Topics on
Computing and Mathematical Sciences I
Graph Theory
(2) Cycles and Extremality

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Today's contents

- A characterization of bipartite graphs
- Euler circuits
- Hamiltonian cycles
- Extremal problems
- Review of useful proof techniques

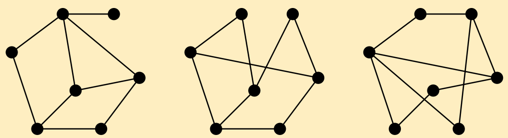
How can we tell the graph is bipartite?

Definition (recap): Bipartite graph

A graph $G = (V, E)$ is bipartite if
V can be partitioned into two parts V_1, V_2 s.t.
$\{u, v\} \in E \Rightarrow u \in V_1, v \in V_2$

Question

Among these graphs, which is bipartite?



A characterization of bipartite graphs

Theorem 2.1 (A characterization of bipartite graphs, Kőnig '36)

A graph $G = (V, E)$ is bipartite $\iff$ G contains no odd cycle

Proof idea.

We may assume G is connected, without loss of generality
[$\Rightarrow$] Every cycle should alternately visit different partite set...
[$\Leftarrow$]

- $u \in V$ an arbitrary vertex (fixed)
- For all $v \in V$ let $f(v)$ = the minimum length of a u, v-path in G
- $A = \{v \in V \mid f(v) \text{ even}\}$ and $B = \{v \in V \mid f(v) \text{ odd}\}$ give rise to partite sets of G □

Lemma 2.2 : Every closed odd walk contains an odd cycle

Good characterization

How can we convince others that a given graph is bipartite?

- We can just exhibit a bipartition of the vertex set

How can we convince others that a given graph is *not* bipartite?

- We can exhibit all candidates of bipartitions and show any of them do not fit the definition; This is too inefficient!!!
- The theorem gives an efficient way; Just exhibit an odd cycle

In this sense, the theorem is good; it gives an efficiently verifiable certificate for non-bipartiteness

Concept (Edmonds '65)

A characterization is good if it provides an efficiently verifiable certificate

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Euler circuits and Eulerian graphs

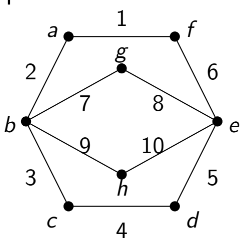
Definition (Euler circuit)

An Euler circuit in G is a circuit in G in which every edge of G appears exactly once

Definition (Eulerian graph)

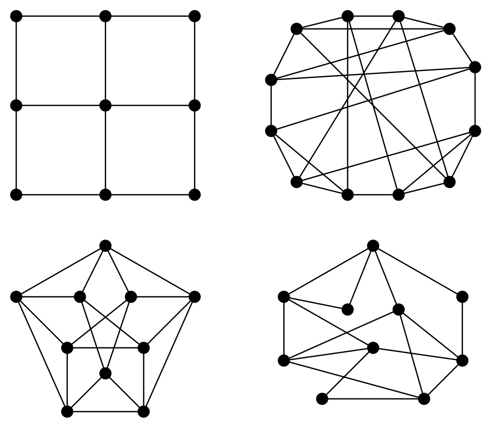
G is Eulerian if it contains an Euler circuit

Example:



$a, b, g, e, h, b, c, d, e, f, a$

Which graph is Eulerian?



A characterization of Eulerian graphs

Theorem 2.3 (A characterization of Eulerian graphs, Euler 1736)

A graph G is Eulerian $\Leftrightarrow$ G has the following properties

1. All edges of G lie in the same component of G
2. The degree of each vertex of G is even

Proof idea.

[$\Rightarrow$] Easy direction
[$\Leftarrow$] Useful proof technique: Extremality

- Choose a *longest trail* t of G...
- t starts at some vertex v, and should return to v because of the assumption; So t is a circuit
- If t isn't an Euler circuit, then $\exists$ an edge e that doesn't participate in t; ... □

Determining whether a graph is Eulerian or not

Problem (Eulerian Graph)

Input: A graph G
Question: Is G Eulerian?

Theorem 2.3 is a good characterization for a graph to be Eulerian, leading to an $\mathrm{O}(n+m)$-time algorithm

"Finding one" is another issue, but this can also be done in $\mathrm{O}(n+m)$-time (Hierholzer 1873)

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Hamiltonian cycles

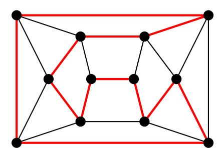
Definition (Hamiltonian cycle (path))

A cycle (or a path) in G is Hamiltonian if it contains all vertices of G

Definition (Hamiltonian graph)

G is Hamiltonian if it contains a Hamiltonian *cycle*

Example:



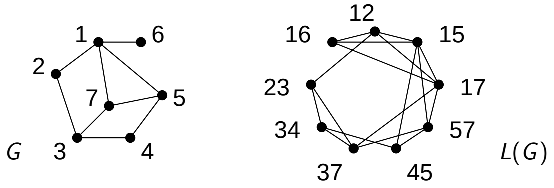
Relationship of Eulerian and Hamiltonian graphs

Definition (Line graph)

The line graph of G is a graph $L(G)$ defined as

- $V(L(G)) = E(G)$
- $E(L(G)) = \{\{e, f\} \mid e, f \in E(G), e \cap f \neq \emptyset\}$

Example:



Proposition 2.4 (Eulerian graphs have Hamiltonian line graphs)

G is Eulerian $\Longrightarrow$ $L(G)$ is Hamiltonian

An attempt for a characterization of the Hamiltonian graphs

To determine whether a given graph G is Hamiltonian or not, it suffices to find H s.t. $G = L(H)$ and determine whether H is Eulerian or not

Problem

Such a graph H might not exist

Determining the Hamiltonicity

Problem (Hamiltonian Cycle)

Input: a graph G
Question: Is G Hamiltonian?

Karp '72

Hamiltonian Cycle is NP-complete

$\therefore$ We can't hope for a poly-time algo for Hamiltonian Cycle
$\therefore$ We can't hope for a good characterization for the existence of Hamiltonian cycles

Is a graph the line graph of some graph?

Problem (Line Graph)

Input: a graph G
Question: Does there exist H s.t. $G = L(H)$?

Known facts

- Good characterizations (van Rooij, Wilf '65; Beineke '68)
- An $O(n+m)$-time algorithm (Lehot '74)

Intuition? 1

Question: Is the following true?

If G has many edges, then G surely contains a Hamiltonian cycle

What if $e(G) \geq n(G)^2/4$?

Theorem 2.5 (Density doesn't help for Hamiltonicity)

$\forall\, \varepsilon < 1\ \exists\, n\ \exists$ a non-Hamiltonian n-vertex graph G s.t. $e(G) \geq \varepsilon \binom{n}{2}$

Proof idea.

Given ε, construct such a graph G □

Intuition? 2

Question: Is the following true?

If $\delta(G)$ is large, then G surely contains a Hamiltonian cycle

What if $\delta(G) \geq n(G)/4$?

Dirac's theorem

Theorem 2.6 (Min-degree condition for Hamiltonicity, Dirac '52)

$n(G) \geq 3$, $\delta(G) \geq \lceil n(G)/2 \rceil \implies G$ Hamiltonian

Before a proof

Observation

$\delta(G) \geq \lceil n(G)/2 \rceil \Rightarrow G$ connected

Proof: By the averaging argument

Dirac's theorem: proof

Proof idea.

Useful techniques: Extremality and Pigeonhole principle (PhP)

- Suppose $\exists$ a non-Hamiltonian graph G with $\delta(G) \geq n(G)/2$
- Consider a *longest* path $P = v_1, \ldots, v_k$ of G ($k \leq n(G)$)
- Consider boxes $1, \ldots, k-1$
- Put a red ball in box i if v_1 and v_{i+1} are adjacent;
  Put a blue ball in box i if v_i and v_k are adjacent
- $\#$ red balls $\geq n(G)/2$, $\#$ blue balls $\geq n(G)/2$
- $\exists\, i$ s.t. box i contains a red and a blue balls (by PhP)
- $\therefore v_1, v_2, \ldots, v_i, v_k, v_{k-1}, \ldots, v_{i+1}, v_1$ is a cycle
- This should be a Hamiltonian cycle (why?) □

Can a coefficient be made smaller?

We saw

- "$\delta(G) \geq n(G)/4$" is not sufficient for Hamiltonicity
- "$\delta(G) \geq \lceil n(G)/2 \rceil$" is sufficient for Hamiltonicity

Question

Can we make "$\lceil n(G)/2 \rceil$" smaller to ensure G to be Hamiltonian?

Observation

For every $n \geq 3$, there exists an n-vertex non-Hamiltonian graph G with $\delta(G) = \lceil n/2 \rceil - 1$

In this sense, the bound "$\delta(G) \geq n(G)/2$" is tight

This is an example of Extremal Problems

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What's an extremal problem?: Example

We saw, assuming $n(G) \geq 3$,

- $\exists$ G s.t. $\delta(G) = \lceil n(G)/2 \rceil - 1$ and G is not Hamiltonian
- $\forall$ G s.t. $\delta(G) \geq \lceil n(G)/2 \rceil$: G is Hamiltonian

Therefore

$$\max\{\delta(G) \mid G \text{ } n\text{-vertex and non-Hamiltonian}\} = \lceil n/2 \rceil - 1$$

What's an extremal problem?: Informal definition

Informal definition: Extremal problem

Determination of the max (or min) value of a parameter over some class of objects (n-vertex graphs in this course)

For the example in the previous slide

objective	maximization
parameter	minimum degree
class	non-Hamiltonian graphs

A typical extremal problem

Question

Given a natrual number n and a graph H,
What is the maximum number of edges in an n-vertex graph that doesn't contain H?

This question has made graph theory quite fruitful;
We'll look at this question more thoroughly later in the course
Today, only when $H = K_3$ (triangle)

Graphs having no K_3: Mantel's theorem

Theorem 2.7 (Extremality for having no K_3, Mantel 1907)

The maximum number of edges in an n-vertex graph that contains no K_3 is $\lfloor n^2/4 \rfloor$

Wrong Proof Idea.

[max $\geq \lfloor n^2/4 \rfloor$]

- Consider $K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$

[max $\leq \lfloor n^2/4 \rfloor$] Induction on n

- When $n \leq 2$, consider K_n
- Suppose the claim is valid when $n = k$ $(k \geq 2)$
- $K_{\lceil k/2 \rceil, \lfloor k/2 \rfloor}$ is a largest k-vertex graph containing no K_3
- Add one vertex to obtain a largest $(k+1)$-vertex graph...

What's wrong with this proof?? □

Correct proof of Mantel's theorem

Proof idea.

[max $\geq \lfloor n^2/4 \rfloor$]

- Consider $K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$

[max $\leq \lfloor n^2/4 \rfloor$]

- $d(x) + d(y) \leq n(G)$ for every $\{x, y\} \in E$ (since $G \not\supseteq K_3$)
- $\displaystyle\sum_{\{x,y\} \in E} (d(x) + d(y)) = \sum_{v \in V} d(v)^2$ (double counting)
- $\displaystyle\left(\sum_{v \in V} d(v)\right)^2 \leq n(G) \sum_{v \in V} d(v)^2$ (Cauchy-Schwarz)
- $\displaystyle\sum_{v \in V} d(v) = 2e(G)$ (Handshaking lemma)

□

More extremal problems to come

Here, we consider graphs containing no K_3

Questions

- What about graphs containing no K_4
- What about graphs containing no K_r (r fixed)
- What about graphs containing no $K_{r,r}$
- What about graphs containing no Petersen graph
- ...

These questions will be answered (completely or partially) later in this course ⇝ Extremal Graph Theory

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Useful proof techniques

- Double Counting
  - Count a set of objects in two different ways
- Principle of Induction
  - Assume the validity for smaller objects
- Pigeonhole Principle
  - When many objects are put in few boxes, at least one box has two objects
- Extremality
  - Consider an object maximal (or minimal) w.r.t. a certain property

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- Open problems

Approximate counting of the Euler circuits in graphs

Problem

Is it possible to count the number of Euler circuits in a given n-vertex graph with relative error ε in $\mathrm{poly}(n, \varepsilon)$ time?

Known facts

- Exact counting is #P-complete (no poly-time algorithm expected) (Brightwell, Winkler '05)
- Poly-time approximate counting when $\Delta(G) \leq 6$ using MCMC method (Tetali, Vempala '01)
- For directed version the exact counting can be done in poly-time using determinant computation (van Aardenne-Ehrenfest, de Bruijn '51; Smith, Tutte '41 + Matrix-Tree Theorem)

Extension of Dirac's theorem

Definition: Power of a graph

The k-th power of a graph G is a graph G^k defined as

- $V(G^k) = V(G)$
- $E(G^k) = \{\{u, v\} \mid \exists$ a u, v-walk of length $k\}$

Especially, $G^1 = G$

Conjecture (Seymour '74)

$k \geq 1$, $n \geq 3$, $\delta(G) \geq \dfrac{k}{k+1} n(G)$
$\Rightarrow G$ contains a Hamiltonian cycle H s.t. $H^k \subseteq G$

- $k = 1$: Dirac's theorem
- $k = 2$: Pósa's conjecture ('63), yet unsettled
  - Proved asymptotically (Komlós, Sárközy, Szemerédi '96)

Hamiltonicity of graphs of bounded polyhedra

Open Problem (Barnette)

Is the edge-graph of a 3-dimensional bounded polyhedron Hamiltonian, when it is 3-regular and bipartite?

If not bipartite, it can be non-Hamiltonian. (Tutte '46)

Open Problem (Barnette)

Is the edge-graph of a 4-dimensional bounded polyhedron Hamiltonian, when it is 4-regular?