Topics on Computing and Mathematical Sciences I Graph Theory (10) Random Graphs

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Today's contents

- Random graphs
- Probabilistic methods
- The Galton-Watson process

Probability space

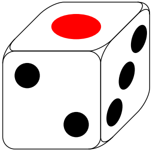
Definition (Probability space)

A pair $(\Omega, \mathbb{P})$ is a finite probability space if Ω is a finite set and $\mathbb{P}$ is a map from Ω into $\mathbb{R}_{\geq 0}$ with $\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$

Example: An unbiased dice

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

$$\mathbb{P}(1) = \mathbb{P}(2) = \mathbb{P}(3) = \mathbb{P}(4) = \mathbb{P}(5) = \mathbb{P}(6) = \frac{1}{6}$$



Random graphs

$V = \{1, \ldots, n\}$ a set of vertices

$N = \binom{n}{2} = \frac{n(n-1)}{2}$

p a number with $0 \le p \le 1$

Definition (Random graph I)

- We select the edges of K_n independently, with probability p
- $\mathcal{G}(n, p) = (\mathcal{G}_n, \mathbb{P}_p)$ is a probability space, where $\mathcal{G}_n$ is the set of all 2^N graphs on V and

$$\mathbb{P}_p(H) = p^m(1-p)^{N-m}$$

  if the graph H on V has precisely m edges
- $G_{n,p}$ denotes a random graph in the space $\mathcal{G}(n, p)$

Random graphs

$V = \{1, \ldots, n\}$ a set of vertices

$N = \binom{n}{2} = \frac{n(n-1)}{2}$

M an integer with $0 \leq M \leq N$

Definition (Random Graph II)

- $\mathcal{G}(n, M) = (\mathcal{G}_{n,M}, \mathbb{P}_M)$ is a probability space, where $\mathcal{G}_{n,M}$ is the set of all $\binom{N}{M}$ subgraphs of K_n with M edges and

$$\mathbb{P}_M(H) = \binom{N}{M}^{-1}$$

  for all $H \in \mathcal{G}_{n,M}$
- $G_{n,M}$ denotes a random graph in the space $\mathcal{G}(n, M)$

Tools from probability: Expectation

$(\Omega, \mathbb{P}) = \mathcal{G}(n, p)$ or $\mathcal{G}(n, M)$

A random variable X on Ω is a mapping $X : \Omega \to \mathbb{R}$

The expectation $\mathbb{E}(X)$ of X is $\mathbb{E}(X) = \sum_{G \in \Omega} \mathbb{P}(G) \cdot X(G)$

Lemma 10.1

For two random variables X and Y on Ω, $\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y)$

Proof

$$\begin{aligned}\mathbb{E}(X + Y) &= \sum_{G \in \Omega} \mathbb{P}(G) \cdot (X(G) + Y(G)) \\ &= \sum_{G \in \Omega} \mathbb{P}(G) \cdot X(G) + \sum_{G \in \Omega} \mathbb{P}(G) \cdot Y(G) = \mathbb{E}(X) + \mathbb{E}(Y)\end{aligned}$$

$\square$

Tools from probability: Markov's inequality

$(\Omega, \mathbb{P}) = \mathcal{G}(n, p)$ or $\mathcal{G}(n, M)$

$X = X(G)$ a non-negative random variable on Ω

$a > 0$ a number

Lemma 10.2 (Markov's inequality)

$$\mathrm{Prob}(X \geq a) \leq \frac{\mathbb{E}(X)}{a}$$

Proof

$$\mathbb{E}(X) = \sum_{G \in \Omega} \mathbb{P}(G) \cdot X(G) \geq \sum_{\substack{G \in \Omega \\ X(G) \geq a}} \mathbb{P}(G) \cdot X(G)$$

$$\geq \sum_{\substack{G \in \Omega \\ X(G) \geq a}} \mathbb{P}(G) \cdot a = \mathrm{Prob}(X \geq a) \cdot a$$

$\square$

Structure of random graphs

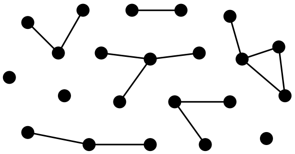
What does $G_{n,\,p}$ look like?

For further information, refer to "*Random Graphs*" by Bollobás (1985)

An isolated tree is a connected component without cycles
Denote by $T(G)$ the number of vertices contained in isolated trees of G
Clearly $T(G) \leq n$

Suppose $p = c/n$, where c is a constant with $0 < c < 1$; Then

- $\mathbb{E}(T(G_{n,\,p})) = n + \mathrm{O}(1)$
- For almost every $G_{n,p}$, the size of the largest component is $\mathrm{O}(\ln n)$



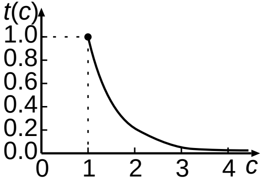
Structure of random graphs

$T(G)$ # of vertices contained in isolated trees of G

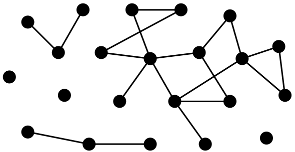
Suppose $p = c/n$, where c is a constant with $c > 1$; Then

- $\mathbb{E}(T(G_{n,\,p})) = t(c) \cdot n + \mathrm{O}(1)$, where

$$t(c) = \frac{1}{c} \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} (c \cdot e^{-c})^k$$



- For almost every $G_{n,p}$, it has a unique giant component, and all other vertices form trees of size $\mathrm{O}(\ln n)$



Structure of random graphs

The following is the most celebrated theorem in the theory of random graphs:

Erdős-Ŕenyi theorem (1960)

Suppose $p = c \cdot \dfrac{\ln n}{n}$. Then

$$\lim_{n \to +\infty} \mathrm{Prob}(G_{n,\,p} \text{ is connected}) = \begin{cases} 1 & \text{if } c > 1 \\ 0 & \text{if } 0 < c < 1 \end{cases}$$

In other words, $\dfrac{\ln n}{n}$ is a sharp threshold for the connectivity of $G_{n,\,p}$

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A theorem of Erdős

Theorem 10.3 (Erdős '59) (This fact was stated in Lecture 7)

For every $k \geq 2$, there exists a graph G with $\chi(G) > k$ and $g(G) > k$.

Reminder:

$\chi(G) =$ the chromatic number of G
$g(G) =$ the girth (the length of a shortest cycle) of G

Let us see that this theorem can be proved using random graphs
This approach is called a probabilistic method

Proof outline of the theorem of Erdős

Proof outline of Theorem 10.3

Choose $p := n^{-\frac{k}{k+1}} = \frac{1}{n} \cdot n^{\frac{1}{k+1}}$ ($> \frac{1}{n}$, and $> \frac{\ln n}{n}$ for sufficiently large n) and consider $\mathcal{G}(n, p)$ for all n

We have that as n goes to infinity

- $\alpha(G_{n,p}) < \frac{n}{2k}$ holds with high probability (Lemma 10.4)
- $\#$ of all cycles of length $\leq k$ of $G_{n,p}$ is at most $\frac{n}{2}$ with high probability (Lemma 10.5)

As a result, there must exist a graph with desired properties

Reminder:

$\alpha(G)$ = the size of a maximum independent set of G

$\chi(G) \cdot \alpha(G) \geq n$

Properties of random graphs

Lemma 10.4

If $p := n^{-\frac{k}{k+1}}$, $\exists n_1 \in \mathbb{N}$ such that for all $n \geq n_1$

$$\mathrm{Prob}\left(\alpha(G_{n,p}) \geq \frac{n}{2k}\right) < \frac{1}{2}$$

Lemma 10.5

If $p := n^{-\frac{k}{k+1}}$, $\exists n_2$ such that for all $n \geq n_2$

$$\mathrm{Prob}\left((\#\text{ of all cycles of length} \leq k \text{ of } G_{n,p}) \geq \frac{n}{2}\right) < \frac{1}{2}$$

By Lemmas 10.4 and 10.5, there exists a graph G on V with

- $\alpha(G) < \frac{n}{2k}$
- fewer than $\frac{n}{2}$ cycles of length $\leq k$.

Properties of random graphs

Proof of Lemma 10.4

$p := n^{-\frac{k}{k+1}}$; Suppose $2 \leq r := \lceil \frac{n}{2k} \rceil$

- $\mathrm{Prob}(\text{a fixed } r\text{-set} \subseteq V \text{ is independent}) = (1-p)^{\binom{r}{2}}$
- $$\begin{aligned} \mathrm{Prob}(\alpha(G_p) \geq r) &\leq \binom{n}{r}(1-p)^{\binom{r}{2}} \\ &\leq n^r(1-p)^{r(r-1)/2} = (n(1-p)^{(r-1)/2})^r \\ &\leq (ne^{-p(r-1)/2})^r && (1-p \leq e^{-p}) \\ &\leq (ne^{-pr/2} \cdot e^{\frac{1}{2}})^r && (p \leq 1) \end{aligned}$$
- $pr \geq \frac{n}{2k} \cdot n^{-\frac{k}{k+1}} = \frac{1}{2k} \cdot n^{\frac{1}{k+1}} \geq 3 \ln n$ for all $n \geq n'_1$
- $ne^{-pr/2} \cdot e^{\frac{1}{2}} \leq ne^{-3\ln n/2} \cdot e^{\frac{1}{2}} = (\frac{e}{n})^{1/2}$ for all $n \geq n'_1$
- $\mathrm{Prob}(\alpha(G_p) \geq r) \leq (\frac{e}{n})^{r/2}$ for all $n \geq n'_1$

$\frac{e}{n}$ converges to 0 as n goes to $+\infty$ $\square$

Properties of random graphs

Proof of Lemma 10.5

$i \in \mathbb{N}$ an integer with $3 \leq i \leq k$

- (# of possible i-cycles on V) $= \binom{n}{i}\frac{(i-1)!}{2}$
- Every such cycle C appears with probability p^i

X a random variable that counts # of all cycles of length $\leq k$ of $G_{n,\,p}$

- $\mathbb{E}(X) = \sum_{i=3}^{k} \binom{n}{i}\frac{(i-1)!}{2} p^i \leq \frac{1}{2}\sum_{i=3}^{k} n^i p^i \leq \frac{1}{2}(k-2)n^k p^k$
- $\mathrm{Prob}(X \geq \frac{n}{2}) \leq \frac{\mathbb{E}(X)}{n/2}$ (Markov's inequality)

$$\leq (k-2)\frac{(np)^k}{n} \quad (p = n^{-\frac{k}{k+1}})$$

$$= (k-2)n^{-\frac{1}{k+1}}$$

$n^{-\frac{1}{k+1}}$ converges to 0 as n goes to $+\infty$ □

Proof of the theorem of Erdős

Proof of Theorem 10.3

- By Lemmas 10.4 and 10.5, there exists a graph G on V with $\alpha(G) < \frac{n}{2k}$ and fewer than $\frac{n}{2}$ cycles of length $\leq k$.
- Delete one vertex from each of cycles of length $\leq k$
  Let H be the resulting graph
- We have $g(H) > k$
- $\alpha(H) \leq \alpha(G) < \frac{n}{2k}$ and $\chi(H) \cdot \alpha(H) \geq n(H) \geq \frac{n}{2}$
  $\Rightarrow \quad \chi(H) > k$
- Thus, H is a graph with desired properties

$\square$

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The Galton-Watson process

Definition (The Galton-Watson process $\{Z_i : i = 0, 1, \ldots\}$ (around 1873))

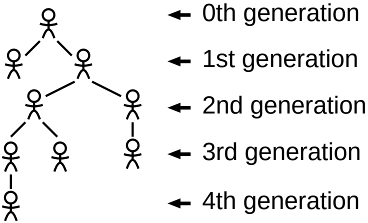
- Z_i denotes the number of people in the i-th generation
- $Z_0 = 1$ (There exists a unique root)
- Z_1 has a fixed distribution: $\mathrm{Prob}(Z_1 = k) = p_k, \;\; k \geq 0$
- Each child produces offspring according to the same distribution

$p_0 = 2/5$

$p_1 = 3/10$

$p_2 = 3/10$

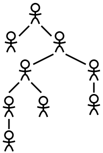
$p_k = 0, \;\; k > 2$



Probability distribution $\{p_k : k \geq 0\}$ is called an offspring distribution

The Galton-Watson process

$\{p_k : k \geq 0\}$ an offspring distribution



$\mathrm{Prob}(\text{extinction}) = \mathrm{Prob}(Z_k = 0 \text{ for some } k \geq 0)$

We want to determine whether $\mathrm{Prob}(\text{extinction}) = 1$ or not

$\longrightarrow$ Basically, it only depends on $\mu := \mathbb{E}(Z_1) = \sum_{k=0}^{+\infty} k \cdot p_k$

Suppose $p_0 + p_1 = 1$; Clearly, $\mu = p_1$
If $p_1 < 1$, then $\mathrm{Prob}(\text{extinction}) = 1$
If $p_1 = 1$, then $\mathrm{Prob}(\text{extinction}) = 0$

Thus, it suffices to consider the case where $p_0 + p_1 < 1$

The Galton-Watson process

$\{p_k : k \geq 0\}$ an offspring distribution

Theorem 10.6

Suppose $p_0 + p_1 < 1$. Then

$$\mathrm{Prob}(\text{extinction}) \begin{cases} = 1 & \text{if } \mu \leq 1 \\ < 1 & \text{if } \mu > 1, \end{cases}$$

where $\mu = \mathbb{E}(Z_1) = \sum_{k=0}^{+\infty} k \cdot p_k$

Note that the population becomes extinct with probability one even if $\mu = 1$

The Galton-Watson process

Intuitive explanation of Theorem 10.6 (i)

$q = \mathrm{Prob}(\text{extinction})$; We have $0 \leq q \leq 1$

- $\mathrm{Prob}(\text{extinction} | Z_1 = k) = q^k$
- $q = \sum_{k=0}^{\infty} p_k \cdot \mathrm{Prob}(\text{extinction} | Z_1 = k) = \sum_{k=0}^{\infty} p_k q^k$

Define $f(s) = \sum_{k=0}^{\infty} p_k s^k \ (s \in \mathbb{R})$

- $f(q) = q$ and $f(1) = 1$
- $f'(s) = \sum_{k=1}^{\infty} k p_k s^{k-1}$ and $f'(1) = \sum_{k=1}^{\infty} k p_k = \mu$
- s^k is strictly convex on $[0, 1]$ if $k \geq 2$, and $p_k > 0$ for some $k \geq 2$
  $\Rightarrow \ f(s)$ is strictly convex on $[0, 1]$

The Galton-Watson process

Intuitive explanation of Theorem 10.6 (ii)

Suppose $f'(1) = \mu \leq 1$

- Equation $f(s) = s$ has a unique solution $s = 1$ on $[0, 1]$
- Thus, we obtain $q = 1$

Suppose $f'(1) = \mu > 1$; Remark that $f(0) = p_0 \geq 0$

- Equation $f(s) = s$ has two distinct solutions $s = 1$ and $\alpha \in [0, 1)$
- We can expect that $q = \alpha$ $\square$