Topics on
Computing and Mathematical Sciences I
Graph Theory
(5) Connectivity

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Today's contents

- Vertex connectivity and edge connectivity
- Local vertex connectivity and local edge connectivity
- Contraction of an edge
- Menger's theorem

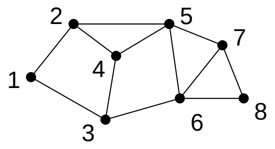
Vertex cuts

$G = (V, E)$ a graph

Definition (Vertex cut)

$S \subseteq V$ is a vertex cut (or a separating set) of G if $G-S$ is disconnected

Example



$\{2, 3, 4\}$ is a vertex cut
$\{6, 7\}$ is a vertex cut
$\{4, 6, 8\}$ is not a vertex cut

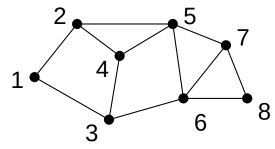
k-Vertex-connectedness

$G = (V, E)$ a graph; k a natural number

Definition (k-Vertex connectedness)

G is k-vertex-connected (or k-connected) if $n(G) > k$ and there exists *no* vertex cut of size *less than* k

Example



This graph is 2-vertex-connected
This graph is not 3-vertex-connected

Notice

G k-vertex-connected $\Rightarrow n(G) \geq k+1$

Vertex connectivity

$G = (V, E)$ a graph

Definition (Vertex connectivity)

The vertex connectivity of G is the maximum k such that G is k-vertex-connected; Denoted by $\kappa(G)$

Example:

- $\kappa(K_n) = ?$
- $\kappa(K_{m,n}) = ?$
- $\kappa(P_n) = ?$
- $\kappa(C_n) = ?$

Notice

G k-vertex-connected $\Leftrightarrow \kappa(G) \geq k$

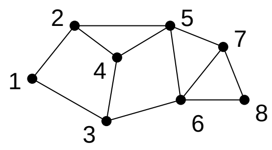
Disconnecting sets

$G = (V, E)$ a graph

Definition (Disconnecting set)

$D \subseteq E$ is a disconnecting set of G if $G - D$ is disconnected

Example



$\{\{1,3\},\{2,4\},\{2,5\}\}$ is a disconnecting set
$\{\{5,6\},\{5,7\}\}$ is not a disconnecting set
$\{\{6,8\},\{7,8\}\}$ is a disconnecting set

Edge cut

$G = (V, E)$ a graph

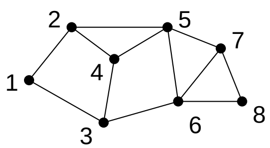
Definition (Edge cut)

$D \subseteq E$ is an edge cut of G if $\exists\, A \subseteq V$ s.t. $D = \{\{u, v\} \in E \mid u \in A, v \in V \setminus A\}$; Then D is denoted by $[A, \overline{A}]$

Remark:

- D an edge cut $\substack{\Longrightarrow \\ \not\Longleftarrow}$ D a disconnecting set
- D an edge cut $\Longleftarrow$ D a minimal disconnecting set

Example



$\{\{1,3\},\{2,4\},\{2,5\}\}$ is a disconnecting set, and also an edge cut $[\{1,2\},\{3,4,5,6,7,8\}]$; $\{\{6,7\},\{6,8\},\{7,8\}\}$ is a disconnecting set, but not an edge cut

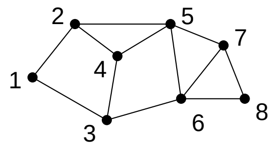
k-Edge-connectedness

$G = (V, E)$ a graph

Definition (k-Edge-connectedness)

G is k-edge-connected if $e(G) \geq k$ and there exists *no* disconnecting set of size *less than* k

Example



This graph is 2-edge-connected
This graph is not 3-edge-connected

Edge connectivity

$G = (V, E)$ a graph

Definition (Edge connectivity)

The edge connectivity of G is the maximum k such that G is k-edge-connected; Denoted by $\kappa'(G)$

Example:

- $\kappa'(K_n) = ?$
- $\kappa'(K_{m,n}) = ?$
- $\kappa'(P_n) = ?$
- $\kappa'(C_n) = ?$

Notice

G k-edge-connected $\Leftrightarrow \kappa'(G) \geq k$

Relationship

Proposition 5.1 (Whitney '32)

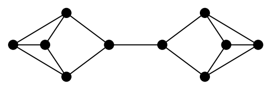
$n(G) \geq 2 \Longrightarrow \kappa(G) \leq \kappa'(G) \leq \delta(G)$

Exercise: The gaps can be arbitrarily large

Proof idea.

$[\kappa'(G) \leq \delta(G)]$

- The edges incid to a vertex of min deg form an edge cut of G

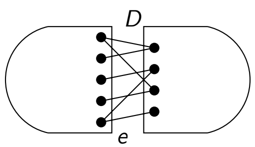


Proof of Whitney '32 (continued)

Proof idea (continued).

$[\kappa(G) \leq \kappa'(G)]$

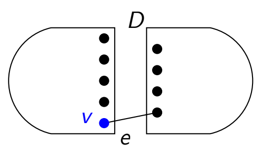
- D a smallest disconnecting set of G (i.e., $|D| = \kappa'(G)$)
- Choose an arbitrary edge $e \in D$; Consider $G_e = G - (D \setminus \{e\})$
- e is a cut-edge of G_e
- For each edge $f \in D \setminus \{e\}$, choose an endpoint $v_f \notin e$ of f
- Let $S = \{v_f \mid f \in D \setminus \{e\}\}$

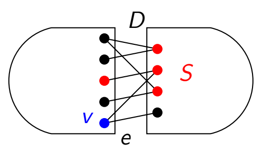


Proof of Whitney '32 (further continued)

Proof idea (further continued).

- Case 1: $G - S$ is disconnected: $\kappa'(G) = |D| > |S| \geq \kappa(G)$
- Case 2-1: $G - S$ is connected and has only one edge
  - $\kappa'(G) = |D| \geq |S| + 1 = n(G) - 1 \geq \kappa(G)$
- Case 2-2: $G - S$ is connected and has more than one edge
  - An endpoint v of e is a cut-vertex of $G - S$
  - $S \cup \{v\}$ is a vertex cut of G
  - $\kappa'(G) = |D| \geq |S \cup \{v\}| \geq \kappa(G)$ □





Computing the vertex connectivity and the edge connectivity

State of the art

Computation of $\kappa(G)$

- $\mathrm{O}((\kappa^{5/2}+n)\kappa n)$ (Gabow '00)
- $\mathrm{O}((\kappa+n^{1/4})\kappa n^{7/4})$ (Gabow '00)

Computation of $\kappa'(G)$

- $\mathrm{O}(m+\kappa' n \log(n/\kappa'))$ (Gabow '95)
- $\mathrm{O}(m \log^3 n)$ randomized (Karger '00)

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- Contraction of an edge
- Menger's theorem

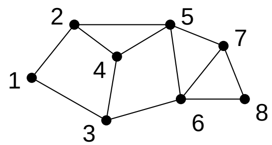
Vertex cut

$G=(V,E)$ a graph; $u,v \in V$ two distinct vertices

Definition (Vertex cut)

$S \subseteq V \setminus \{u,v\}$ is a u,v-vertex cut (or a u,v-separating set) of G if $G-S$ has no u,v-walk (or u,v-path)
In this case, we also say *S separates u and v*

Example



$\{3,5\}$ is a $1,8$-vertex cut
$\{3,5\}$ is not a $1,4$-vertex cut
$\{2,3\}$ is a $1,4$-vertex cut
$\{2,4,6\}$ is a $3,5$-vertex cut

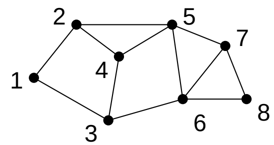
Local vertex connectivity

$G=(V,E)$ a graph; $u,v \in V$ two distinct *non-adjacent* vertices

Definition (Local vertex connectivity)

The local vertex connectivity of G between u,v is the minimum size of a u,v-vertex cut of G;
Denoted by $\kappa_G(u,v)$ (or $\kappa(u,v)$ when G is clear from the context)

Example



$\kappa(\{1,8\})=2$
$\kappa(\{1,4\})=2$
$\kappa(\{3,5\})=3$

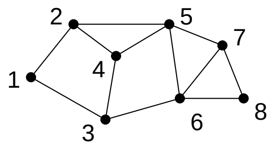
Disconnecting set

$G = (V, E)$ a graph; $u, v \in V$ two distinct vertices

Definition (Disconnecting set)

$D \subseteq E$ is a u, v-disconnecting set of G if $G - D$ has no u, v-walk (or u, v-path)

Example



$\{\{1, 2\}, \{1, 3\}\}$ is a $1, 8$-disconnecting set
$\{\{1, 2\}, \{3, 4\}, \{3, 6\}\}$ is a $3, 5$-disconnecting set

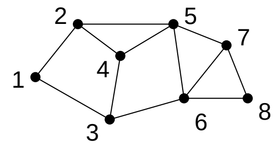
Local edge connectivity

$G = (V, E)$ a graph; $u, v \in V$ two distinct vertices (not necessarily non-adjacent)

Definition (Local edge connectivity)

The local edge connectivity of G between u, v is the minimum size of a u, v-disconnecting set of G;
Denoted by $\kappa'_G(u, v)$ (or $\kappa(u, v)$ when G is clear from the context)

Example



$\kappa'(\{1, 8\}) = 2$
$\kappa'(\{1, 4\}) = 2$
$\kappa'(\{3, 5\}) = 3$

Global vs local connectivities

$G = (V, E)$ a graph

Proposition 5.2 (Global vs local vertex connectivities)

$\kappa(G) = \min\{\kappa_G(u, v) \mid u, v \in V, \{u, v\} \notin E\}$
when G is not complete

Proposition 5.3 (Global vs local edge connectivities)

$\kappa'(G) = \min\{\kappa'_G(u, v) \mid u, v \in V\}$

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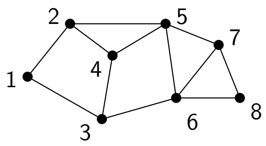
Contraction of an edge

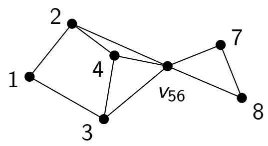
$G = (V, E)$ a graph; $e\{x, y\} \in E$ an edge

Definition (Contraction)

The contraction of e is the following graph denoted by G/e;

- $V(G/e) = (V \setminus \{x, y\}) \cup \{v_{xy}\}$ where $v_{xy} \notin V$
- $E(G/e) = (E \setminus \{f \in E \mid f \text{ incident to } x \text{ or } y\})$
  $\cup \{\{u, v_{xy}\} \mid u \text{ adj to } x \text{ or } y \text{ in } G, u \notin \{x, y\}\}$





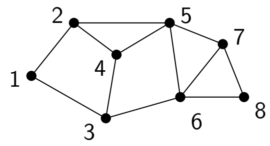
Contraction of a connected vertex subset

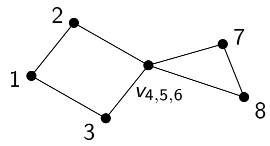
$G = (V, E)$ a graph; $S \subseteq V$ a subset s.t. $G[S]$ is connected

Definition (Contraction)

The contraction of S is the following graph denoted by G/S;

- $V(G/S) = (V \setminus S) \cup \{v_S\}$ where $v_S \notin V$
- $E(G/S) = (E \setminus \{f \in E \mid f \text{ incident to a vertex in } S\})$
  $\cup \{\{u, v_S\} \mid u \text{ adj to a vtx of } S \text{ in } G, u \notin S\}$





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WANTED: A good characterization for k-connectedness

How can we certify a graph is not k-vertex-connected

Enough to exhibit a vertex-cut of size $k-1$

How can we certify a graph is k-vertex-connected

Going through all vertex subsets of size $k-1$;
This is too inefficient

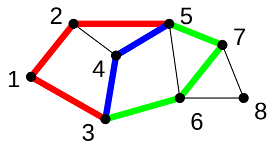
We need a good characterization!!

Internally disjoint paths

$G = (V, E)$ a graph; $u, v \in V$ two distinct vertices

Definition (Internally disjoint path)

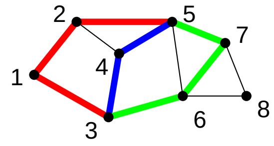
Two u, v-paths P and Q are internally disjoint if
$V(P) \cap V(Q) = \{u, v\}$,
namely P and Q do not share any vertex other than u and v;
Some u, v-paths $P_1, \dots, P_k$ are pairwise internally disjoint if
for any i, j, $i \neq j$, P_i and P_j are internally disjoint



Weak duality

Definition ($\lambda_G(u, v)$)

$\lambda_G(u, v) = \max\{k \mid \exists\ k \text{ pairwise internally disjoint } u, v\text{-paths in } G\}$



$\lambda_G(3, 5) = 3$

Lemma 5.4 (Weak duality)

$G = (V, E)$ a graph; $u, v \in V$ two distinct non-adjacent vertices
$\implies \lambda_G(u, v) \leq \kappa_G(u, v)$

Proof idea.

Double counting □

Menger's theorem: Strong duality

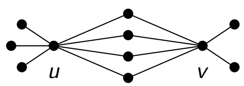
Theorem 5.5 (Menger's theorem; Menger '27)

$G = (V, E)$ a graph; $u, v \in V$ two distinct non-adjacent vertices
$\implies \lambda_G(u, v) = \kappa_G(u, v)$

Enough to prove $\lambda_G(u, v) \geq \kappa_G(u, v)$

Proof idea.

Induction on $e(G)$
If $e(G) = 0$, then $\lambda_G(u, v) = 0 = \kappa_G(u, v)$
Otherwise, two cases
Case 1: Every edge of G is incident to u or v $\rightsquigarrow$ easy

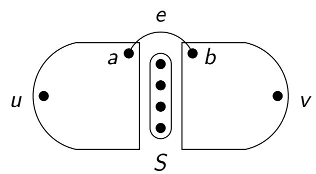


Proof of Menger's theorem (2)

Proof idea (continued)

Case 2: $\exists$ an edge $e = \{a, b\}$ incid. to neither u nor v

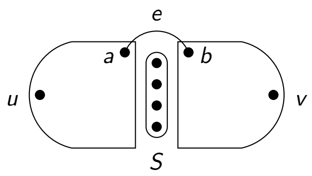
- Consider $G - e$
- $\lambda_G(u, v) \geq \lambda_{G-e}(u, v)$ (since $G - e \subseteq G$)
- $\lambda_{G-e}(u, v) = \kappa_{G-e}(u, v)$ (induction)
- Let $S \subseteq V \setminus \{u, v\}$ a minimum u, v-vertex cut of $G - e$



Proof of Menger's theorem (3)

Proof idea (continued)

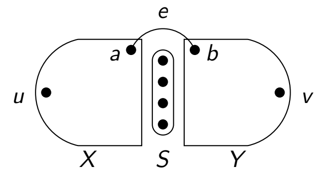
- $S \cup \{a\}$ is a u, v-vertex cut of G
- $\therefore\ \kappa_G(u, v) \le |S \cup \{a\}| \le |S|+1 = \kappa_{G-e}(u, v)+1$
- $\therefore\ \lambda_G(u, v) \ge \kappa_{G-e}(u, v) \ge \kappa_G(u, v)-1$
- If $\lambda_G(u, v) \ge \kappa_G(u, v)$ we're done;
  So assume $\lambda_G(u, v) = \kappa_{G-e}(u, v) = \kappa_G(u, v)-1$



Proof of Menger's theorem (4)

Proof idea (continued)

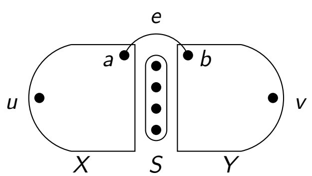
- Let $X = \{x \in V \mid \exists$ a u, x-path in $(G-e)-S\}$
- Let $Y = \{y \in V \mid \exists$ a y, v-path in $(G-e)-S\}$
- Situation: $|S| = \kappa_G(u, v)-1$; $a \in X$, $b \in Y$

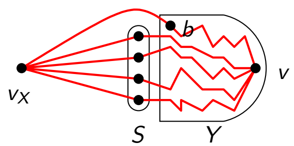


Proof of Menger's theorem (5)

Proof idea (continued)

- Consider G/X
  - $\kappa_{G/X}(v_X, v) \ge \kappa_G(u, v)$ (why?)
  - $\kappa_{G/X}(v_X, v) \le \kappa_G(u, v)$ ($\because$ $S \cup \{b\}$ is a v_X, v-vtx cut of G/X)
  - $\lambda_{G/X}(v_X, v) = \kappa_{G/X}(v_X, v) = \kappa_G(u, v)$ (induction)
  - Each of $\kappa_G(u, v)$ paths goes through one of $S \cup \{b\}$

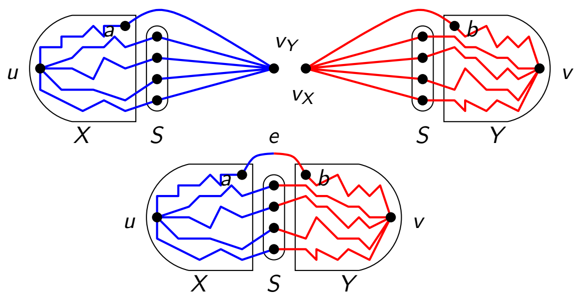




Proof of Menger's theorem (6)

Proof idea (continued)

- Same for G/Y
  - $\lambda_{G/Y}(u, v_Y) = \kappa_{G/Y}(u, v_Y) = \kappa_G(u, v)$ (induction)
  - Each of $\kappa_G(u, v)$ paths goes through one of $S \cup \{a\}$
- Combining these paths gives $\kappa_G(u, v)$ intern. disj. paths in G □

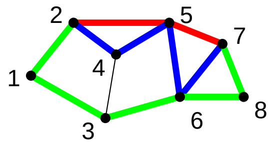


How about the edge connectivity?

$G = (V, E)$ a graph; $u, v \in V$ two distinct vertices

Definition (Edge-disjoint path)

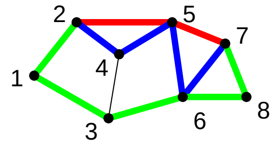
Two u, v-paths P and Q are edge-disjoint if $E(P) \cap E(Q) = \emptyset$, namely P and Q do not share any edge;
Some u, v-paths $P_1, \dots, P_k$ are pairwise edge-disjoint if for any i, j, $i \neq j$, P_i and P_j are edge disjoint



Weak duality: Edge version

Definition ($\lambda'_G(u, v)$)

$\lambda'_G(u, v) = \max\{k \mid \exists\ k$ pairwise edge disjoint u, v-paths in $G\}$



$\lambda'_G(2, 7) = 3$

Lemma 5.6 (Weak duality)

$G = (V, E)$ a graph; $u, v \in V$ two distinct vertices
$\Longrightarrow \lambda'_G(u, v) \leq \kappa'_G(u, v)$

Proof idea.

Double counting □

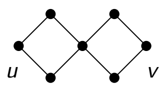
Menger's theorem: Edge version

Theorem 5.7 (Menger's theorem; Menger '27)

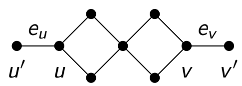
$G = (V, E)$ a graph; $u, v \in V$ two distinct vertices
$\Longrightarrow \lambda'_G(u, v) = \kappa'_G(u, v)$

Proof idea.

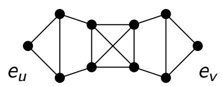
- Form G' from G by adding two vertices u', v' and two edges $e_u = \{u, u'\}, e_v = \{v, v'\}$
- Consider the line graph $L(G')$



G



G'



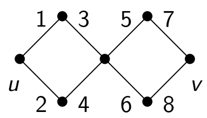
$L(G')$

Proof of Menger's theorem (edge version), continued

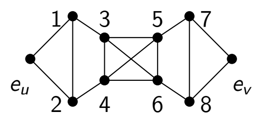
Proof idea (continued).

G		$L(G')$
edge-disj. u, v-paths	$\longrightarrow$	intern. disj. e_u, e_v-paths
a u, v-disconnecting set	$\longleftrightarrow$	an e_u, e_v-vtx cut

- $\therefore \kappa'_G(u, v) = \kappa_{L(G)}(e_u, e_v) = \lambda_{L(G)}(u, v) \leq \lambda'_G(u, v)$ □



G



$L(G')$

Global Menger

Theorem 5.8 (Global Vertex Menger; Whitney '32)

G is k-vertex-connected $\Leftrightarrow$ $\exists$ k internally disjoint paths between any two distinct vertices of G

Proof is in the next slide

Theorem 5.9 (Global Edge Menger; Whitney '32)

G is k-edge-connected $\Leftrightarrow$ $\exists$ k edge-disjoint paths between any two distinct vertices of G

Proof.

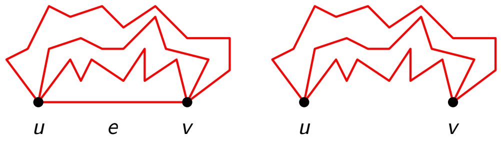
Directly from Edge Menger (Thm 5.7) and Prop 5.3 □

Proof of Global Vertex Menger

Prood idea.

- $\kappa(G) = \min\{\kappa_G(u,v) \mid u,v \in V, \{u,v\} \notin E\}$ (Prop 5.2)
- $\therefore \kappa(G) = \min\{\lambda_G(u,v) \mid u,v \in V, \{u,v\} \notin E\}$ (by Thm 5.6)
- Enough to show: $\lambda_G(u,v) \geq \kappa(G)$ when $\{u,v\} \in E$

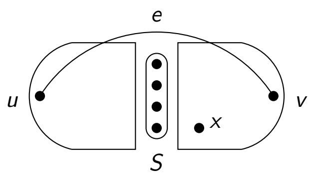
- Suppose $\lambda_G(u,v) < \kappa(G)$ for some $\{u,v\} \in E$
- $\lambda_G(u,v) = \lambda_{G-e}(u,v)+1 = \kappa_{G-e}(u,v)+1$
- $\therefore \kappa_{G-e}(u,v) = \lambda_G(u,v)-1 \leq \kappa(G)-2$



Proof of Global Vertex Menger (continued)

Prood idea (continued).

- Let S a minimum u,v-vtx-cut of $G-e$ ($|S| \leq \kappa(G)-2$)
- Note: $n(G) \geq \kappa(G)+1$ (Notice on Page 4)
- $(G-e)-S$ contains a vertex x different from u, v
- $S \cup \{u\}$ or $S \cup \{v\}$ is a vtx-cut of G
- $|S \cup \{u\}| \leq \kappa(G)-1$; A contradiction □



An application: Kőnig-Egerváry theorem

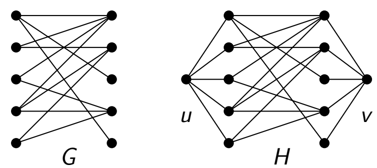
Theorem 4.4

For any bipartite graph G, $\alpha'(G) = \beta(G)$

Proof by Menger's theorem

Construct the following graph H from a bipartite G

- $\kappa_H(u,v) = \lambda_H(u,v)$ (Menger)
- $\lambda_H(u,v) = \alpha'(G)$
- $\kappa_H(u,v) = \beta(G)$ □



Today's contents

- Vertex connectivity and edge connectivity
- Local vertex connectivity and local edge connectivity
- Contraction of an edge
- Menger's theorem
- Open problems

k-Linked graphs

Definition (k-Linked graph)

G is k-linked if $n(G) \geq 2k$ and for any choice of k pairs of vertices there exist k pairwise vertex-disjoint paths between the pairs

Remark: G k-linked $\Rightarrow$ G k-vertex-connected

Jung '70; Larman, Mani '70

$\forall\, k\ \exists\, f : \mathbb{N} \to \mathbb{N}$: Every $f(k)$-vertex-connected graph is k-linked

Thomas, Wollan '05

$f(k) \leq 10k$

Open problem

Determine $f(k)$

Determining a graph is k-linked (1)

Problem (Linkedness)

Input: A graph G, a natural number k
Question: Is G k-linked?

Karp '75

Problem Linkedness is NP-complete

Determining a graph is k-linked (2)

Problem (k-Linkedness)

Pre-input: A natural number k
Input: A graph G
Question: Is G k-linked?

Robertson, Seymour '95

Problem k-Linkedness can be solved in $O(n^3)$ time

This is a *fixed-parameter algorithm* for Linkedness w.r.t. k

Open problem

Implement the algorithm, or provide an practical polynomial-time algorithm for k-Linkedness

Connectivity augmentation problems

Problem (Connectivity augmentation)

Input: A graph G, a natural number k
Output: A set of pair of vertices F s.t. $G+F$ is k-vtx-connected
Objective: Minimize $|F|$

Open problem

Design a polynomial-time algorithm for Connectivity augmentation, or prove it is NP-hard

Remarks

- Poly-time solvable when k is not a part of the input (Jackson, Jordán '05)
- Poly-time solvable for Edge version (multiple edges allowed) (Watanabe, Nakamura '87)
- Poly-time solvable for Directed version (Frank, Jordán '95)