# Topics on Computing and Mathematical Sciences I Graph Theory (2) Cycles and Extremality

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# Today's contents

- A characterization of bipartite graphs
- Euler circuits
- Hamiltonian cycles
- Extremal problems
- Review of useful proof techniques

# How can we tell the graph is bipartite?

**Definition (recap): Bipartite graph**

A graph $G = (V, E)$ is bipartite if
$V$ can be partitioned into two parts $V_1, V_2$ s.t.
$\{u, v\} \in E \Rightarrow u \in V_1, v \in V_2$

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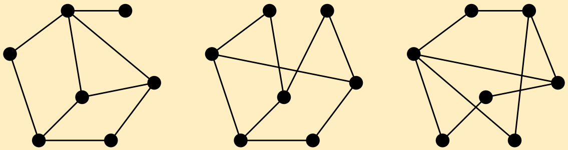
## Question

Among these graphs, which is bipartite?



# A characterization of bipartite graphs

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[$\Leftarrow$]

- $u \in V$ an arbitrary vertex (fixed)
- For all $v \in V$ let $f(v) =$ the minimum length of a $u, v$-path in $G$
- $A = \{v \in V \mid f(v) \text{ even}\}$ and $B = \{v \in V \mid f(v) \text{ odd}\}$ give rise to partite sets of $G$ □

Lemma 2.2 : Every closed odd walk contains an odd cycle

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- The theorem gives an efficient way; Just exhibit an odd cycle

In this sense, the theorem is good; it gives an efficiently verifiable certificate for non-bipartiteness

**Concept (Edmonds '65)**

A characterization is good if it provides an efficiently verifiable certificate

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# Euler circuits and Eulerian graphs

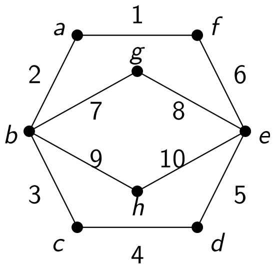
**Definition (Euler circuit)**

An Euler circuit in $G$ is a circuit in $G$ in which every edge of $G$ appears exactly once

**Definition (Eulerian graph)**

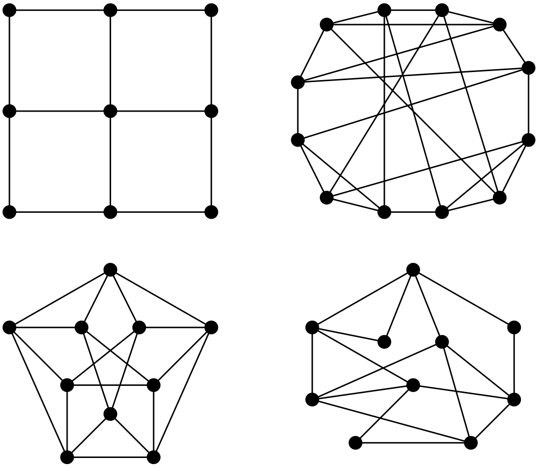
$G$ is Eulerian if it contains an Euler circuit

Example:




$a, b, g, e, h, b, c, d, e, f, a$

# Which graph is Eulerian?



# A characterization of Eulerian graphs

**Theorem 2.3 (A characterization of Eulerian graphs, Euler 1736)**

A graph $G$ is Eulerian $\Leftrightarrow$ $G$ has the following properties

1. All edges of $G$ lie in the same component of $G$
2. The degree of each vertex of $G$ is even

**Proof idea.**

[$\Rightarrow$] Easy direction

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[$\Leftarrow$] Useful proof technique: Extremality

- Choose a *longest trail* $t$ of $G$...
- $t$ starts at some vertex $v$, and should return to $v$ because of the assumption; So $t$ is a circuit
- If $t$ isn't an Euler circuit, then $\exists$ an edge $e$ that doesn't participate in $t$; ... □

# Determining whether a graph is Eulerian or not

**Problem (EULERIAN GRAPH)**

Input: A graph $G$
Question: Is $G$ Eulerian?

Theorem 2.3 is a good characterization for a graph to be Eulerian, leading to an $\mathrm{O}(n+m)$-time algorithm

"Finding one" is another issue, but this can also be done in $\mathrm{O}(n+m)$-time (Hierholzer 1873)

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# Hamiltonian cycles

**Definition (Hamiltonian cycle (path))**

A cycle (or a path) in $G$ is Hamiltonian if it contains all vertices of $G$

**Definition (Hamiltonian graph)**

$G$ is Hamiltonian if it contains a Hamiltonian *cycle*

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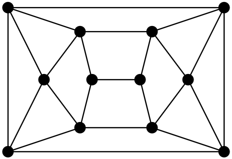

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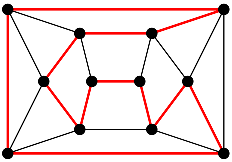

# Relationship of Eulerian and Hamiltonian graphs

## Definition (Line graph)

The line graph of $G$ is a graph $L(G)$ defined as

- $V(L(G)) = E(G)$
- $E(L(G)) = \{\{e, f\} \mid e, f \in E(G), e \cap f \neq \emptyset\}$

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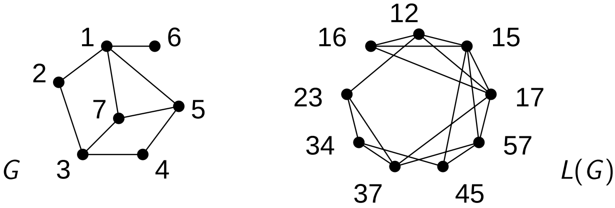

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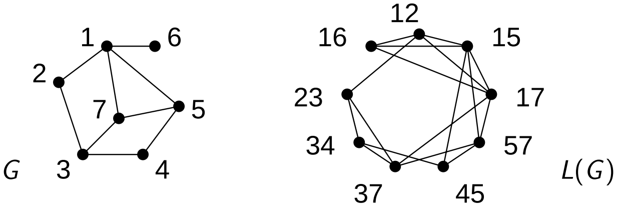


## Proposition 2.4 (Eulerian graphs have Hamiltonian line graphs)

$G$ is Eulerian $\Longrightarrow$ $L(G)$ is Hamiltonian

# An attempt for a characterization of the Hamiltonian graphs

To determine whether a given graph $G$ is Hamiltonian or not, it suffices to find $H$ s.t. $G = L(H)$ and determine whether $H$ is Eulerian or not

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**Problem**

Such a graph $H$ might not exist

# Determining the Hamiltonicity

**Problem (Hamiltonian Cycle)**

Input: a graph $G$
Question: Is $G$ Hamiltonian?

**Karp '72**

Hamiltonian Cycle is NP-complete

$\therefore$ We can't hope for a poly-time algo for Hamiltonian Cycle
$\therefore$ We can't hope for a good characterization for the existence of Hamiltonian cycles

# Is a graph the line graph of some graph?

## Problem (Line Graph)

Input: a graph $G$
Question: Does there exist $H$ s.t. $G = L(H)$?

## Known facts

- Good characterizations (van Rooij, Wilf '65; Beineke '68)
- An $\mathrm{O}(n+m)$-time algorithm (Lehot '74)

# Intuition? 1

**Question: Is the following true?**

If $G$ has many edges, then $G$ surely contains a Hamiltonian cycle

What if $e(G) \geq n(G)^2/4$?

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**Theorem 2.5 (Density doesn't help for Hamiltonicity)**

$\forall\, \varepsilon < 1\ \exists\, n\ \exists$ a non-Hamiltonian $n$-vertex graph $G$ s.t. $e(G) \geq \varepsilon \binom{n}{2}$

**Proof idea.**

Given $\varepsilon$, construct such a graph $G$ □

# Intuition? 2

**Question: Is the following true?**

If $\delta(G)$ is large, then $G$ surely contains a Hamiltonian cycle

What if $\delta(G) \geq n(G)/4$?

# Dirac's theorem

**Theorem 2.6 (Min-degree condition for Hamiltonicity, Dirac '52)**

$n(G) \geq 3$, $\delta(G) \geq \lceil n(G)/2 \rceil \Longrightarrow G$ Hamiltonian

Before a proof

**Observation**

$\delta(G) \geq \lceil n(G)/2 \rceil \Rightarrow G$ connected

Proof: By the averaging argument

# Dirac's theorem: proof

## Proof idea.

Useful techniques: Extremality and Pigeonhole principle (PhP)

- Suppose $\exists$ a non-Hamiltonian graph $G$ with $\delta(G) \geq n(G)/2$
- Consider a *longest* path $P = v_1, \ldots, v_k$ of $G$ ($k \leq n(G)$)
- Consider boxes $1, \ldots, k-1$
- Put a red ball in box $i$ if $v_1$ and $v_{i+1}$ are adjacent;
  Put a blue ball in box $i$ if $v_i$ and $v_k$ are adjacent
- $\#$ red balls $\geq n(G)/2$, $\#$ blue balls $\geq n(G)/2$
- $\exists$ $i$ s.t. box $i$ contains a red and a blue balls (by PhP)
- $\therefore v_1, v_2, \ldots, v_i, v_k, v_{k-1}, \ldots, v_{i+1}, v_1$ is a cycle
- This should be a Hamiltonian cycle (why?) □

# Can a coefficient be made smaller?

## We saw

- "$\delta(G) \geq n(G)/4$" is not sufficient for Hamiltonicity
- "$\delta(G) \geq \lceil n(G)/2 \rceil$" is sufficient for Hamiltonicity

## Question

Can we make "$\lceil n(G)/2 \rceil$" smaller to ensure $G$ to be Hamiltonian?

## Observation

For every $n \geq 3$, there exists an $n$-vertex non-Hamiltonian graph $G$ with $\delta(G) = \lceil n/2 \rceil - 1$

In this sense, the bound "$\delta(G) \geq n(G)/2$" is tight

This is an example of Extremal Problems

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## What's an extremal problem?: Example

We saw, assuming $n(G) \geq 3$,

- $\exists$ $G$ s.t. $\delta(G) = \lceil n(G)/2 \rceil - 1$ and $G$ is not Hamiltonian
- $\forall$ $G$ s.t. $\delta(G) \geq \lceil n(G)/2 \rceil$: $G$ is Hamiltonian

Therefore

$$\max\{\delta(G) \mid G \text{ } n\text{-vertex and non-Hamiltonian}\} = \lceil n/2 \rceil - 1$$

# What's an extremal problem?: Informal definition

**Informal definition: Extremal problem**

Determination of the max (or min) value of a parameter over some class of objects ($n$-vertex graphs in this course)

For the example in the previous slide

| | |
|---|---|
| objective | maximization |
| parameter | minimum degree |
| class | non-Hamiltonian graphs |

# A typical extremal problem

**Question**

Given a natrual number $n$ and a graph $H$,
What is the maximum number of edges in an $n$-vertex graph that doesn't contain $H$?

This question has made graph theory quite fruitful;
We'll look at this question more thoroughly later in the course
Today, only when $H = K_3$ (triangle)

# Graphs having no $K_3$: Mantel's theorem

**Theorem 2.7 (Extremality for having no $K_3$, Mantel 1907)**

The maximum number of edges in an $n$-vertex graph that contains no $K_3$ is $\lfloor n^2/4 \rfloor$

**Proof Idea.**

[max $\geq \lfloor n^2/4 \rfloor$]

- Consider $K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$

□

## Graphs having no $K_3$: Mantel's theorem

### Theorem 2.7 (Extremality for having no $K_3$, Mantel 1907)

The maximum number of edges in an $n$-vertex graph that contains no $K_3$ is $\lfloor n^2/4 \rfloor$

### Wrong Proof Idea.

[max $\geq \lfloor n^2/4 \rfloor$]

- Consider $K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$

[max $\leq \lfloor n^2/4 \rfloor$] Induction on $n$

- When $n \leq 2$, consider $K_n$
- Suppose the claim is valid when $n = k$ ($k \geq 2$)
- $K_{\lceil k/2 \rceil, \lfloor k/2 \rfloor}$ is a largest $k$-vertex graph containing no $K_3$
- Add one vertex to obtain a largest ($k$+1)-vertex graph...

What's wrong with this proof?? □

# Correct proof of Mantel's theorem

## Proof idea.

[max $\geq \lfloor n^2/4 \rfloor$]

- Consider $K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$

[max $\leq \lfloor n^2/4 \rfloor$]

- $d(x) + d(y) \leq n(G)$ for every $\{x, y\} \in E$ (since $G \not\supseteq K_3$)
- $\displaystyle\sum_{\{x,y\} \in E} (d(x) + d(y)) = \sum_{v \in V} d(v)^2$ (double counting)
- $\displaystyle\left(\sum_{v \in V} d(v)\right)^2 \leq n(G) \sum_{v \in V} d(v)^2$ (Cauchy-Schwarz)
- $\displaystyle\sum_{v \in V} d(v) = 2e(G)$ (Handshaking lemma)

□

## More extremal problems to come

Here, we consider graphs containing no $K_3$

### Questions

- What about graphs containing no $K_4$
- What about graphs containing no $K_r$ ($r$ fixed)
- What about graphs containing no $K_{r,r}$
- What about graphs containing no Petersen graph
- ...

These questions will be answered (completely or partially) later in this course $\rightsquigarrow$ Extremal Graph Theory

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# Useful proof techniques

- Double Counting
  - Count a set of objects in two different ways
- Principle of Induction
  - Assume the validity for smaller objects
- Pigeonhole Principle
  - When many objects are put in few boxes, at least one box has two objects
- Extremality
  - Consider an object maximal (or minimal) w.r.t. a certain property

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- Open problems

# Approximate counting of the Euler circuits in graphs

## Problem

Is it possible to count the number of Euler circuits in a given $n$-vertex graph with relative error $\varepsilon$ in $\mathrm{poly}(n, \varepsilon)$ time?

## Known facts

- Exact counting is #P-complete (no poly-time algorithm expected) (Brightwell, Winkler '05)
- Poly-time approximate counting when $\Delta(G) \leq 6$ using MCMC method (Tetali, Vempala '01)
- For directed version the exact counting can be done in poly-time using determinant computation (van Aardenne-Ehrenfest, de Bruijn '51; Smith, Tutte '41 + Matrix-Tree Theorem)

# Extension of Dirac's theorem

## Definition: Power of a graph

The *k-th power* of a graph $G$ is a graph $G^k$ defined as

- $V(G^k) = V(G)$
- $E(G^k) = \{\{u, v\} \mid \exists$ a $u, v$-walk of length $k\}$

Especially, $G^1 = G$

## Conjecture (Seymour '74)

$k \geq 1$, $n \geq 3$, $\delta(G) \geq \dfrac{k}{k+1} n(G)$
$\Rightarrow G$ contains a Hamiltonian cycle $H$ s.t. $H^k \subseteq G$

- $k = 1$: Dirac's theorem
- $k = 2$: Pósa's conjecture ('63), yet unsettled
  - Proved asymptotically (Komlós, Sárközy, Szemerédi '96)

# Hamiltonicity of graphs of bounded polyhedra

## Open Problem (Barnette)

Is the edge-graph of a 3-dimensional bounded polyhedron Hamiltonian, when it is 3-regular and bipartite?

If not bipartite, it can be non-Hamiltonian. (Tutte '46)

## Open Problem (Barnette)

Is the edge-graph of a 4-dimensional bounded polyhedron Hamiltonian, when it is 4-regular?