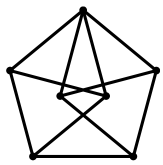
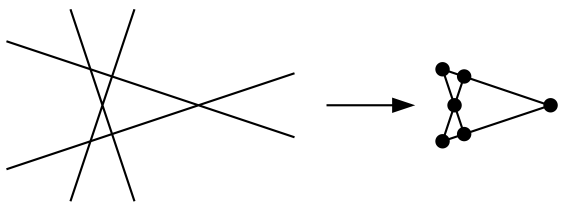
Due Date: June 4, 2008
Legend: (−) easy; (+) hard
Warning: Your solutions have to be substantiated, namely, you have to provide proofs for the given answers.

Exercise 6.1 (−) Determine the clique number, the independence number and the chromatic number of the graph below. Is the graph color-critical?



Exercise 6.2 Let G be a graph whose odd cycles are pairwise intersecting, meaning that every two odd cycles in G have a common vertex. Prove that $\chi(G) \leq 5$.

Exercise 6.3 Given a set of line in the plane with no three meeting at a point, form a graph G whose vertices are the intersections of the lines, with two vertices adjacent if they appear consecutively on one of the lines. Prove that $\chi(G) \leq 3$. (Comment: The conclusion may fail when three lines are allowed to share a point.)



Exercise 6.4 Prove that the complement of a bipartite graph is perfect. (This implies the weak perfect graph conjecture for bipartite graphs.)

Exercise 6.5

1. (−) Construct a graph G that is neither a complete graph nor an odd cycle but has a vertex ordering relative to which greedy coloring uses $\Delta(G)+1$ colors.
2. Prove that every graph G has a vertex ordering relative to which greedy coloring uses $\chi(G)$ colors.
3. For all natural numbers k, construct a tree T_k with maximum degree k and an ordering σ of $V(T_k)$ such that greedy coloring relative to the ordering σ uses $k+1$ colors. (Hint: Use induction and construct the tree and ordering simultaneously.)
4. Let G be a graph having no induced subgraph isomorphic to P_4. Prove that for every vertex ordering, greedy coloring produces an optimal coloring of G.

Exercise 6.6

1. Prove that $\chi(G) \cdot \chi(\overline{G}) \geq n(G)$, use this to prove that $\chi(G) + \chi(\overline{G}) \geq 2\sqrt{n(G)}$, and provide a construction achieving these bounds whenever $\sqrt{n(G)}$ is an integer.
2. Prove that $\chi(G) + \chi(\overline{G}) \leq n(G) + 1$. (Hint: Use induction on $n(G)$.)