Topics on Computing and Mathematical Sciences I Graph Theory (6) Coloring I

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Today's contents

- Coloring, chromatic number
- Lower bounds, perfect graphs
- Upper bounds, greedy coloring

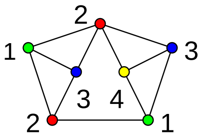
Coloring

$G = (V, E)$ a graph; k a natural number

Definition (Coloring)

A *k-coloring* of G is a map $c\colon V \to \{1, \dots, k\}$;
The vertices of one color form a color class;
A k-coloring of G is proper if $c(u) \neq c(v)$ for all $\{u, v\} \in E$

Each element of the range of a coloring is called a color

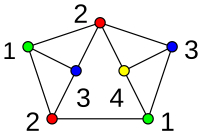


Colorability

$G = (V, E)$ a graph; k a natural number

Definition (Colorability)

G is k-colorable if $\exists$ a proper k-coloring of G



not 3-colorable
but 4-colorable

Note: G k-colorable $\Rightarrow$ G ℓ-colorable for all $\ell \geq k$

Chromatic numbers

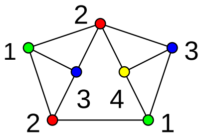
$G = (V, E)$ a graph

Definition (Chromatic number)

The chromatic number of G is the min k for which G is k-colorable

Notation

$\chi(G) =$ the chromatic number of G



$\chi(G) = 4$

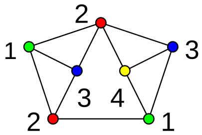
k-Chromatic graphs

$G = (V, E)$ a graph; k a natural number

Definition (k-Chromatic graph)

G is *k-chromatic* if $\chi(G) = k$

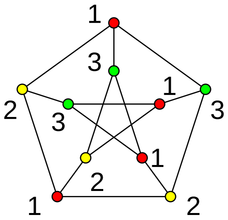
Remark: G k-colorable $\Leftrightarrow \chi(G) \leq k$



4-chromatic,
but not 3-chromatic,
not 5-chromatic

Chromatic numbers of some graphs

- $\chi(K_n) = ??$
- $\chi(K_{m,n}) = ??$
- $\chi(P_n) = ??$
- $\chi(C_n) = ??$
- $\chi(\text{Petersen}) = ??$



Remark

$H \subseteq G \Rightarrow \chi(H) \leq \chi(G)$

Color-critical graphs

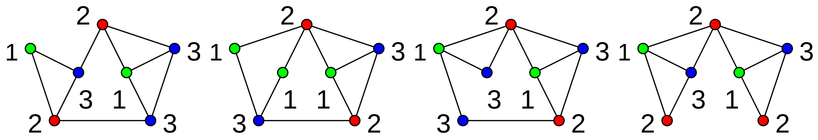
$G = (V, E)$ a graph; $\chi(G) = k$

Definition (Color-critical graph)

G is k-critical if $\chi(H) < \chi(G)$ for every proper subgraph H of G

Observation

- For G without isolated vertex:
  G k-critical $\Leftrightarrow$ $\chi(G-e) < \chi(G)$ for all $e \in E$
- G 2-critical $\Leftrightarrow$ $G \simeq K_2$
- G 3-critical $\Leftrightarrow$ G an odd cycle



Proper coloring and independent sets

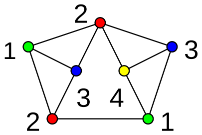
$G = (V, E)$ a graph

Definition (Independent set (recap))

A set $S \subseteq V$ is independent if no two vertices of S are adjacent

Observation

c is a proper k-coloring of $G \Rightarrow$ each color class is independent



Multi-partite graphs

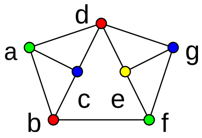
$G = (V, E)$ a graph; r a natural number

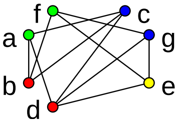
Definition (Multi-partite graph)

G is r-partite if $\exists$ a partition $V_1 \cup \cdots \cup V_r$ of V s.t. $\{u, v\} \in E \Rightarrow \{u, v\} \not\subseteq V_i$ for any i

Observation

G k-colorable $\Leftrightarrow$ G k-partite





Deciding k-colorability

Problem k-Colorability

Pre-input: A natural number k

Input: A graph G

Question: Is G k-colorable?

Facts

- $k \leq 2 \Rightarrow k$-Colorability is poly-time solvable
- $k \geq 3 \Rightarrow k$-Colorability is NP-complete (Karp '72)

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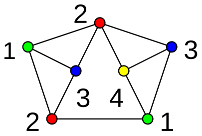
Easy lower bound (1)

Definition (clique, clique number (recap))

A set $S \subseteq V$ is a clique if every pair of vertices of S are adjacent;
$\omega(G) =$ the size of a largest clique of G

Proposition 6.1 (Easy lower bound for the chromatic number)

$\chi(G) \geq \omega(G)$ for every graph G



$$\chi(G) = 4$$
$$\omega(G) = 3$$

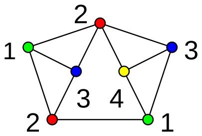
Easy lower bound (2)

Definition (independence number (recap))

$\alpha(G)$ = the size of a largest independent set of G

Proposition 6.2 (Easy lower bound for the chromatic number)

$\chi(G) \geq n(G)/\alpha(G)$ for every graph G

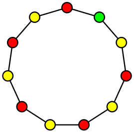


$$\chi(G) = 4$$
$$\alpha(G) = 2$$
$$n(G) = 7$$

Is the lower bounds tight?

Consider an odd cycle C_{2k+1} of length at least 5

- $n(C_{2k+1}) = 2k+1$
- $\omega(C_{2k+1}) = 2$
- $\alpha(C_{2k+1}) = k$
- $\chi(C_{2k+1}) = 3$



We will see the bound $\chi(G) \geq \omega(G)$ can be arbitrarily bad (in the next lecture)

Lesson

Difficulty of optimization problems lies in certifying the optimality; Efficient algorithms require good lower bounds (for minimization problems)

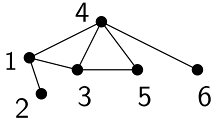
When is it tight?

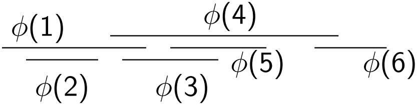
Graphs G with $\chi(G) = \omega(G)$

- Complete graphs, bipartite graphs, interval graphs, ...

Definition (Interval graph)

G is an interval graph if $\exists$ a set $\mathcal{I}$ of (closed) intervals and a bijection $\phi\colon V(G) \to \mathcal{I}$ s.t. u, v adjacent iff $\phi(u) \cap \phi(v) \neq \emptyset$





Will prove later: G an interval graph $\Rightarrow \chi(G) = \omega(G)$

Perfect graphs

Definition (Perfect graph)

G is perfect if $\chi(H) = \omega(H)$ for all *induced* subgraphs H of G

Weak Perfect Graph Theorem (Lovász '72)

G is perfect $\iff$ $\overline{G}$ is perfect

Strong Perfect Graph Theorem (Chudnovsky, Robertson, Seymour, Thomas '06)

G is perfect $\iff$ G contains no induced subgraph iso. to an odd cycle of length at least 5 or its complement

Both conjectured by Berge ('61)

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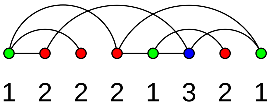
Greedy coloring

To give an upper bound for $\chi(G)$, we construct a proper coloring

Definition (Greedy coloring)

Fix a linear order $\prec$ on $V(G)$;
The greedy coloring of G with respect to $\prec$ is the following coloring $c\colon V(G) \to \{1, 2, \ldots\}$

$$c(v) = \begin{cases} 1 & v \text{ the min w.r.t. } \prec \\ \min(\{1,2,\ldots\} \setminus \{c(u) \mid u \prec v, \{u,v\} \in E\}) & \text{otherwise} \end{cases}$$



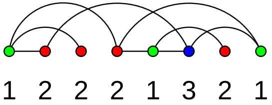
An easy upper bound

Proposition 6.3 (Greedy coloring)

$\chi(G) \leq \Delta(G) + 1$ for every graph G

Proof idea.

- Look at an arbitrary vertex v
- $\#$ vertices preceding v wrt $\prec \leq \Delta$
- $\therefore$ at least one color in $\{1, \ldots, \Delta{+}1\}$ is available for v $\square$



Degenerate graphs

Definition (Degenerate graph)

A graph G is *d-degenerate* if $\delta(H) \leq d$ for every subgraph H of G

Proposition 6.4 (chromatic number of degenerate graph)

G d-degenerate $\Longrightarrow \chi(G) \leq d+1$

Prood idea.

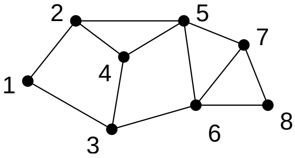
Consider the following linear order $\prec$ on $V(G)$

- The largest vertex wrt $\prec$ is one with min degree, denote it by v; Then, consider $G-v$ and the second largest vertex w.r.t. $\prec$ is one with min degree in $G-v$, denote it by v'; Then, consider $G-\{v, v'\}$, ...

On this order $\prec$, try the greedy coloring $\square$

Example of a proposed linear order

In other words, the order $\prec$ satisfies: $u \prec v$ if $d_H(u) \geq d_H(v)$ where H is the subgraph induced by $V(G) \setminus \{w \mid v \prec w\}$



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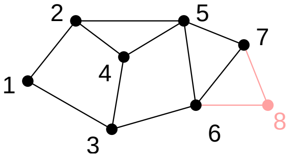

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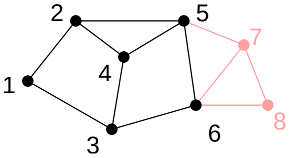

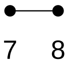

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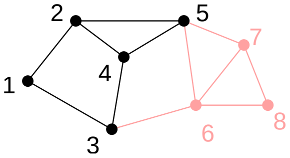

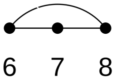

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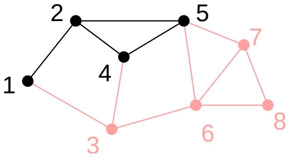

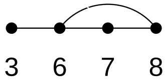

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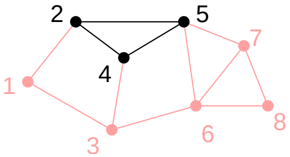

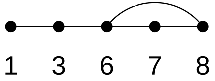

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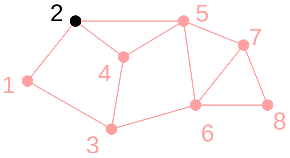

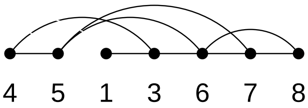

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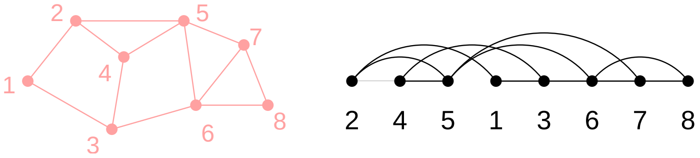

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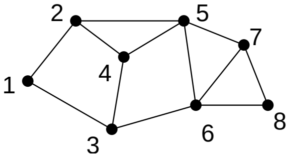
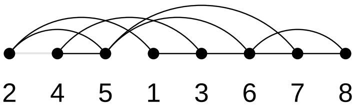




Non-regular graphs

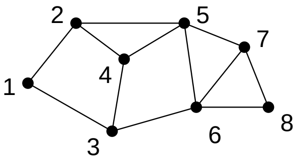
If a graph G is not regular, we can improve $\Delta(G)+1$ a bit

Proposition 6.5 (An improvement of an easy upper bound)

No component of G is regular $\Longrightarrow \chi(G) \le \Delta(G)$

Proof idea.

- WLOG, G is connected
- G contains a spanning tree T (Prop. 3.6)
- v a min degree vertex of G (Rem: $\delta(G) < \Delta(G)$)



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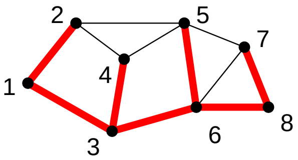
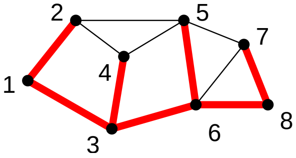
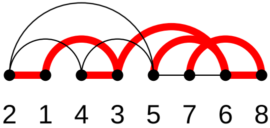


Non-regular graphs (cont'd)

Proof idea (continued).

- Define a linear order $\prec$ on V as the reverse order of the length of a unique path to v in T
- Property 1: $\forall\ u \in V \setminus \{v\}\ \exists\ w \in V$ s.t. $u \prec w$ and $\{u, w\} \in E(T)\ (\subseteq E)$
- Property 2: $d(v) \leq \Delta(G) - 1$
- These properties lead to our upper bound $\square$





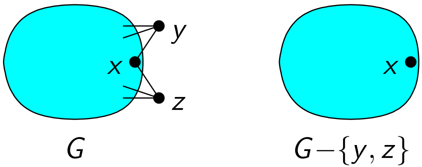
What about regular graphs?: Brooks' theorem

Theorem 6.6 (Brooks '41)

No component of G is complete or an odd cycle $\Rightarrow \chi(G) \leq \Delta(G)$

Proof idea.

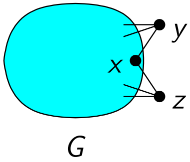
- WLOG $G = (V, E)$ is 2-vertex-connected
- WLOG G is k-regular, $k \geq 3$
- Use Exer 5.3
  - G non-complete 2-vtx-connected k-reg ($k \geq 3$) $\Rightarrow \exists\, x, y, z \in V$ s.t. $\{x, y\}, \{x, z\} \in E$, $\{y, z\} \notin E$ and $G - \{y, z\}$ connected

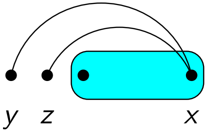


Brooks' theorem (cont'd)

Proof idea (continued).

- A linear order $\prec$:
  - The smallest two are y and z
  - Order the vertices of $G - \{y, z\}$ from a spanning tree (as before)
  - But this time, consider paths to x (so x is the largest in $\prec$)
- Greedy coloring wrt $\prec$ gives the desired bound





Chromatic number of an interval graph

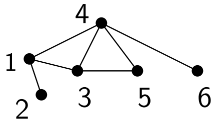
Theorem 6.7 (Chromatic number of an interval graph)

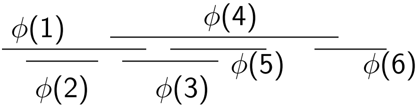
G an interval graph $\Rightarrow \chi(G) = \omega(G)$

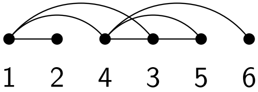
Proof idea.

$\phi\colon V(G) \to \mathcal{I}$ a bijection to a set of intervals

- A linear order $\prec$: An ascending order of the left endpoints of the corresponding intervals (tie is broken arbitrarily)



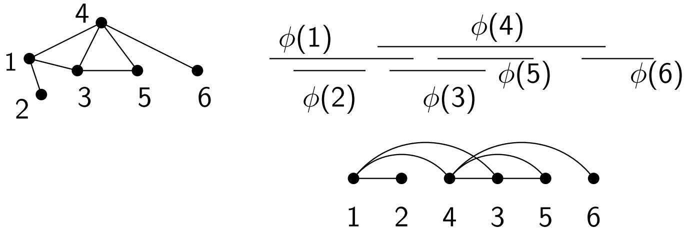




Chromatic number of an interval graph (continued)

Proof idea (cont'd).

- Consider a vertex v with the largest color k
- $\phi(v)$ intersects k other intervals at the left endpoint of $\phi(v)$; They form a clique
- $\omega(G) \geq k = \chi(G)$ $\square$



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- Open problems

Coping with NP-completeness

Fact

Computing the chromatic number of a graph is NP-hard (Karp '72);
Hence, no polynomial-time algorithm is unlikely to exist

Question

Then, what can we do for this problem?

We need to compromise somehow

What to compromise (three-dimensional view)

Basic ways to cope with NP-hardness

- Restriction Approach
  Require our algorithm to output the chromatic number in poly-time for a restricted class of graphs
- Exact Approach
  Require our algorithm to output the chromatic number for every graph, but not necessarily in poly-time
- Approximate Approach
  Require our algorithm to output a value close to the chromatic number for every graph in poly-time

Chromatic number: Restriction approach

Fact

We can compute the chromatic number of a perfect graph in polynomial time (Grötschel, Lovász, Schrijver '84)

- A milestone in the history of combinatorial optimization
- Use of semidefinite programming and the ellipsoid method (from continuous optimization)
- Note: The ellipsoid method is inefficient

Open problem

Design a practical algorithm (that does not rely on techniques in continuous optimization too) to compute the chromatic number of a perfect graph

Chromatic number: Exact approach

Fact

We can compute the chromatic number of a graph in $\mathrm{O}(2^n \mathrm{poly}(n))$ time (Björklund, Husfeldt, Koivisto '06)

- This is one of the milestones in the research of exponential-time exact algorithms
- The algorithm is based on a simple principle "inclusion-exclusion"

Open problem

Design a faster algorithm to compute the chromatic number; For example, can we do it in $\mathrm{O}(1.7^n \mathrm{poly}(n))$?

Chromatic number: Approximate approach

Definition (Approximation factor)

An algorithm for the chromatic # problem is an r-approximation if it always outputs a value at most r times the chromatic # of the input; $r \geq 1$ is called an approximation ratio

Facts for the chromatic number approximation

- $\exists$ a poly-time alg w/ apx ratio $\mathrm{O}(n(\log\log n)^2/\log^3 n)$ (Halldórsson '93)
- no poly-time alg w/ apx ratio
  - n^{1-c} for *some* const $c > 0$ if $\mathrm{P} \neq \mathrm{NP}$ (Lund, Yannakakis '94)
  - $n^{1-\varepsilon}$ for *any* const $\varepsilon > 0$ if $\mathrm{NP} \neq \mathrm{ZPP}$ (Feige, Kilian '98)
  - $n^{1-\mathrm{O}((\log\log n)^{-1/2})}$ if $\mathrm{NP} \not\subseteq \mathrm{ZPTIME}(2^{\mathrm{O}(\log n(\log\log n)^{3/2})})$ (Engebretsen, Holmerin '03)

Remark: $\mathrm{O}(n(\log\log n)^2/\log^3 n) = n^{1-\mathrm{O}(\log\log n/\log n)}$

Reed's ω-Δ-χ conjecture

Conjecture (ω-Δ-χ conjecture; Reed '98)

$$\chi(G) \leq \left\lceil \frac{1}{2}(\omega(G) + \Delta(G) + 1) \right\rceil \text{ for every graph } G$$

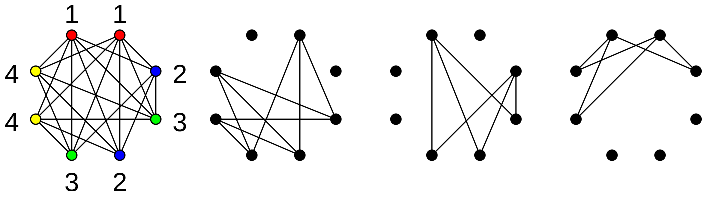
Status

- True when $\omega(G) \geq \Delta(G)$ (easy)
- True when $\omega(G) = \Delta(G) - 1$ (Brooks)
- True when $\Delta(G) = n(G) - 1$ (Reed '98)
- Asympototically true (Reed '98)
  - $\exists\ \Delta_0\ \forall\ \Delta \geq \Delta_0\ \exists\ \varepsilon < 1\ \exists\ \omega \geq (1-\varepsilon)\Delta$: $\omega(G) \leq \omega \Rightarrow$ $\chi(G) \leq \left\lceil \frac{1}{2}(\omega(G) + \Delta(G) + 1) \right\rceil$
- True for line graphs (King, Reed, Vetta '07)

Alon-Saks-Seymour conjecture

Conjecture (Alon, Saks, Seymour '94)

G can be decomposed into k complete bipartite graphs $\Rightarrow$ $\chi(G) \leq k+1$



Status

- True when G complete (Graham, Pollak '72)
  - using linear algebra (the spectral method)