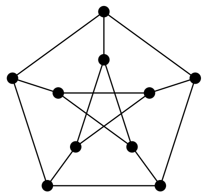
Topics on Computing and Mathematical Sciences I — **August 6, 2008**
Graph Theory Lecture: Exam — **Yoshio Okamoto**

9:30–12:00, an A4 sheet of paper with both sides filled by anything can be brought, no other notes, books, or no electronic devices are allowed
You have six problems. Complete solutions to five problems will be enough for 100 points to you. To pass the exam, you need to solve at least two problems completely.
You are allowed to use the theorems, lemmas, propositions that have been proved during the course of lectures. You are also allowed to use Szemerédi's regularity lemma and the embedding lemma, if needed.
Warning: Your solutions have to be substantiated, namely, you have to provide proofs for the given answers.

Problem 1 let k, n be natural numbers and G be an n-vertex planar graph with girth k. Prove that G has at most $\frac{k(n-2)}{k-2}$ edges. Use this to prove that the Petersen graph (shown below) is non-planar.



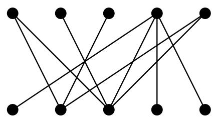
Problem 2 Let $k \geq 2$ be a natural number, and G be a connected graph with $\delta(G) = k$ and $n(G) > 2k$.

1. Let P be a maximal path in G (namely P cannot be extended to a longer path). Prove that if $n(P) \leq 2k$ then the induced subgraph $G[V(P)]$ has a Hamiltonian cycle (this cycle need not have its vertices in the same order as P).
2. Use the first part to prove that G has a path with at least $2k+1$ vertices.

Problem 3 Let k be a natural number and G be a k-chromatic graph (namely, $\chi(G) = k$). Prove that G has at least $\binom{k}{2}$ edges.

Problem 4 In this problem, we have four independent subproblems. Explicitly choose two subproblems out of four, and answer them. If you choose more than two subproblems, your answers will not be counted.

1. For any real number $\varepsilon > 0$, show that an ε-regular partition $\{V_0, V_1, \ldots, V_k\}$ of a graph G is also an ε-regular partition of its complement $\overline{G}$.
2. Find a maximum matching in the graph below. Prove that it is a maximum matching.



3. Prove that every 3-regular Hamiltonian graph is 3-edge-colorable.
4. Prove $R(3,4) \leq 9$. Here $R(k, \ell)$ is the minimum r for which every r-vertex graph G contains K_k or its complement $\overline{G}$ contains K_ℓ. (Hint: First prove $R(3,4) \leq 10$. Then examine the proof carefully to deduce $R(3,4) \leq 9$.)

Problem 5 Let k, n be natural numbers and G be an n-vertex graph with maximum degree k. Prove that every *maximal* (not necessarily maximum) independent set of G contains at least $n/(k+1)$ vertices.

Problem 6 Let n be even and $r \geq 1$. Prove that if G is an n-vertex r-vertex-connected graph having no $K_{1,r+1}$ as an induced subgraph, then G has a perfect matching. (Hint: Use Tutte's theorem on the existence of a perfect matching.)