Topics on Computing and Mathematical Sciences I — **April 9, 2008**
Graph Theory Lecture 1: Exercises — **Yoshio Okamoto**

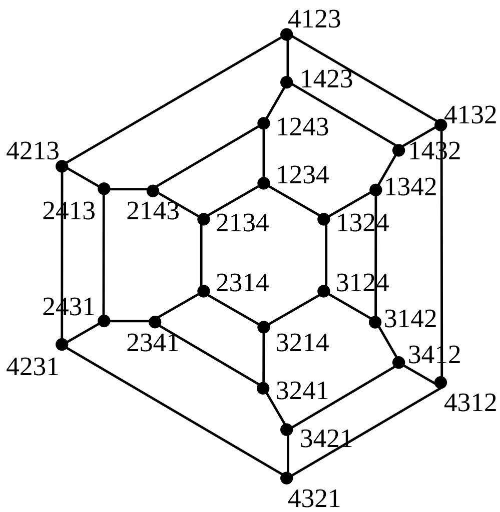
Due Date: April 16, 2008
Legend: (−) easy; (+) hard
Warning: Your solutions have to be substantiated, namely, you have to provide proofs for the given answers.

Exercise 1.1 Let G be a graph with girth five. Prove that if every vertex of G has degree at least k, then G has at least k^2+1 vertices. For $k=2$ and $k=3$, find one such graph with exactly k^2+1 vertices.

Exercise 1.2 A graph is *self-complementary* if it is isomorphic to its complement. Prove that a self-complementary graph with n vertices exists if and only if n or $n-1$ is divisible by four.

Exercise 1.3 (−) Let G be a regular bipartite graph with partite sets X and Y. Prove that $|X|=|Y|$.

Exercise 1.4 Let G_n be the graph whose vertices are the permutations of $\{1,\dots,n\}$, with two permutations $a_1,\dots,a_n$ and $b_1,\dots,b_n$ adjacent if and only if they differ by interchanging a pair of adjacent entries (G_4 shown below). Prove that G_n is connected for all $n \geq 1$.



Exercise 1.5 Let $k \geq 1$ be an integer and G be the graph whose vertex set is the set of ordered k-tuples with elements in $\{0,1\}$ (namely, the vertex set is $\{0,1\}^k$) with x adjacent to y if and only if x and y differ in exactly two positions. Determine the number of connected components of G.

Exercise 1.6 Let G be a 57-regular graph with girth 5 such that each non-adjacent pair of vertices has a common neighbor. Prove that $n(G)=3250$ and $e(G)=92625$. (Hint: Count "something" in two ways.)

Exercise 1.7 (+) For $k \geq 2$ and $g \geq 3$, prove that there exists a k-regular graph with girth g. (Hint: To construct such a graph inductively, make use of a $(k-1)$-regular graph H with girth g and an $n(H)$-regular graph with girth $\lceil g/2 \rceil$.)

Exercise 1.8 An *automorphism* of a graph G is an isomorphism from G to G itself. Construct a 3-regular graph with 12 vertices in which the only automorphism is the identity.