Topics on Computing and Mathematical Sciences I Graph Theory (7) Coloring II

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Plan changed

- 4/09 Definition of Graphs; Paths and Cycles
- 4/16 Cycles; Extremality
- 4/23 Trees; Matchings in Bipartite Graphs
- 4/30 Matchings and Factors
- 5/14 Connectivity
- 5/21 Coloring I
- 6/04 Coloring II
- 6/11 Planarity
- 6/18 Extremal Graph Theory I (Turán's theorem)
- 6/25 Guest lecture (by K. Nagano)
- 7/02 Extremal Graph Theory II (Erdős-Stone's theorem)
- 7/09 Ramsey Theory
- 7/16 TBA
- 7/23 No class

Today's contents

- Mycielski's construction
- Edge coloring
- Other concepts and open problems

Chromatic numbers can be arbitrarily far from clique numbers

$G = (V, E)$ a graph

Definition (recap)

A set $S \subseteq V$ is a clique if every pair of edges are adjacent;
$\omega(G)$ = the size of a largest clique of G

Proposition 6.1 (recap)

$\chi(G) \geq \omega(G)$

Today, we show that this lower bound can be arbitrarily bad

Mycielski's construction

$G = (V, E)$ a graph

Definition (Mycielski's construction)

From G, Mycielski's construction produces a graph $M(G)$ containing G, as follows:

- Let $V = \{v_1, \ldots, v_n\}$
- $V(M(G)) = V \cup \{u_1, \ldots, u_n, w\}$
- $E(M(G)) = E \cup \{\{u_i, v\} \mid v \in N_G(v_i) \cup \{w\}\}$

Reminder: $N_G(v_i)$ = the set of vertices in G adjacent to v_i

Iterative application of Mycielski's construction

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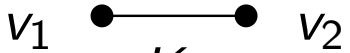
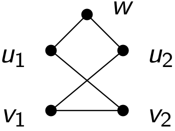
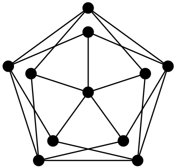


K_2



$M(K_2) \simeq C_5$



$M^2(K_2)$
(Grötzsch graph)

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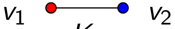
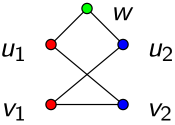


K_2



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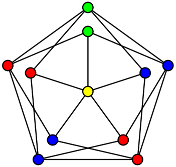

$M^2(K_2)$
(Grötzsch graph)

Properties of Mycielski's construction

Theorem 7.1 (Mycielski '55)

1. $G \not\supseteq K_3 \Rightarrow M(G) \not\supseteq K_3$
2. $\chi(G) = k \Rightarrow \chi(M(G)) = k+1$

Let $U = \{u_1, \ldots, u_n\}$ in the Mycielski's construction

Prood idea of (1).

- U is independent (i.e., no pair of vertices in U is adjacent)
- $\therefore$ no K_3 in $M(G)$ containing w
- $\therefore$ Any K_3 in $M(G)$ has its vertex set as $\{u_i, v_j, v_k\}$
- Then, $\{v_j, v_k\}, \{v_i, v_j\}, \{v_i, v_k\} \in E$; A contradiction □

Properties of Mycielski's construction (continued)

Prood idea of (2).

- We can color $M(G)$ with $k+1$ colors
  - Let c be a proper k-coloring of G, then $\tilde{c}$ def'ed as $\tilde{c}(v) = c(v)$ for $v \in V$, $\tilde{c}(u_i) = c(v_i)$ for $u_i \in U$, and $\tilde{c}(w) = k+1$ is a proper $k+1$-coloring of $M(G)$
- Suppose we can color $M(G)$ with k colors; Let $\tilde{c}\colon V(M(G)) \to \{1, \dots, k\}$ be a proper k-coloring of $M(G)$
- WLOG $\tilde{c}(w) = k$; then $\tilde{c}(u_i) \in \{1, \dots, k-1\}\ \forall\ u_i \in U$
- Define $c\colon V \to \{1, \dots, k\}$ by $c(v_i) = \tilde{c}(v_i)$ if $\tilde{c}(v_i) \neq k$ and $c(v_i) = \tilde{c}(u_i)$ if $\tilde{c}(v_i) = k$, then c is a proper $k-1$-coloring of G; A contradiction □

Consequence of Mycielski's construction

Corollary 7.2

1. $\forall\ k \geq 2\ \exists$ a graph G: $\omega(G) = 2$ and $\chi(G) = k$
2. $\forall\ k \geq \ell \geq 2\ \exists$ a graph G: $\omega(G) = \ell$ and $\chi(G) = k$

Remarks on Mycielski's construction (1)

Start from $G = K_2$

- $M^0(G) = K_2$; $n(M^0(G)) = 2$
- $M^1(G) = C_5$; $n(M^1(G)) = 5$
- $M^2(G) =$ Grötzsch graph; $n(M^2(G)) = 11$
- $M^k(G) = \ldots$; $n(M^k(G)) = \Theta(2^k)$

This is an exponential growth; Too large!?

$f(k) =$ the min # of vertices of a k-chromatic graph $\not\supseteq K_3$

Facts

- $f(k) = \mathrm{O}(k^2 \log k)$ (Ajtai, Komlós, Szemerédi '80)
- $f(k) = \Omega(k^2 \log k)$ (Kim '95; Fulkerson Prize Winner)

Remarks on Mycielski's construction (2)

Another view of a consequence of Mycielski's construction

$\forall\, k \geq 2\ \exists$ a graph G: $g(G) \geq 4$ and $\chi(G) = k$

Reminder: $g(G) =$ the girth (the length of a shortest cycle) of G

Intuition?

Graphs with large girths would be colored by few colors??
(Large girths should make the graph locally tree-like)

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Intuition?

Graphs with large girths would be colored by few colors??
(Large girths should make the graph locally tree-like)

Fact (rejecting the intuition above; Erdős '59)

$\forall\ g, k \geq 3\ \exists$ a graph G: $g(G) \geq g$ and $\chi(G) \geq k$

This result was obtained by the "first" application of the so-called probabilistic method, which we don't touch upon

Today's contents

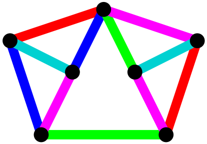
- Mycielski's construction
- Edge coloring
- Other concepts and open problems

Edge coloring

$G = (V, E)$ a graph; k a natural number

Definition (Edge-coloring, Proper edge-coloring)

A k-edge-coloring of G is a map $c\colon E \to \{1, \ldots, k\}$;
The edges of one color form a color class;
A k-edge-coloring of G is proper if $c(e) \neq c(e')$ for all adjacent edges $e, e' \in E$



Chromatic index

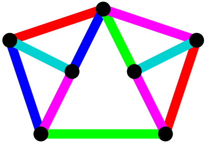
$G = (V, E)$ a graph; k a natural number

Definition (Edge-colorability, Chromatic index)

G is k-edge-colorable if $\exists$ a proper k-edge-coloring of G;
The chromatic index (or edge-chromatic number) of G is the min k for which G is k-edge-colorable

Notation

$\chi'(G)$ = the chromatic index of G



$\chi'(G) = 5$

Edge coloring and line graphs

$G = (V, E)$ a graph

Observation

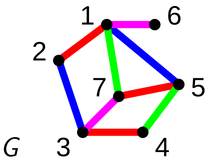
$\chi'(G) = k \iff \chi(L(G)) = k$

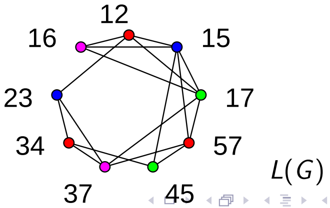
Definition (Line graph (recap))

The line graph of G is a graph $L(G)$ defined as

- $V(L(G)) = E(G)$
- $E(L(G)) = \{\{e, f\} \mid e, f \in E(G), e \cap f \neq \emptyset\}$

Example:





Proper edge-coloring and matchings

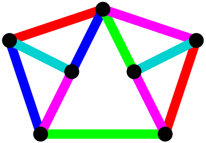
$G = (V, E)$ a graph

Definition (Matching (recap))

A set $M \subseteq E$ is a matching of G if no two edge of M are adjacent

Observation

c is a proper k-edge-coloring of G $\Rightarrow$ each color class is a matching of G



Easy lower and upper bounds

Proposition 7.3 (Easy lower and upper bounds for χ')

1. $\chi'(G) \geq \Delta(G)$
2. $\chi'(G) \leq 2\Delta(G) - 1$

Proof idea.

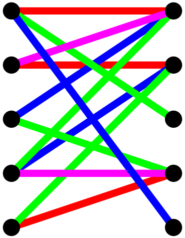
- $\omega(L(G)) \leq \chi(L(G)) \leq \Delta(L(G)) + 1$ (Props 6.1 & 6.3)
- $\chi(L(G)) = \chi'(G)$ (as observed before)
- $\omega(L(G)) \geq \Delta(G)$
- $\Delta(L(G)) \leq 2\Delta(G) - 2$ □

Chromatic index of a bipartite graph

When does $\chi'(G) = \Delta(G)$ hold?

Theorem 7.4 (Chromatic index of a bipartite graph; Kőnig '16)

G bipartite $\Longrightarrow \chi'(G) = \Delta(G)$

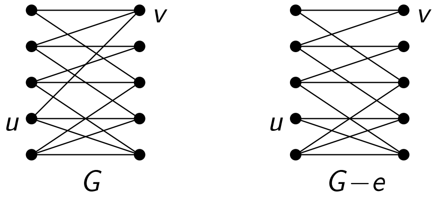


Proof of Kőnig's theorem

Prood idea.

Induction on $e(G)$; Easy if $e(G) = 0$; Assume $e(G) \geq 1$

- Let $e = \{u, v\} \in E$ and consider $G - e$ $\qquad (\Delta(G - e) \leq \Delta(G))$
- $G - e$ has a proper $\Delta(G)$-edge-coloring; Let $M_1, \ldots, M_{\Delta(G)}$ be the color classes (each being a matching)
- Let $C_u = \{i \mid \text{ no edge of } M_i \text{ incid. to } u\}$;
  Let $C_v = \{i \mid \text{ no edge of } M_i \text{ incid. to } v\}$
- $|C_u|, |C_v| \geq 1$ $\qquad (\because d_{G-e}(u), d_{G-e}(v) \leq \Delta(G) - 1)$



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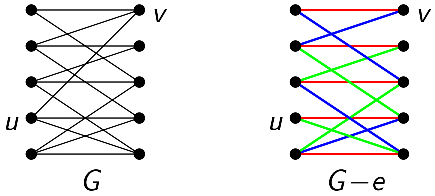

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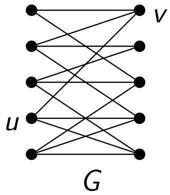
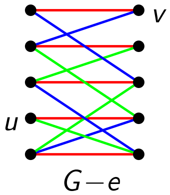


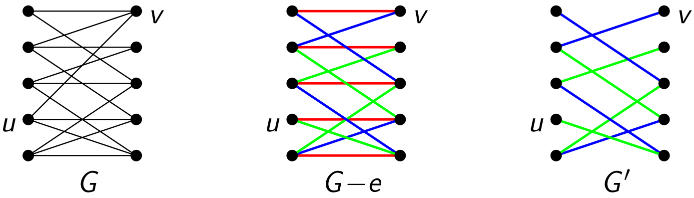


$C_u = \{\text{blue}\}$, $C_v = \{\text{green}\}$

Proof of Kőnig's theorem (continued)

Prood idea (continued).

- If $C_u \cap C_v \neq \emptyset$, color e by $i \in C_u \cap C_v$
- If not, consider $i \in C_u$ and $j \in C_v$; Let $G' = (V, M_i \cup M_j)$
- Each component of G' is a path or a cycle
- v belongs to some path P of G'; u does not belong to P (why?)
- To obtain a proper edge-coloring of G, we alter the colors on P, and color e by i □



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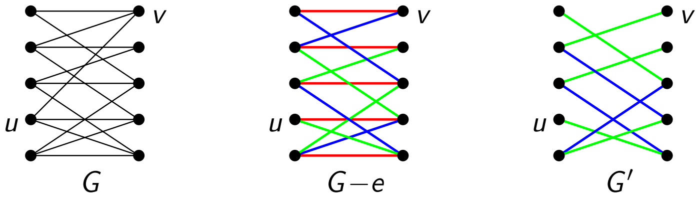

Proof of Kőnig's theorem (continued)

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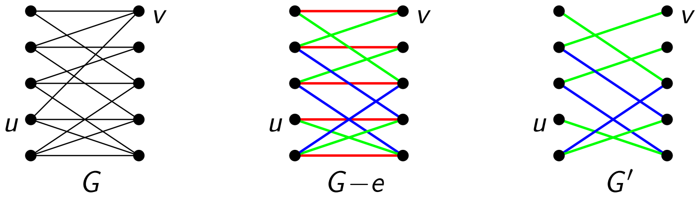


Only two values are possible: Vizing's theorem

We saw $\Delta(G) \leq \chi'(G) \leq 2\Delta(G) - 1$, but we can improve the upper bound

Vizing's theorem ('64)

For every graph G, $\chi'(G) \leq \Delta(G) + 1$

Consequence

$\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$ for every graph G

We skip the proof in the lecture

Deciding the k-edge-colorability

Problem k-Edge-Colorability

Pre-input: An integer k
Input: A graph G
Question: Is G k-edge-colorable?

Facts

- The problem 2-Edge-Colorability can be solved in polynomial time (Easy)
- The problem 3-Edge-Colorability is NP-complete (Holyer '81)

Remark: A proof of Vizing's theorem will give a poly-time algorithm to edge-color G with $\Delta(G)$ colors

Today's contents

- Mycielski's construction
- Edge coloring
- Other concepts and open problems
  - Hajós conjecture and Hadwiger conjecture
  - Total coloring conjecture
  - List coloring conjecture

Forced substructures in graphs with high chromatic numbers

We saw

- G contains $K_r \Longrightarrow \chi(G) \geq r$
- $\chi(G) \geq r \not\Longleftarrow G$ contains K_r

Question

Is there a class $\mathcal{C}$ of graphs such that

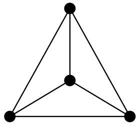
- $\chi(G) \geq r \Longrightarrow G$ contains a graph in $\mathcal{C}$

Subdivisions of a graph

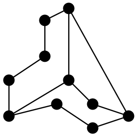
G, H graphs

Definition (Subdivision)

G is an *H-subdivision* if G can be constructed from H by replacing some edges by paths



K_4



a K_4-subdivision

Hajós' conjecture

Conjecture (Hajós '61)

$\chi(G) \geq r \implies G$ contains a K_r-subdivision

Facts

- $r = 2, 3$: True (Easy; Exercise)
- $r = 4$: True (Dirac '52)
- $r = 5, 6$: Still open
- $r \geq 7$: False (Catlin '79; Exercise)
- Almost all graphs: False (Erdős, Fajtlowicz '81)
- $g(G) \geq 186$: True (Kühn, Osthus '02)
- G a line graph: True (Thomassen '07)

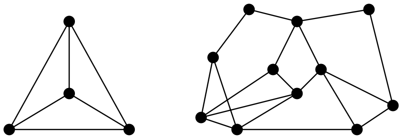
Minors

G, H graphs

Definition (Minor)

G contains an *H-minor* if G can be made isomorphic to H by successively contracting/deleting edges and deleting vertices

Note: G contains an H-subdivision $\Rightarrow$ G contains an H-minor



K_4 containing a K_4-minor

Minors

G, H graphs

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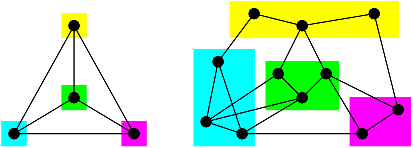

K_4 containing a K_4-minor

Hadwiger's conjecture

Conjecture (Hadwiger '43)

$\chi(G) \geq r \implies G$ contains a K_r-minor

Facts

- $r = 2, 3, 4$: True (from Hajós conj)
- $r = 5$: True (equiv. to Four Color Thm; Wagner '37)
- $r = 6$: True (Robertson, Seymour, Thomas '93)
- $r \geq 7$: Open

For $r = 7$, if $\chi(G) \geq r$ then G contains an H-minor, where...

- $H = K_7$ with two edges missing (Jakobsen '71)
- $H = K_7$ or $K_{4,4}$ (Kawarabayashi, Toft '05)
- $H = K_7$ or $K_{3,5}$ (Kawarabayashi)

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- Mycielski's construction
- Edge coloring
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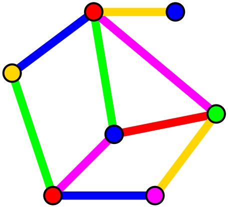
  - Hajós conjecture and Hadwiger conjecture
  - Total coloring conjecture
  - List coloring conjecture

Color the vertices and the edges at the same time

$G = (V, E)$ a graph, k a natural number

Definition (Total coloring)

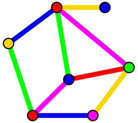
A *k-total-coloring* of G is a labeling $c\colon V \cup E \to \{1, \dots, k\}$;
A k-total-coloring is *proper* if $c|_V$ is a proper k-coloring, $c|_E$ is a proper k-edge-coloring and $c(v) \neq c(e)$ if $v \in V, e \in E, v \in e$



Total chromatic number

Definition (Total chromatic number)

The total chromatic number of G is the min k for which G has a proper k-total-coloring; denoted by $\chi''(G)$



Easy bounds

- $\chi''(G) \geq \chi(G)$, $\chi''(G) \geq \chi'(G)$ (By definitions)
- $\Delta(G) + 1 \leq \chi''(G) \leq 2\Delta(G)$ (Exercise)

Total coloring conjecture

Conjecture (Behzad '64; Vizing '68)

$\chi''(G) \leq \Delta(G) + 2$

Known facts

- $\chi''(G) \leq \Delta(G) + C$ for some const C (Molloy, Reed '98)
- $\chi''(G) \leq \Delta(G) + 3$ if the list coloring conjecture is true (Exercise)
- True for 3-regular graphs (Rosenfeld '71)
- True for interval graphs (Bojarshinov '01)

Deciding the k-total-colorability

Problem k-TOTAL-COLORABILITY

Pre-input: An integer k

Input: A graph G

Question: Does G have a proper k-total-coloring?

Facts

- The problem 3-TOTAL-COLORABILITY can be solved in polynomial time (Easy)
- The problem 4-TOTAL-COLORABILITY is NP-complete (Sanchez-Arroyo '89)

Today's contents

- Mycielski's construction
- Edge coloring
- Other concepts and open problems
  - Hajós conjecture and Hadwiger conjecture
  - Total coloring conjecture
  - List coloring conjecture

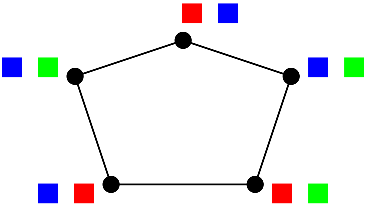
When sets of available colors are given

Setting

- $G = (V, E)$ a graph
- $L(v)$ a set of colors, $v \in V$

Definition (List coloring)

A list coloring of G with lists $\{L(v) \mid v \in V\}$ is a labeling $c \colon V \to \bigcup_v L(v)$ such that $c(v) \in L(v)$ for all $v \in V$;
A list coloring c is proper if $c(u) \neq c(v)$ for all $\{u, v\} \in E$



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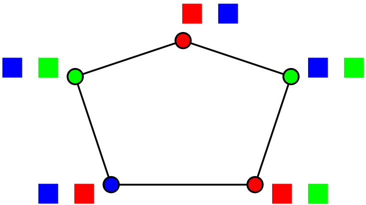
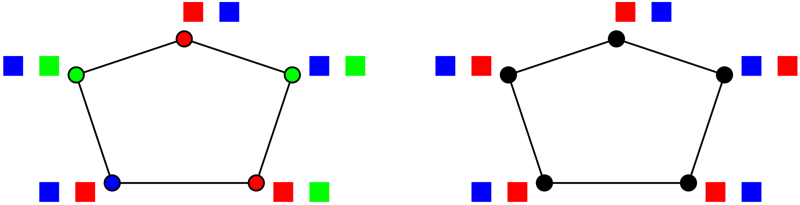


List colorability

$G = (V, E)$ a graph, k a natural number

Definition (List k-colorability)

G is list k-colorable (or k-choosable) if for *any* list $L(v)$ of colors with $|L(v)| = k$ there exists a proper list coloring of G with lists $\{L(v)\}$



C_5 is not list 2-colorable, but list 3-colorable

List chromatic number

$G = (V, E)$ a graph

Definition (List chromatic number)

The list chromatic number (or the choosability) of G is the min k such that G is list k-colorable

Notation

$\chi_\ell(G)$ = the list chromatic number of G

Easy bound

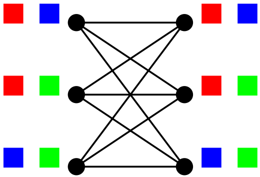
- $\chi_\ell(G) \geq \chi(G)$

Chromatic numbers and list chromatic numbers can be quite different

Facts

- $\chi(K_{n,n}) = 2$
- $\chi_\ell(K_{m,m}) \geq k$ when $m = \binom{2k-1}{k}$ (Erdős, Rubin, Taylor '79)

Example for $k = 2$



List edge-chromatic number

We may define the corresponding concepts for edge coloring:

- List edge-coloring
- List *k*-edge-colorability
- List chromatic index (List edge-chromatic number)

Notation

$\chi'_\ell(G)$ = the list chromatic index of G

Easy bound

- $\chi'_\ell(G) \geq \chi'(G)$

List coloring conjecture

Conjecture (List coloring conjecture)

For every graph $\chi'(G) = \chi'_\ell(G)$

Known facts

- True for G bipartite (Galvin '95)
- True asymptotically $(\chi'_\ell(G) \leq (1 + \mathrm{o}(1))\chi'(G))$ (Kahn '00)

Deciding the list-k-colorability

Problem LIST-k-COLORABILITY

Pre-input: An integer k
Input: A graph G
Question: Is G list-k-colorable?

Facts

- The problem LIST-2-COLORABILITY can be solved in poly-time (from Erdős, Rubin, Talyor '79)
- The problem LIST-3-COLORABILITY is Π_2^P-complete (Gutner, Tarsi '94)

For the class Π_2^P refer to lectures/textbooks on Computational Complexity