Topics on Computing and Mathematical Sciences I Graph Theory (8) Planarity

Yoshio Okamoto

Tokyo Institute of Technology

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Today's contents

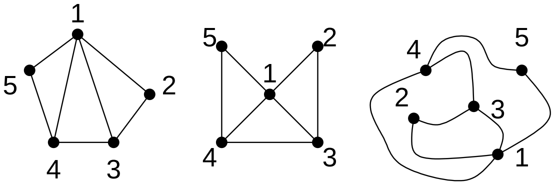
- Plane graphs, planar graphs
- Euler's formula
- Coloring a planar graph
- Kuratowski's theorem

Plane graphs

Definition (Plane graph)

A plane graph is an ordered pair $G = (P, \mathcal{C})$ satisfying the following

- P is a set of points on $\mathbb{R}^2$, called the vertices of G,
- $\mathcal{C}$ is a set of non-selfintersecting curves on $\mathbb{R}^2$, called the edges of G,
- The endpoints of each edge $e \in \mathcal{C}$ belong to P,
- The interior of each edge $e \in \mathcal{C}$ contains no point of P or no point on the other edges $e' \in \mathcal{C} \setminus \{e\}$



Planar graphs

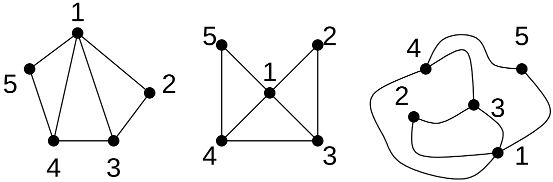
$G = (V, E)$ a graph

Definition (Planar graph)

G is planar if $\exists$ a plane graph $G_p = (V_p, E_p)$ and a bijection $f: V \to V_p$ such that

- $\{u, v\} \in E \iff \exists$ a curve $\in E_p$ connecting $f(u)$ and $f(v)$

$V = \{1, 2, 3, 4, 5\}$
$E = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{3, 4\}, \{4, 5\}\}$

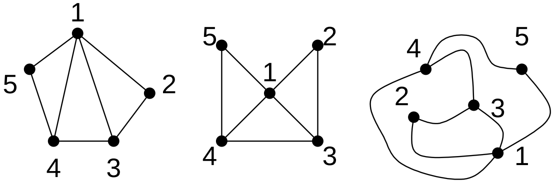


Plane embeddings

Definition (Plane embedding)

With the notation in the previous slide, G_p is a plane embedding of G

$V = \{1, 2, 3, 4, 5\}$
$E = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{3, 4\}, \{4, 5\}\}$



Convention

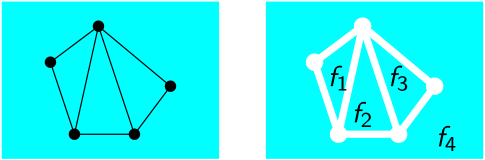
We also regard a plane graph G_p as a graph. For example, we say a plane graph G_p is 3-regular if the associated planar graph G is 3-regular

Faces of a plane graph

$G = (V, E)$ a plane graph

Definition (Face)

A face of P is a connected component (in the topological sense) of $\mathbb{R}^2 \setminus (V \cup E)$



Remark: There is always a unique *unbounded* face

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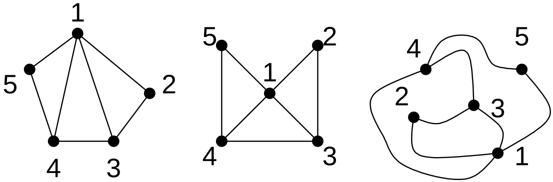
Euler's formula

$G = (V, E)$ a plane graph

Theorem 8.1 (Euler' formula; Euler 1758)

G has n vertices, e edges, f faces and k connected components $\Rightarrow$ $n - e + f = 1 + k$



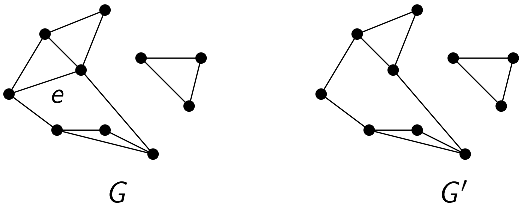
$$n = 5, e = 7, f = 4, k = 1$$
$$n - e + f = 2 = 1 + k$$

Proof of Euler's formula

Proof idea.

Induction on e

- When $e = 0$, $n = k$ and $f = 1$; Suppose $e \geq 1$
- Case 1: G has a cycle
  - Delete one edge from a cycle to obtain a new graph G'
  - $n' = n$, $e' = e-1$, $f' = f-1$, $k' = k$

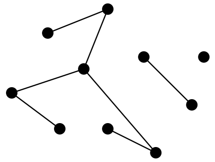


Proof of Euler's formula (continued)

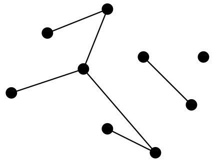
Proof idea (continued).

Induction on e

- Case 2: G has no cycle (hence a forest)
  - Delete a vertex of degree one to obtain a new graph G'
  - $n' = n-1$, $e' = e-1$, $f' = f$, $k' = k$ □



G



G'

Number of edges in a planar graph

G a planar graph

Proposition 8.2 (Number of edges in a planar graph)

1. $n(G) \geq 3 \implies e(G) \leq 3n(G) - 6$
2. $n(G) \geq 3,\ G \not\supseteq K_3 \implies e(G) \leq 2n(G) - 4$

Proof idea.

1. Double counting and apply Euler's formula
2. Exercise □

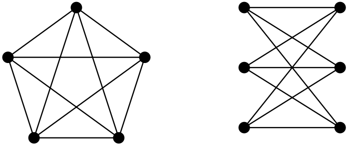
Application: Non-planar graphs

Proposition 8.3

K_5 and $K_{3,3}$ are non-planar

Proof.

Apply Proposition 8.2 □



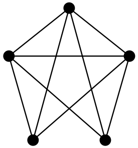
Maximal planar graphs and triangulations

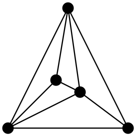
Definition (Maximal planar graph)

A planar graph G is maximal if adding any other edge will destroy the planarity

Definition (Triangulation)

A plane graph G is a triangulation if all faces are triangles (incident to three edges)



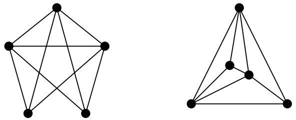


Characterizations of maximal planar graphs

Proposition 8.4 (maximality and triangulations)

For an n-vertex planar graph G the following are equivalent

1. G has $3n - 6$ edges
2. G is a maximal planar graph
3. Every plane graph associated to G is a triangulation



Proof idea.

[(1)⇒(2)] From Prop 8.2
[(2)⇒(3)] Add an edge inside a non-triangle face
[(3)⇒(1)] Remember the proof of Prop 8.2 □

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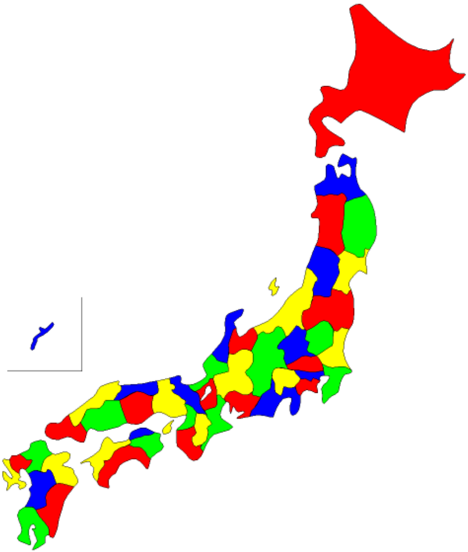
Coloring a map of Japan



From a map to a planar graph

Easy bound for the chromatic number of planar graphs

Proposition 8.5 ("Six color theorem")

G planar $\Rightarrow$ G is 6-colorable

Lemma 8.6 (min degree of a planar graph)

G planar $\Rightarrow \delta(G) \leq 5$

Proof.

Immediate from Proposition 8.2 □

Proof idea of Proposition 8.5.

- $\delta(H) \leq 5$ for all subgraphs H of G (Lem 8.6)
- $\therefore$ G is 5-degenerate
- G is 6-colorable (Prop 6.4)

Five color theorem

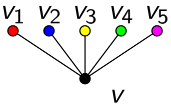
Theorem 8.7 (Five color theorem; Heawood 1890)

G planar $\Rightarrow$ G is 5-colorable

Proof idea.

Induction on $n(G)$; Fix a plane embedding of G

- $n(G) \leq 5 \Rightarrow G$ 5-colorable; Suppose $n(G) \geq 5$
- Let v be such that $d(v) \leq 5$ (cf. Lem 8.6)
- $G - v$ is 5-colorable (induction)
- If G is not 5-colorable, $d(v) = 5$ and all five colors appear in $N(v) = \{v_1, \ldots, v_5\}$; Let i be the color of v_i

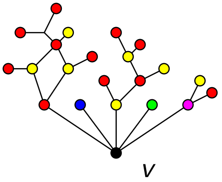


Five color theorem (continued)

Proof idea (continued).

Assume $\{v_1, \ldots, v_5\}$ are placed clockwise around v

- Consider the subgraph G_{13} of $G - v$ induced by those vertices colored 1 or 3 by c
- If a component of G_{13} containing v_1 does not contain v_3, then we switch the colors in that component
- v can be colored 1 and we're done



Five color theorem (continued)

Proof idea (continued).

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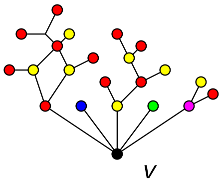

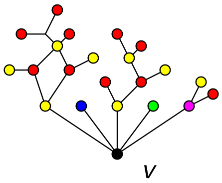

Five color theorem (continued)

Proof idea (continued).

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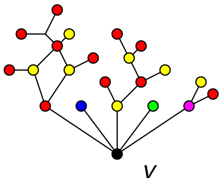
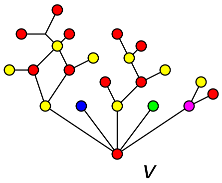
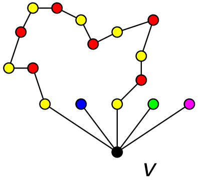




Five color theorem (further continued)

Proof idea (continued).

- We may suppose v_1 and v_3 in the same component of G_{13}
- $\therefore$ G contains a cycle C_{13} through v, v_1, v_3
- By the same argument, we may suppose G contains a cycle C_{24} through v, v_2, v_4
- This is impossible since G is planar □



Five color theorem (further continued)

Proof idea (continued).

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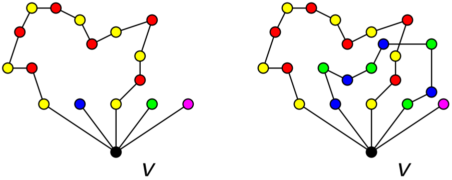

Four color theorem

Four color theorem (Appel, Haken '77; Appel, Haken, Koch '77)

G planar $\Rightarrow$ G is 4-colorable

Remarks

- One of the hardest theorem in Graph Theory; We don't prove in the class
- Their proof is computer-assisted
- A shorter proof (Robertson, Sanders, Seymour, Thomas '97) is available, but still computer-assisted (but people think it could be turned to a manual proof)
- Lots of equivalent formulations (in terms of graphs, Lie algebra, polynomials over a finite field, number theory, probability, ...)

Coloring a planar graph with four colors

Problem FOUR-COLORING OF A PLANAR GRAPH

Input: a planar graph G

Output: a proper 4-coloring of G

Known fact

The proof of Robertson, Sanders, Seymour, and Thomas ('97) gives rise to an $\mathrm{O}(n^2)$-time algorithm

Open problem

Can we do it in $\mathrm{O}(n)$ time? How about in $\mathrm{O}(n \log n)$ time?

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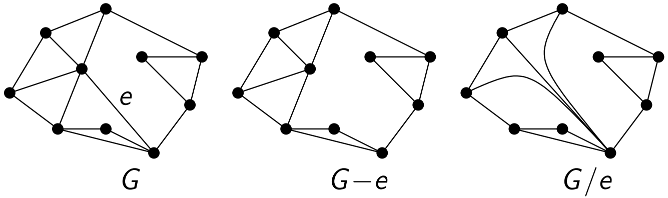
Planarity preserved under deletion and contraction

Observation

$G = (V, E)$ a planar graph, $e \in E$ an edge $\Rightarrow$ $G - e$ is planar and G/e is planar



Consequence

- G is planar, G contains an H-minor $\Rightarrow$ H is planar
- H is not planar, G contains an H-minor $\Rightarrow$ G is not planar

Wagner's theorem and Kuratowski's theorem

Consequence (of the previous slide and Prop 8.3)

G planar $\Rightarrow$ G contains no K_5-minor or no $K_{3,3}$-minor

Wagner's theorem ('37)

G planar $\Leftrightarrow$ G contains no K_5-minor or no $K_{3,3}$-minor

One can deduce Wagner's theorem from the following theorem by Kuratowski (Exercise)

Kuratowski's theorem ('30)

G planar $\Leftrightarrow$ G contains no K_5-subdivision or no $K_{3,3}$-subdivision

We're not going to prove Kuratowski's theorem, but it's important to notice Kuratowski's theorem gives a *good characterization* for planarity

Testing planarity

Problem Planarity

Input: a graph G

Question: Is G planar?

Known fact

- Can be solved in $\mathrm{O}(n+m)$ time (Hopcroft, Tarjan '74)
- Some simpler linear-time algorithms are also available (Booth, Luecker '76; Boyer, Myrvold '99, '04)

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- Open problems

Hamiltonicity of 3-connected planar graphs

Open Problem (Barnette)

Is every 3-regular 3-connected bipartite planar graph Hamiltonian?

If not bipartite, it can be non-Hamiltonian. (Tutte '46)

Fact (Steinitz '22)

Every 3-connected planar graph is the edge graph of a 3-dimensional convex polytope (i.e., bounded polyhedron)

Relaxed colorings of planar graphs

$G = (V, E)$ a graph; k a natural number

Definition (k-Relaxed coloring)

A coloring of G is k-relaxed if every connected component of the subgraph of G induced by any color class is of size at most k

Question (Alon, Ding, Oporowski, Vertigan '03)

$\exists$ a function $f : \mathbb{N} \to \mathbb{N}$ $\forall$ $d \in N$: G a planar graph of max degree Δ $\Rightarrow$ G has a $f(\Delta)$-relaxed 3-coloring?

Notes

- Every planar graph has a 1-relaxed 4-coloring (Four Color Thm)
- The function f in the Question cannot be constant (Alon, Ding, Oporowski, Vertigan '03)