Topics on Computing and Mathematical Sciences I — **April 16, 2008**
Graph Theory Lecture 2: Exercises — **Yoshio Okamoto**

Due Date: April 23, 2008
Legend: (−) easy; (+) hard
Warning: Your solutions have to be substantiated, namely, you have to provide proofs for the given answers.

Exercise 2.1 Let G be a graph. Prove that G is bipartite if and only if G contains no odd cycle as an induced subgraph.

Exercise 2.2 (−) Prove or disprove:

1. Every Eulerian bipartite graph has an even number of edges.
2. Every connected Eulerian graph with an even number of vertices has an even number of edges.

Exercise 2.3 Let P and Q be paths of maximum length in a connected graph G. Prove that P and Q have a common vertex.

Exercise 2.4 Let G be a graph and $L(G)$ be the line graph of G.

1. Prove that the number of edges in $L(G)$ is $\sum_{v \in V(G)} \binom{d(v)}{2}$.
2. Prove that G is isomorphic to $L(G)$ if and only if G is 2-regular.

Exercise 2.5 Let G be a connected graph with $\delta(G) = k \geq 2$ and $n(G) > 2k$.

1. Let P be a maximal path in G (namely P cannot be extended to a longer path). Prove that if $n(P) \leq 2k$ then the induced subgraph $G[V(P)]$ has a Hamiltonian cycle (this cycle need not have its vertices in the same order as P).
2. Use the first part to prove that G has a path with at least $2k+1$ vertices.

Exercise 2.6 Denote by $t(G)$ the number of triangles (subgraphs isomorphic to K_3) in G, and by $t(e)$ the number of triangles in G containing a given edge e.

1. Show that $d(x) + d(y) \leq n(G) + t(e)$ for a graph $G = (V, E)$ and an edge $e = \{x, y\} \in E$.
2. By summing this inequality over all $e \in E$, deduce that $t(G) \geq \dfrac{e(G)(4e(G) - n(G)^2)}{3n(G)}$.
3. Deduce that if $k \geq 3$ and $e(G) \geq \dfrac{1}{2}\left(1 - \dfrac{1}{k}\right) n(G)^2$, then $t(G) \geq \binom{k}{3}\left(\dfrac{n(G)}{k}\right)^3$.
4. For $k \geq 3$ and $n = 0 \pmod k$, construct a graph G with $e(G) = \dfrac{1}{2}\left(1 - \dfrac{1}{k}\right) n(G)^2$ and $t(G) = \binom{k}{3}\left(\dfrac{n(G)}{k}\right)^3$.