(A, B)| \le \varepsilon$ (by ε-regularity of (A, B))
- In both cases, we have a contradiction □

Regular pairs look like random bipartite graphs (2)

Lemma 11.2

Setup

- $G = (V, E)$ a tripartite graph with its partite sets V_1, V_2, V_3
- $|V_1| = |V_2| = |V_3| = n$
- (V_i, V_j) is ε-regular with density d, $\forall\, 1 \le i < j \le 3$, for some $\varepsilon > 0$, $0 < d < 1$
- $\varepsilon \le d/2$

Statement

- $\#\, K_3$'s in $G \ge (1-4\varepsilon)(d-\varepsilon)^3 n^3$

Note: the exact coefficient "$(1-4\varepsilon)(d-\varepsilon)^3$" is not important, but it's important to note the coefficient only depends on ε and d

Proof idea of the lemma 11.2 (1/2)

First, estimate the number of vertices in V_1 that have many neighbors in both of V_2 and V_3

Let $X_i = |\{v \in V_1 \mid (d-\varepsilon)n \le |N_G(v) \cap V_i| \le (d+\varepsilon)n\}|$ $(i \in \{2, 3\})$

$$\begin{aligned} |X_2 \cap X_3| &= |X_2| + |X_3| - |X_2 \cup X_3| \\ &\ge (1-2\varepsilon)n + (1-2\varepsilon)n - n \\ &\qquad \text{(by Lem 11.1)} \\ &= (1-4\varepsilon)n \end{aligned}$$

Proof idea of Lemma 11.2 (2/2)

- Fix an arbitrary $v \in X_2 \cap X_3$
- let $S_i(v) = N_G(v) \cap V_i$ $(i \in \{2,3\})$
- $|S_2(v)| \geq (d-\varepsilon)n \geq (2\varepsilon-\varepsilon)n = \varepsilon n$
- $|S_3(v)| \geq (d-\varepsilon)n \geq (2\varepsilon-\varepsilon)n = \varepsilon n$ (similarly)
- $\therefore$ by the ε-regularity of (V_2, V_3), we have

$$\begin{aligned} e(S_2(v), S_3(v)) &= d(S_2(v), S_3(v))|S_2(v)||S_3(v)| \\ &\geq (d-\varepsilon)\cdot(d-\varepsilon)n\cdot(d-\varepsilon)n \\ &= (d-\varepsilon)^3 n^2 \end{aligned}$$

- Hence

$$\begin{aligned} \#\text{ of } K_3\text{'s in } G &\geq \sum_{v \in X_2 \cap X_3} e(S_2(v), S_3(v)) \\ &\geq (1-4\varepsilon)n \cdot (d-\varepsilon)^3 n^2 \\ &= (1-4\varepsilon)(d-\varepsilon)^3 n^3 \quad \square \end{aligned}$$

Extension of the previous lemma

Lemma 11.3

$\forall\, r \in \mathbb{N}\ \forall\, d \in (0,1)\ \exists\, \varepsilon_0 < 1\ \exists\, c > 0\ \forall\, \varepsilon \leq \varepsilon_0\ \forall$ graph G s.t.

- G an r-partite graph with its partite sets $V_1, \dots, V_r$
- $|V_1| = \cdots = |V_r| = n$
- (V_i, V_j) is ε-regular with density d, $\forall\, i,j \in \{1,\dots,r\}$

it holds that

- $\#$ K_r's in $G \geq cn^r$

Proof: Exercise (Hint: Induction on r)

Note

From the previous lemma, when $r = 3$ we can take $\varepsilon_0 = d/2$ and $c = (1-4\varepsilon_0)(d-\varepsilon_0)^3$

Today's contents

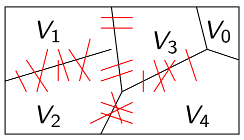
ε-Regular partitions

$G = (V, E)$ a graph; $\varepsilon > 0$;
$\{V_0, V_1, \dots, V_k\}$ a partition of V (w/ possibly $V_0 = \emptyset$)

Definition (ε-Regular partition)

$\{V_0, V_1, \dots, V_k\}$ is ε-regular if

- $|V_0| \leq \varepsilon |V|$
- $|V_1| = \cdots = |V_k|$
- $|\{(i,j) \mid 1 \leq i < j \leq k, (V_i, V_j) \text{ not } \varepsilon\text{-regular}\}| \leq \varepsilon k^2$



V_0 is called an exceptional set

Szemerédi's regularity lemma

Szemerédi's regularity lemma (Szemerédi '76)

$\forall\ \varepsilon > 0$ a real number $\forall\ n_0 \geq 1$ an integer
$\exists\ N_0 \in \mathbb{N}\ \forall\ G$ a graph:

$$n(G) \geq n_0 \implies \begin{array}{l} G \text{ has an } \varepsilon\text{-regular partition} \\ \{V_0, V_1, \ldots, V_k\} \text{ w/ } n_0 \leq k \leq N_0 \end{array}$$

Intuition

Evey dense graph looks like a union of random bipartite graphs

- Often called "Szemerédi's uniformity lemma" too
- We are not going to prove Szemerédi's regularity lemma, but looking at some applications of it

Quantitative version of Mantel's theorem

The following proposition gives an estimate for the number of K_3's contained in an n-vertex graph with more than $n^2/4$ edges

Proposition 11.4 (Quantitative version of Mantel's theorem)

$\forall\ \delta > 0\ \exists n_0 \in \mathbb{N}\ \exists\ c > 0\ \forall\ G = (V, E)$ a graph:

$$\begin{array}{l} n(G) = n \geq n_0, \\ e(G) \geq \left(\frac{1}{4} + \delta\right) n^2 \end{array} \implies \#\ K_3\text{'s in } G \geq cn^3$$

The above proposition "counts" the number of K_3's, not just showing the existence of K_3 as Mantel's theorem

Theorem 2.7 (Mantel 1907)

$e(G) > \lfloor n^2/4 \rfloor \implies K_3 \subseteq G$

Proof outline

1. Apply Szemerédi's regularity lemma to G
2. Consider a regularity graph R of G (next slide)
3. Conclude that R contains a copy of K_3 (by Mantel's theorem) and apply Lemma 11.2

Whole idea

Under some assumptions, to conclude G contains many K_3, it is enough to show R contains K_3

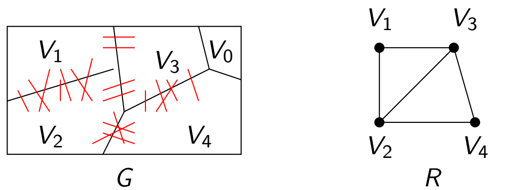
Regularity graphs

G a graph; $\varepsilon > 0$;
$\{V_0, V_1, \ldots, V_k\}$ an ε-regular partition of G; $|V_1| = \cdots = |V_k| = \ell$;
$0 \leq d \leq 1$

Definition (Regularity graph)

The regularity graph R of $\{V_0, V_1, \ldots, V_k\}$ with parameters ε, ℓ, d is defined as follows:

- $V(R) = \{V_1, \ldots, V_k\}$
- $E(R) = \{\{V_i, V_j\} \mid (V_i, V_j)$ an ε-regular pair of density $\geq d\}$



Rephrasing the quantitative version

We actually prove the following

$\forall\, \delta > 0\ \exists n_0 \in \mathbb{N}\ \exists\ N_0 \in \mathbb{N}\ \exists\ \varepsilon > 0\ \exists\ d \in (0,1]\ \exists\ c > 0$
$\forall\ G = (V, E)$ a graph:
$n(G) = n \geq n_0, e(G) \geq \left(\frac{1}{4} + \delta\right) n^2$
$\Longrightarrow \forall\ \{V_0, V_1, \ldots, V_k\}$ an ε-regular partition of G
$(n_0 \leq k \leq N_0, |V_1| = \cdots = |V_k| = \ell)$:
the regularity graph of $\{V_i\}$ with param's ε, ℓ, d contains K_3
$\Longrightarrow \#\ K_3$'s in $G \geq cn^3$

Here, N_0 is a natural number that Szemerédi's regularity lemma gives

Proof idea of Proposition 11.4 (1/11)

Given $\delta > 0$, we specify $\varepsilon = \varepsilon(\delta)$ and $n_0 = n_0(\delta)$ later

- Apply Szemerédi's regularity lemma to G with parameters ε and n_0 to obtain a number $N_0 \in \mathbb{N}$ and an ε-regular partition $V_0, V_1, \ldots, V_k$ $(n_0 \leq k \leq N_0, |V_1| = \cdots = |V_k| = \ell)$
- Let $d = d(\delta, \varepsilon, n_0)$ be constant we specify later, and R be the regularity graph of $\{V_0, V_1, \ldots, V_k\}$ with parameters ε, ℓ, d

Proof idea of Proposition 11.4 (2/11)

To show R has a lot of edges, we separately consider the following five types of edges in G

- $E = E_0 \cup E_1 \cup E_2 \cup E_3 \cup E_4$
- $E_0 = \{e \in E \mid V_0 \cap e \neq \emptyset\}$
- $E_1 = \left\{ e \in E \;\middle|\; \begin{array}{l} V_i \cap e \neq \emptyset, V_j \cap e \neq \emptyset \\ \text{for some } \{V_i, V_j\} \in E(R) \end{array} \right\}$
- $E_2 = \left\{ e \in E \;\middle|\; \begin{array}{l} V_i \cap e \neq \emptyset, V_j \cap e \neq \emptyset \\ \text{for some } (V_i, V_j) \text{ not } \varepsilon\text{-regular} \end{array} \right\}$
- $E_3 = \left\{ e \in E \;\middle|\; \begin{array}{l} V_i \cap e \neq \emptyset, V_j \cap e \neq \emptyset \\ \text{for some } (V_i, V_j) \text{ w/ } d(V_i, V_j) < d \end{array} \right\}$
- $E_4 = \{e \in E \mid e \subseteq V_i \text{ for some } V_i, i \in \{1, \ldots, k\}\}$

Proof idea of Proposition 11.4 (3/11)

$E_0 = \{e \in E \mid V_0 \cap e \neq \emptyset\}$

$$\begin{aligned} |E_0| &\leq \binom{|V_0|}{2} + |V_0||V \setminus V_0| \\ &\leq \binom{\varepsilon n}{2} + \varepsilon n \cdot k\ell \\ &\leq \frac{\varepsilon^2 n^2}{2} + \varepsilon n \cdot k\frac{n}{k} \\ &= \frac{\varepsilon^2}{2} n^2 + \varepsilon n^2 \end{aligned}$$

Proof idea of Proposition 11.4 (4/11)

$$E_1 = \left\{ e \in E \;\middle|\; \begin{array}{l} V_i \cap e \neq \emptyset, V_j \cap e \neq \emptyset \\ \text{for some } \{V_i, V_j\} \in E(R) \end{array} \right\}$$

$$\begin{aligned} |E_1| &= \sum_{\{V_i,V_j\}\in E(R)} e(V_i, V_j) \\ &\leq \sum_{\{V_i,V_j\}\in E(R)} |V_i||V_j| \\ &= \sum_{\{V_i,V_j\}\in E(R)} \ell^2 \\ &= e(R)\ell^2 \\ &\leq e(R)\left(\frac{n}{k}\right)^2 \\ &= \frac{e(R)}{k^2} n^2 \end{aligned}$$

Proof idea of Proposition 11.4 (5/11)

$$E_2 = \left\{ e \in E \;\middle|\; \begin{array}{l} V_i \cap e \neq \emptyset, V_j \cap e \neq \emptyset \\ \text{for some } (V_i, V_j) \text{ not } \varepsilon\text{-regular} \end{array} \right\}$$

$$\begin{aligned} |E_2| &\leq (\#\text{ non } \varepsilon\text{-regular pairs}) \cdot \ell^2 \\ &\leq \varepsilon k^2 \ell^2 \\ &\leq \varepsilon k^2 \left(\frac{n}{k}\right)^2 \\ &= \varepsilon n^2 \end{aligned}$$

Recall

$\#$ non ε-regular pairs in $V_1, \dots, V_k \leq \varepsilon k^2$ in any ε-regular partition

Proof idea of Proposition 11.4 (6/11)

$$E_3 = \left\{ e \in E \;\middle|\; \begin{array}{l} V_i \cap e \neq \emptyset, V_j \cap e \neq \emptyset \\ \text{for some } (V_i, V_j) \text{ w/ } d(V_i, V_j) < d \end{array} \right\}$$

$$\begin{aligned} |E_3| &\leq (\#\text{ pairs}) \cdot d\ell^2 \\ &= \binom{k}{2} \cdot d\ell^2 \leq \frac{k^2}{2} \cdot d \cdot \left(\frac{n}{k}\right)^2 = \frac{d}{2} n^2 \end{aligned}$$

$E_4 = \{e \in E \mid e \subseteq V_i \text{ for some } V_i, i \in \{1, \dots, k\}\}$

$$|E_4| \leq k \cdot \binom{\ell}{2} \leq k \cdot \frac{\ell^2}{2} \leq \frac{k}{2}\left(\frac{n}{k}\right)^2 = \frac{1}{2k} n^2$$

Proof idea of Proposition 11.4 (7/11)

- Therefore,

$$\begin{aligned} |E| &\leq |E_0| + |E_1| + |E_2| + |E_3| + |E_4| \\ &\leq \left(\frac{\varepsilon^2}{2} + \varepsilon\right) n^2 + \frac{e(R)}{k^2} n^2 + \varepsilon n^2 + \frac{d}{2} n^2 + \frac{1}{2k} n^2 \\ &\leq \left(\frac{e(R)}{k^2} + \frac{1}{2}\varepsilon^2 + 2\varepsilon + \frac{d}{2} + \frac{1}{2k}\right) n^2 \end{aligned}$$

- On the other hand, from the assumption we have

$$|E| \geq \left(\frac{1}{4} + \delta\right) n^2$$

Proof idea of Proposition 11.4 (8/11)

- Therefore,

$$\begin{aligned} \frac{1}{4}+\delta &\le \frac{e(R)}{k^2}+\frac{1}{2}\varepsilon^2+2\varepsilon+\frac{d}{2}+\frac{1}{2k} \\ \therefore\quad e(R) &\ge \left(\frac{1}{4}+\delta-\frac{1}{2}\varepsilon^2-2\varepsilon-\frac{d}{2}-\frac{1}{2k}\right)k^2 \end{aligned}$$

- Setting $\varepsilon=\frac{\delta}{16}$, $n_0=\left\lceil\frac{1}{\delta}\right\rceil$, $d=\frac{\delta}{4}$ we have

$$e(R)\ge\left(\frac{1}{4}+\gamma\right)k^2$$

for some γ only depending on δ (calculation in the next slide)

Proof idea of Proposition 11.4 (9/11)

$$\begin{aligned} &\frac{1}{4}+\delta-\frac{1}{2}\varepsilon^2-2\varepsilon-\frac{d}{2}-\frac{1}{2k} \\ &\ge \frac{1}{4}+\delta-\frac{1}{512}\delta^2-\frac{1}{8}\delta-\frac{1}{8}\delta-\frac{1}{2}\delta \qquad (k\ge n_0\ge \tfrac{1}{\delta}) \\ &= \frac{1}{4}+\frac{1}{4}\delta-\frac{1}{512}\delta^2 \\ &= \frac{1}{4}+\frac{\delta}{512}(128-\delta) \\ &\ge \frac{1}{4}+\frac{\delta}{512}(128-1) \qquad (\because \text{ we may assume } \delta\le 1) \\ &= \frac{1}{4}+\frac{127}{512}\delta; \end{aligned}$$

Then set $\gamma=\frac{127}{512}\delta$

Proof idea of Proposition 11.4 (10/11)

We now have

$$e(R)\ge\left(\frac{1}{4}+\gamma\right)k^2$$

for some γ only depending on δ.

- $\therefore$ R contains a copy of K_3 (from Mantel's theorem)
- $\therefore$ The ε-regular partition of G has three parts, say V_1, V_2, V_3, such that (V_i, V_j) are ε-regular $(1\le i<j\le 3)$ with density d
- $\therefore$ V_1, V_2, V_3 contain at least $(1-4\varepsilon)(d-\varepsilon)^3\ell^3$ K_3's (by Lemma 11.2 and $\varepsilon=\delta/16<\delta/8=d/2$)
- Namely, G contains $(1-4\varepsilon)(d-\varepsilon)^3\ell^3$ K_3's

Proof idea of Proposition 11.4 (11/11)

How many K_3's at least in G, eventually?

$$\begin{aligned} &(1-4\varepsilon)(d-\varepsilon)^3\ell^3 \\ &\ge (1-4\varepsilon)(d-\varepsilon)^3\left(\frac{(1-\varepsilon)n}{k}\right)^3 \\ &= (1-4\varepsilon)(d-\varepsilon)^3(1-\varepsilon)^3\frac{n^3}{k^3} \\ &\ge (1-4\varepsilon)(d-\varepsilon)^3(1-\varepsilon)^3\frac{n^3}{N_0^3} \qquad (N_0 \text{ in Regularity lemma}) \\ &= c(\varepsilon, d, N_0)n^3 \end{aligned}$$

and ε, d only depend on δ; N_0 only depends on ε, n_0 which only depend on δ; $\therefore$ c only depends on δ □

Quantitative version of Turán's theorem

Lemma 11.5 (Quantitative version of Turán's theorem)

$\forall\ r \in \mathbb{N}\ \forall\ \delta > 0\ \exists n_0 \in \mathbb{N}\ \exists\ N_0 \in \mathbb{N}\ \exists\ \varepsilon > 0\ \exists\ d \in (0,1]\ \exists\ c > 0$
$\forall\ G = (V,E)$ a graph:
$n(G) = n \geq n_0, e(G) \geq \left(1 + \frac{1}{r-1} + \delta\right)\binom{n}{2}$
$\Longrightarrow \forall\ \{V_0, V_1, \dots, V_k\}$ an ε-regular partition of G
$(n_0 \leq k \leq N_0, |V_1| = \dots = |V_k| = \ell)$:
the regularity graph of $\{V_i\}$ with param's ε, ℓ, d contains K_r
$\Longrightarrow \#\ K_r$'s in $G \geq cn^r$

Here, N_0 is a natural number that Szemerédi's regularity lemma gives

Proof

Exercise

Today's contents

I think we somewhat missed our road map...

We want to show the following

Rephrasing the Erdős-Stone thm

$\forall\ s \geq 1, r \geq 2$ natural numbers $\forall\ \delta > 0\ \exists n_1 \in \mathbb{N}\ \forall n \geq n_1\ \forall\ G$:
$n(G) = n \geq n_1$ and

$$e(G) \geq \left(1 - \frac{1}{r-1} + \delta\right)\binom{n}{2} \Rightarrow G \supseteq T_{rs,r}$$

Proof outline

- Use Szemerédi's regularity lemma to prove the embedding lemma
- Use the embedding lemma to prove Erdős-Stone thm

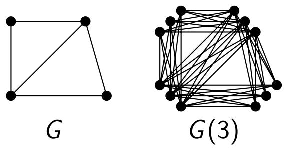
Expanding vertices in a graph

$G = (V,E)$ a graph; $s \in \mathbb{N}$

Notation

The graph $G(s)$ is defined as follows:

- $V(G(s)) = V \times \{1, \dots, s\}$
- $E(G(s)) = \{\{(u,i),(v,j)\} \mid \{u,v\} \in E\}$



Note: $K_r(s) \simeq T_{rs,r}$

Embedding lemma

Embedding lemma (Komlós, Simonovits '96)

- $\forall\, d \in (0,1]\ \forall\, \Delta \geq 1$
- $\exists\, \gamma_0 > 0$
- $\forall\, G$ a graph $\forall\, H$ a graph w/ $\Delta(H) \leq \Delta$
- $\forall\, \gamma < \gamma_0\ \forall\, s \in \mathbb{N}\ \forall\, \ell \geq s/\gamma_0$
- $\forall\, \{V_0, V_1, \ldots, V_k\}$ a γ-reg partition of G ($|V_1| = \cdots = |V_k| = \ell$):
  R is the reg graph of $\{V_i\}$
  w/ param's γ, ℓ, d, and $\implies H \subseteq G$
  $H \subseteq R(s)$

Under some assumptions, to conclude G contains H, instead of showing R contains H, it is enough to show $R(s)$ contains H

Proof idea of the Erdős-Stone thm (1/4)

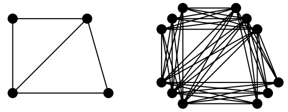
Rephrasing the Erdős-Stone thm

$\forall\, s \geq 1, r \geq 2$ natural numbers $\forall\, \delta > 0\ \exists n_1 \in \mathbb{N}\ \forall n \geq n_1\ \forall\, G$:
$n(G) = n \geq n_1$ and

$$e(G) \geq \left(1 - \frac{1}{r-1} + \delta\right)\binom{n}{2} \Rightarrow G \supseteq T_{rs,r}$$

Proof outline:

- Enough to show $T_{rs,r} \subseteq R(s)$ under the given assumption (by the embedding lemma)
- $\therefore$ enough to show $K_r \subseteq R$ under the assumptions



G $G(3)$

Proof idea of the Erdős-Stone thm (2/4)

We use the following as well

Lemma 11.5 (Quantitative version of Turán's theorem)

$\forall\, r \in \mathbb{N}\ \forall\, \delta > 0\ \exists\, n_2 \in \mathbb{N}\ \exists\, N_2 \in \mathbb{N}\ \exists\, \varepsilon_2 > 0\ \exists\, d_2 \in (0,1]$
$\forall\, G = (V,E)$ a graph:
$n(G) = n \geq n_2, e(G) \geq \left(1 + \frac{1}{r-1} + \delta\right)\binom{n}{2}$
$\implies \forall\, \{V_0, V_1, \ldots, V_k\}$ an ε_2-regular partition of G
($n_2 \leq k \leq N_2$, $|V_1| = \cdots = |V_k| = \ell$):
the regularity graph of $\{V_i\}$ with param's ε_2, ℓ, d_2 contains K_r

Here, N_2 is a natural number that Szemerédi's regularity lemma gives

Proof idea of the Erdős-Stone thm (3/4)

Given $s \geq 1, r \geq 2, \delta > 0$ (as in the Erdős-Stone thm)

- Lem 11.5 gives $n_2(r,\delta)$, $N_2(r,\delta)$, $\varepsilon_2(r,\delta)$, $d_2(r,\delta)$
- Let $d = d_2$
- Let $\Delta = \Delta(T_{rs,r}) = (r-1)s$
- Embedding Lemma gives $\gamma_0 = \gamma_0(d, \Delta)$
- Let $\varepsilon < \min\{\gamma_0, \varepsilon_2\}$ (Note: $\varepsilon < \varepsilon_2$)
- Let $n_1 \geq \max\left\{n_2, \frac{N_2 s}{\gamma_0(1-\varepsilon)}\right\}$

Proof idea of the Erdős-Stone thm (4/4)

Let G satisfy $n(G) = n \geq n_1$, and H satisfy $\Delta(H) \leq \Delta$
(Note: $n \geq n_2$)

- By Regularity Lemma, $\exists$ an ε-reg partition $\{V_0, V_1, \dots, V_k\}$ w/ $|V_1| = \cdots = |V_k| = \ell$ and $n_2 \leq k \leq N_2$
- Note: this ε-reg partition is ε_2-reg as well
- Let R be the regularity graph of $\{V_i\}$ w/ parameters ε, ℓ, d
- By Lem 11.5 (and $\varepsilon < \varepsilon_2$, $d \leq d_2$), $K_r \subseteq R$
- $\therefore$ $T_{rs,r} \subseteq R(s)$
- Let $\gamma = \varepsilon$ (Note $\varepsilon < \min\{\gamma_0, \varepsilon_2\} \leq \gamma_0$)
- Then $\ell \geq \frac{(1-\varepsilon)n}{N_0} \geq \frac{(1-\varepsilon)n_1}{N_0} \geq \frac{1-\varepsilon}{N_0}\frac{N_0 s}{\gamma_0(1-\varepsilon)} = \frac{s}{\gamma_0}$
- $\therefore$ By Embedding Lemma, $T_{rs,r} \subseteq G$ □

Today's contents

Deciding a given pair is ε-regular

Problem ε-Regularity

Input: A graph $G = (V, E)$ with its partite sets A, B and a real number $\varepsilon > 0$
Question: Is $\{A, B\}$ ε-regular?

Fact (Alon, Duke, Lefmann, Rödl, Yuster '94)

Problem ε-Regular is coNP-complete

Finding a ε-regular partition

Now comes a strange phenomenon... What if we want to construct one partition?

Problem ε-Regularity Partition

Input: A graph $G = (V, E)$ and a real number $\varepsilon > 0$
Output: An ε-regular partition of G

Fact (Alon, Duke, Lefmann, Rödl, Yuster '94)

Problem ε-Regular Partition can be solved in polynomial time

This "algorithmic version" of Szemerédi's regularity lemma is quite useful (in theory)

Colorable? or far from being colorable?

The following was proved with the regularity lemma

Fact (Duke, Rödl '85)

$\forall\, \varepsilon > 0\ \forall\, k \in \mathbb{N}\ \exists\, c\ \exists\, \delta\ \forall\, G$ an n-vertex graph:

G contains a c-vertex non-k-colorable subgraph or
G has δn^2 edges whose removal leaves a k-colorable graph

The following is an algorithmic version

Fact (Alon, Duke, Lefmann, Rödl, Yuster '94)

Fix $\varepsilon > 0$, $k \in \mathbb{N}$. For a given n-vertex graph G we can find either

an $\mathrm{O}(1)$-vertex non-k-colorable subgraph of G or
$\mathrm{O}(n^2)$ edges whose removal leaves a k-colorable graph

in polynomial time

Approximately finding a maximum edge cut

Fact (Frieze, Kannan '96)

Fix $\varepsilon > 0$. Given an n-vertex graph G, we can compute in polynomial time an edge cut $[X, \overline{X}]$ of G that satisfies

$$|[X^*, \overline{X^*}]| - |[X, \overline{X}]| \le \varepsilon n^2$$

where $[X^*, \overline{X^*}]$ is an edge cut with maximum number of edges

- Their algorithm uses ideas from Szemerédi's regularity lemma (Another algorithm by Arora, Karger, Karpinski '95 is based on integer linear programming and random sampling)
- If G is "dense" ($|[X^*, \overline{X^*}]| = \Omega(n^2)$) then this algorithm is a polynomial-time approximation scheme (PTAS)
- The "ℓ-way version" is also available (Frieze, Kannan '96)