Topics on Computing and Mathematical Sciences I **April 30, 2008**
Graph Theory Lecture 4: Exercises **Yoshio Okamoto**

Due Date: May 14, 2008
Legend: (−) easy; (+) hard
Warning: Your solutions have to be substantiated, namely, you have to provide proofs for the given answers.

Exercise 4.1 Let $\mathcal{A} = (A_1, \ldots, A_m)$ be a collection of subsets of a set Y. A *system of distinct representatives* (SDR) for $\mathcal{A}$ is a set of distinct elements $a_1, \ldots, a_m$ such that $a_i \in A_i$ for every $i \in \{1, \ldots, m\}$. Prove that $\mathcal{A}$ has an SDR if and only if $|\bigcup_{i \in S} A_i| \geq |S|$ for every $S \subseteq \{1, \ldots, m\}$. (Hint: Transform this to a graph problem.)

Exercise 4.2 Let $G = (V, E)$ be a graph. An edge subset $L \subseteq E$ is an *edge cover* of G if every vertex $v \in V$ is incident to at least one edge of L. The size of minimum edge cover of G is called the *edge covering number* of G, and denoted by $\beta'(G)$.

1. Prove that $\alpha(G) \leq \beta'(G)$ for every graph G.
2. Prove that if G has no isolated vertex then $\alpha'(G) + \beta'(G) = n(G)$.
3. Use Kőnig-Egerváry's theorem to deduce that $\alpha(G) = \beta'(G)$ when G is a bipartite graph without isolated vertex.

Exercise 4.3 Let T be a tree. Prove that T has a perfect matching if and only if $o(T - v) = 1$ for every $v \in V(T)$.

Exercise 4.4 Let G be a graph. Prove that

$$\alpha'(G) = \min \left\{ \frac{1}{2}(n(G) + |S| - o(G-S)) \mid S \subseteq V(G) \right\}.$$

(Hint: Construct another graph H with a perfect matching from G, and apply Tutte's theorem to H.)

Exercise 4.5 Let G be a k-regular graph with even number of vertices. Prove that if G remains connected when any $k-2$ edges are deleted, then G has a perfect matching.

Exercise 4.6 Let n be even and G be an n-vertex graph having a set S of size k such that $o(G - S) > k$. Prove that G has at most $\binom{k}{2} + k(n-k) + \binom{n-2k-1}{2}$ edges, and that this bound is tight. Use this to determine the maximum number of edges in an n-vertex graph with no perfect matching.

Exercise 4.7 A graph is *H-free* (for some graph H) if it does not contain H as an induced subgraph. Prove that every connected $K_{1,3}$-free graph with even number of vertices has a perfect matching.