Matroid representation of clique complexes

Yoshio Okamoto

ETH Zürich

okamotoy@inf.ethz.ch

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CGC
Combinatorics
Geometry
Computation

Background

- An independence system can be represented as the intersection of finitely many matroids.
- The clique complex $\mathrm{cliq}(G)$ of a graph G is an independence system.

Question

- How many matroids do we need for clique complexes?

Result

- We characterize how many we need.

Def.: independence system

V a nonempty finite set

Def.: an independence system on V is
a nonempty family $\mathcal{I}$ of subsets of V satisfying

$$X \in \mathcal{I} \implies Y \in \mathcal{I} \quad \text{for all } Y \subseteq X.$$

Example: $V = \{1, 2, 3, 4\}$;

- an example of independence systems
  $\{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 4\}, \{1, 2, 3\}\}$.
- NOT an example of independence systems
  $\{\emptyset, \{1\}, \{2\}, \{3\}, \quad \{1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 4\}, \{1, 2, 3\}\}$.

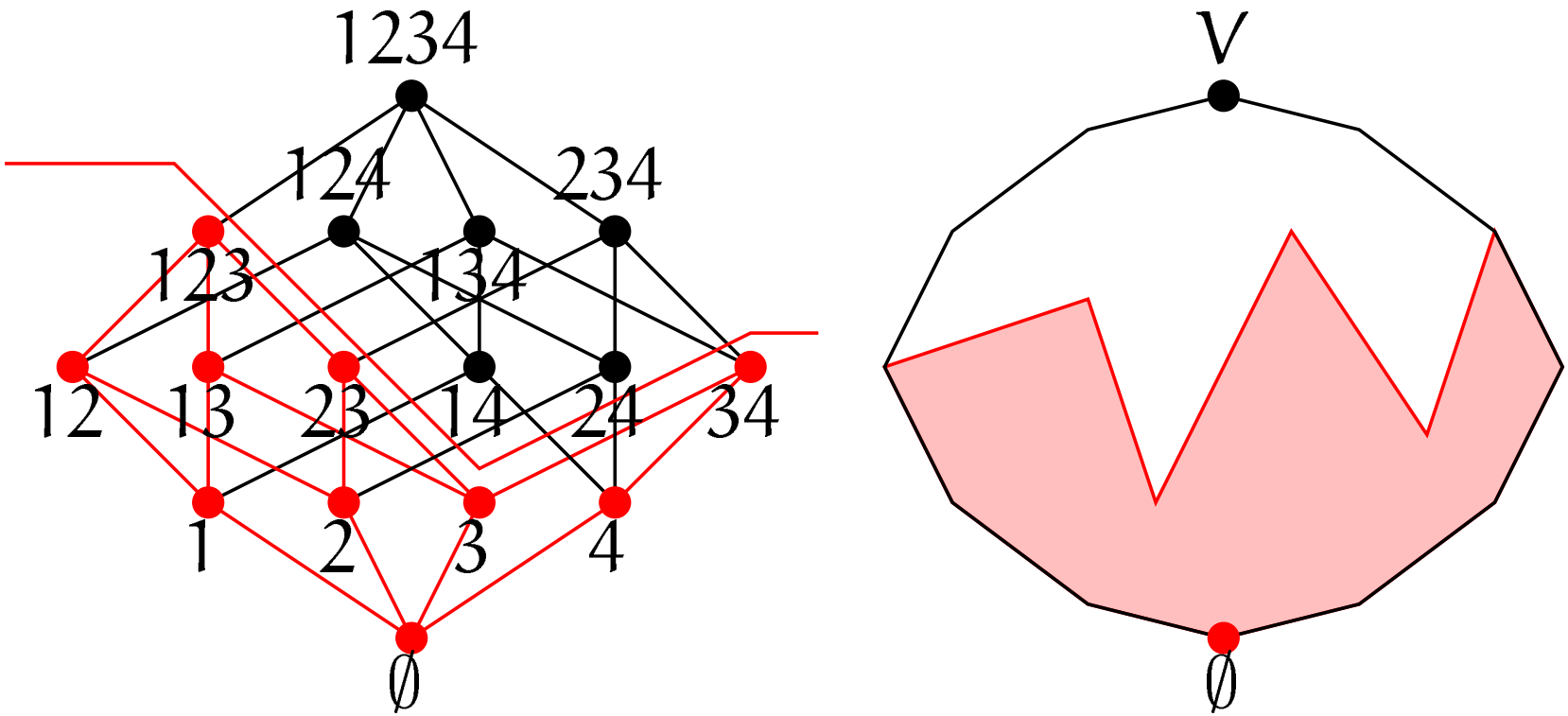
Remark: also called an abstract simplicial complex.

Typical independence systems from ...

- From graphs
  - the family of the cliques
  - the family of the forests
  - the family of the matchings
  - the family of vertices which share a common neighbor
- From partially ordered sets
  - the family of all chains
  - the family of all antichains
- From polytopes
  - the family of all proper faces of a simplicial polytope
- From optimization
  - the set of all feasible solutions of a knapsack problem
- From game theory
  - the family of losing coalitions of a weighted majority game

Understanding independence systems

$\mathcal{I} = \{\emptyset, 1, 2, 3, 4, 12, 13, 23, 34, 123\}$.

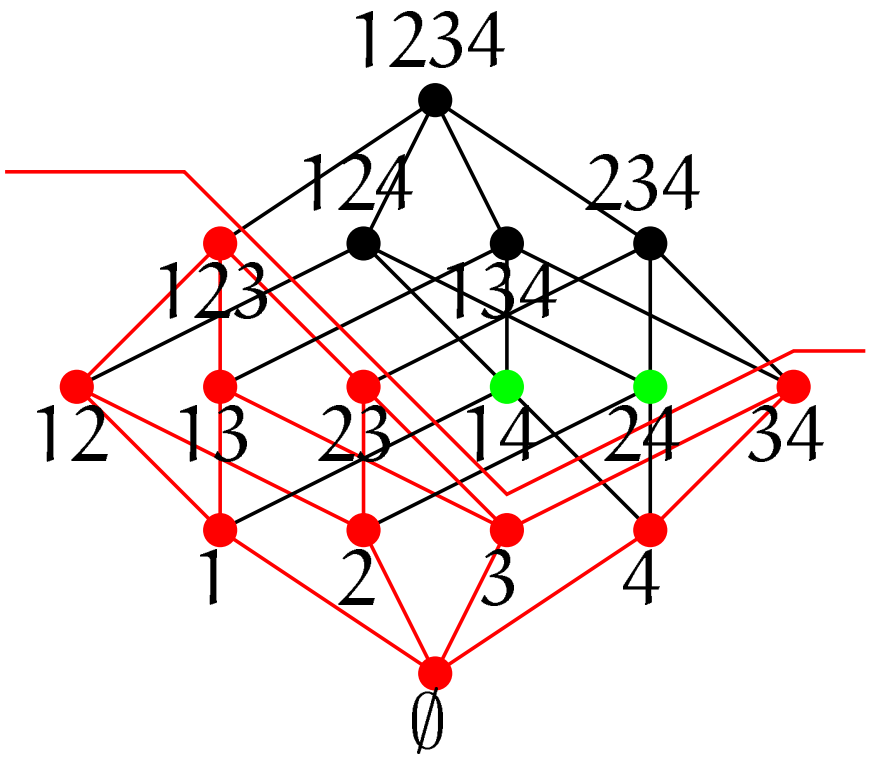


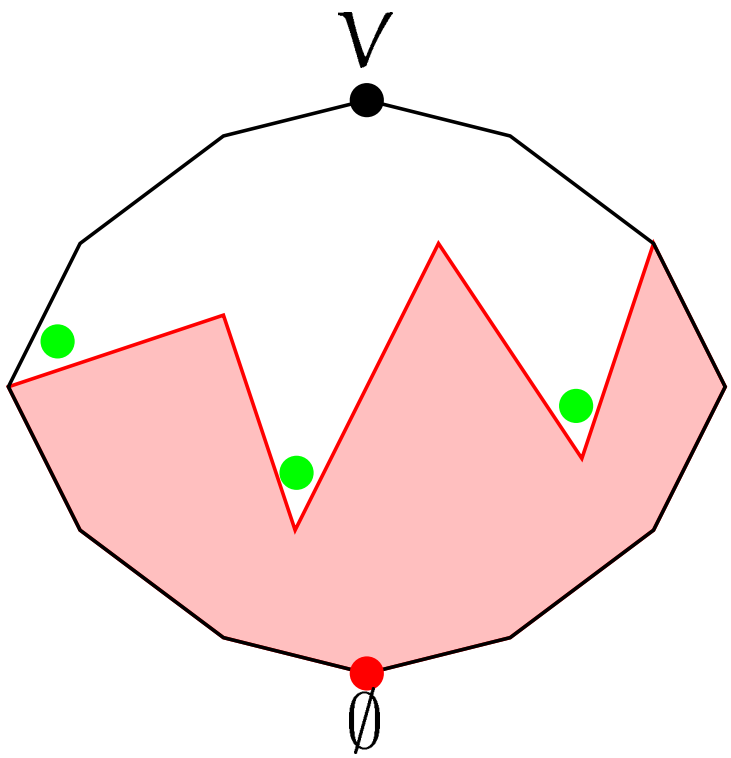
(known as the face poset of $\mathcal{I}$,
when we see $\mathcal{I}$ as an abstract simplicial complex)

Def.: circuits

$\mathcal{I}$ an independence system on V

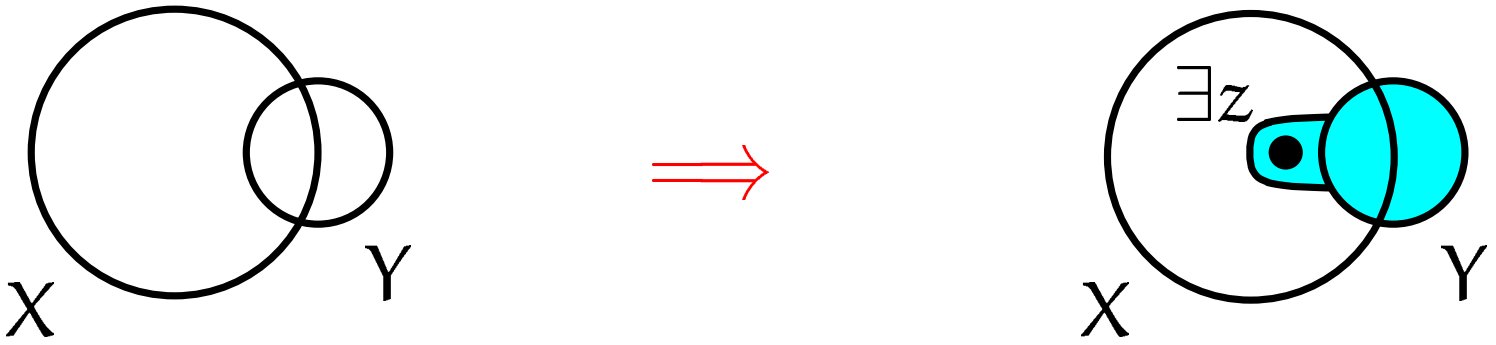
Def.: $C \subseteq V$ is called a circuit if $C \notin \mathcal{I}$ and C is minimal w.r.t. this property.





Def.: An independence system $\mathcal{I}$ is called a matroid if $\mathcal{I}$ satisfies the following augmentation axiom:

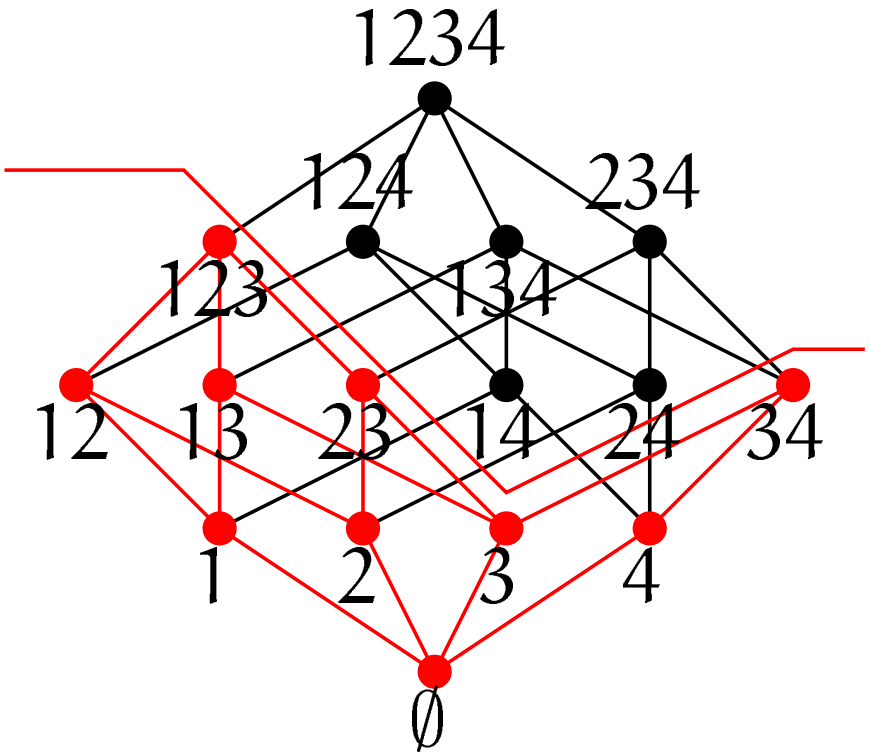
$$X, Y \in \mathcal{I}, |X| > |Y| \Longrightarrow \exists z \in X \setminus Y \text{ s.t. } Y \cup \{z\} \in \mathcal{I}.$$

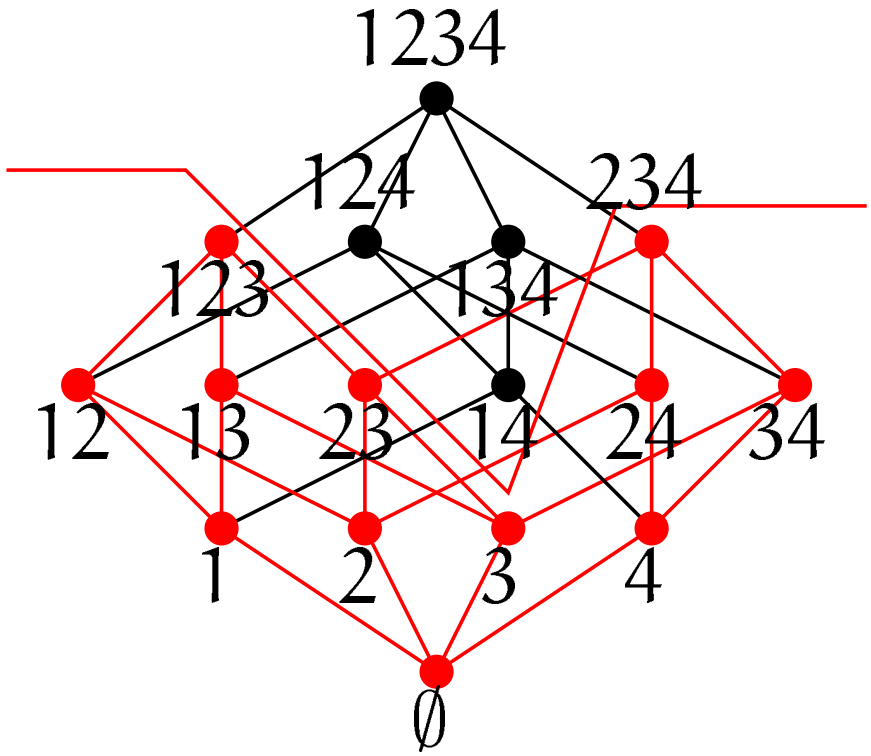


Appearance of matroids:

Combinatorics, Combinatorial optimization, Combinatorial geometry, Combinatorial topology, Coding theory, Design theory, etc.

Which is a matroid?

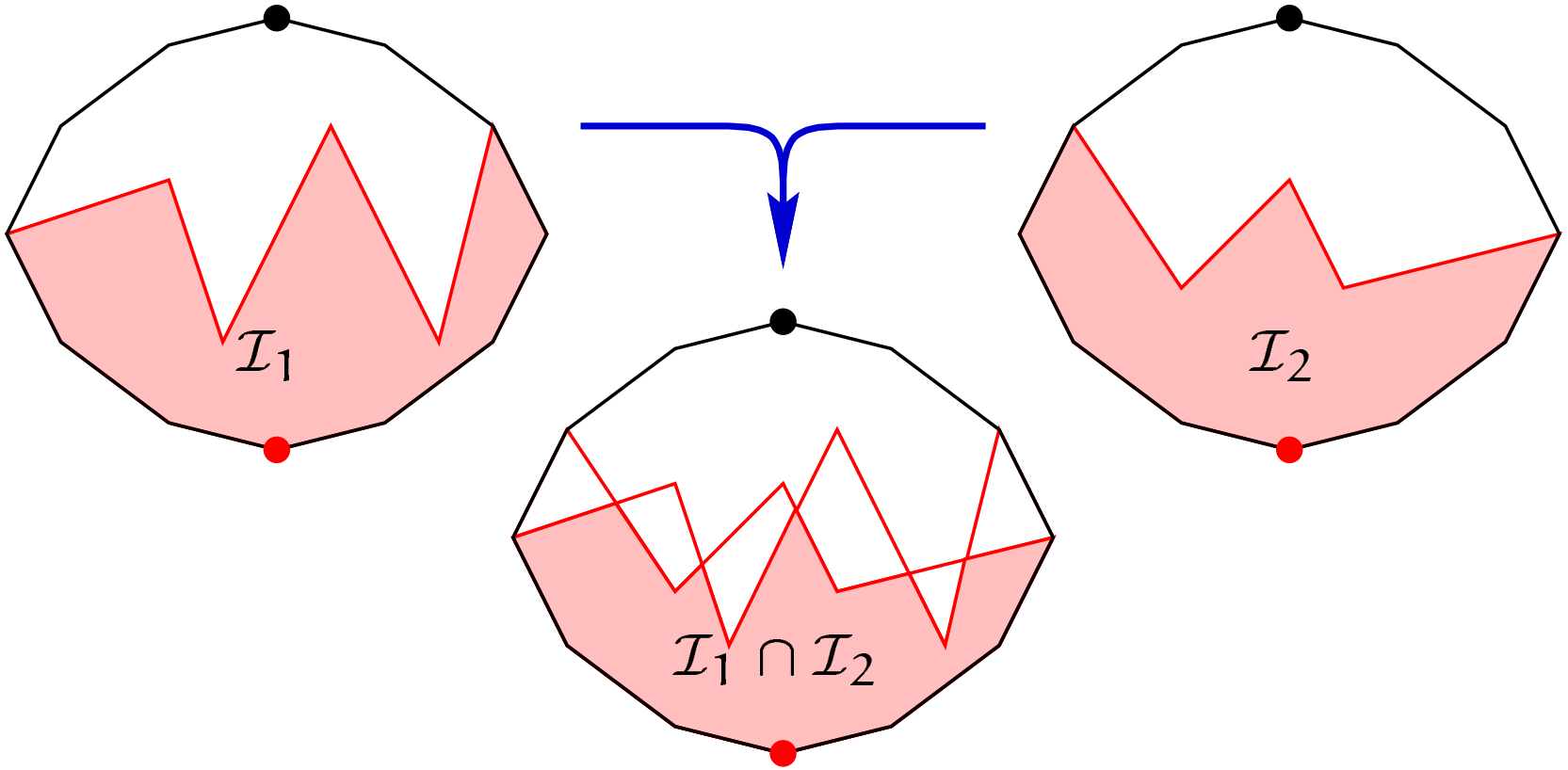




Intersection of indepence systems

The intersection of two independence systems $\mathcal{I}_1$, $\mathcal{I}_2$ on V is

$$\mathcal{I}_1 \cap \mathcal{I}_2 = \{X \subseteq V : X \in \mathcal{I}_1 \text{ and } X \in \mathcal{I}_2\}.$$



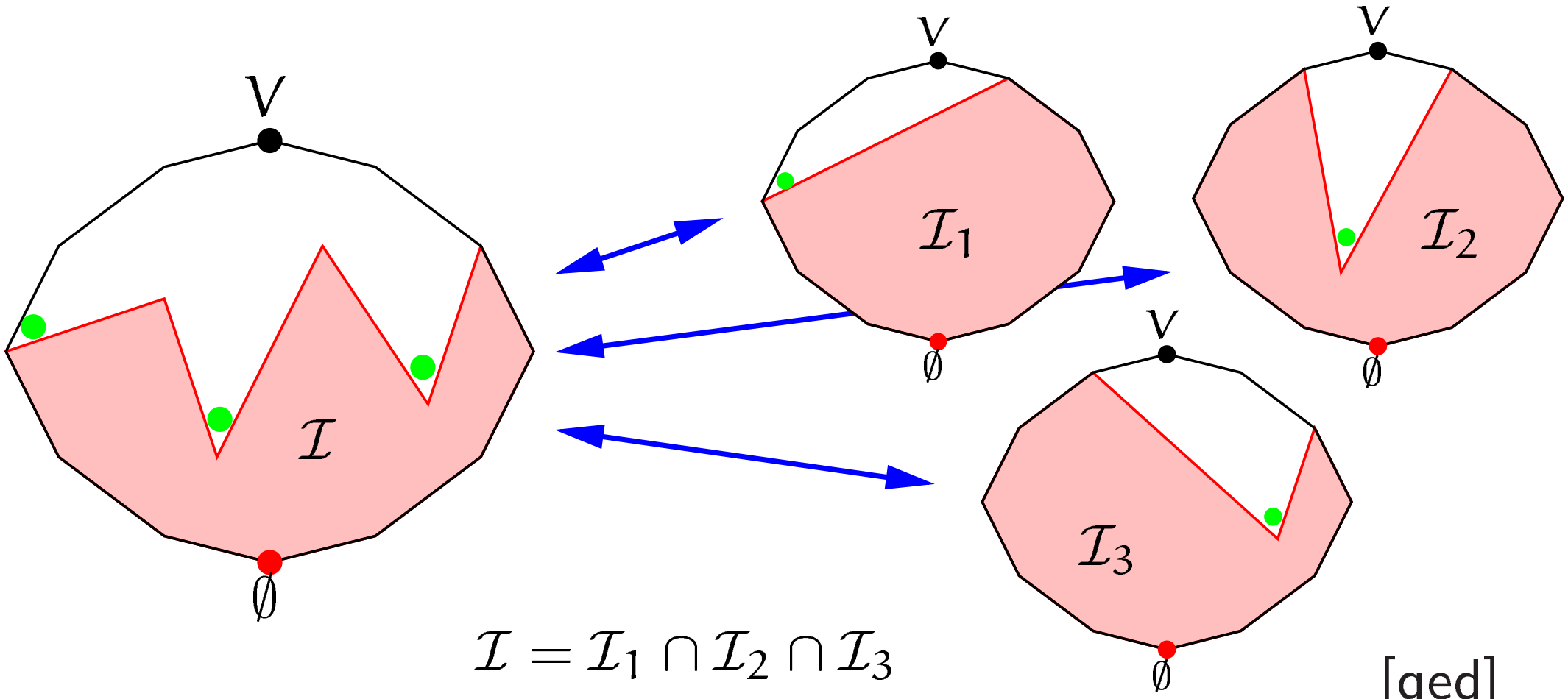
The intersection of more independence systems is defined analogously.

★ The intersection of matroids is an independence system but not necessarily a matroid. However ...

An important fact

Fact: Every independence system can be represented as the intersection of finitely many matroids.

Proof. An indep. sys. with only one circuit is a matroid.



$$\mathcal{I} = \mathcal{I}_1 \cap \mathcal{I}_2 \cap \mathcal{I}_3$$

[qed]

Question:

How many matroids do we need to
represent a given independence system as their intersection??

. . . . . .

The proof in the prev. slide says that:

this number $\leq$ the number of circuits of $\mathcal{I}$.

But, we might do better.

. . . . . .

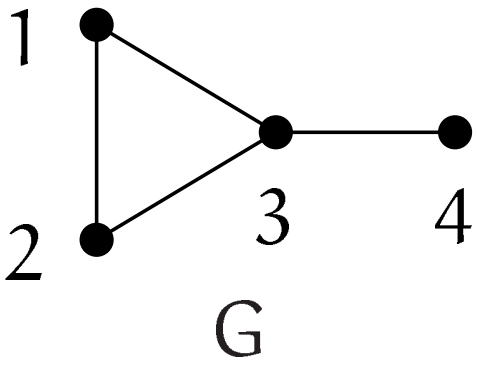
We investigate this number for clique complexes.

Def.: Clique complex

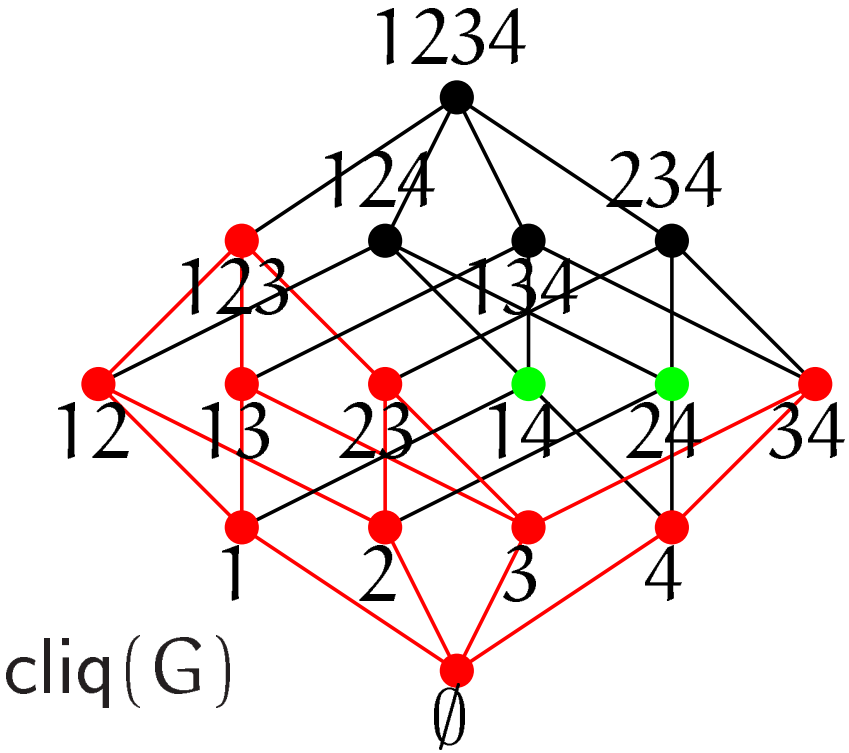
$G = (V, E)$ a graph

Def.: The clique complex of G
is the family of all cliques of G.

Example: $V = \{1, 2, 3, 4\}$



G



cliq(G)

Notation: cliq(G) = the clique complex of G.

Question (again)

Question:

How many matroids do we need to
represent a given clique complex as their intersection??

Result:

We investigate and characterize these numbers.

. . . . . .

However, you may ask...

(1) Why do we consider such representation??

(2) Why do we consider clique complexes??

Motivation (1)

Q. Why do we consider such representation?

Ans. in the following proposition.

Prop.: (Jenkyns, '76; Korte–Hausmann, '78)

(Translation of their prop. to clique complexes)

$\mathrm{cliq}(G)$ is the intersection of k matroids

$\Longrightarrow$ Greedy Algorithm is a k-approximation alg. for MaxWeightedCliqueProblem in G.

Motivation (2)

Q. Why do we consider clique complexes?

Ans. The class of clique complexes includes some important classes like the classes of

- the family of the matchings of a graph G
  (the matching complex of G)
  $=$ cliq(the complement of the line graph of G)
- the family of the chains of a partially ordered set P
  (the order complex of P)
  $=$ cliq(the comparability graph of P)
- ...

The talk plan

In the rest of my talk

(1) clique complex which is a matroid

- Key concept: partition matroid

(2) clique complex which is the intersection of k matroids

- Key concept: stable-set partition

Def.: Partition matroid

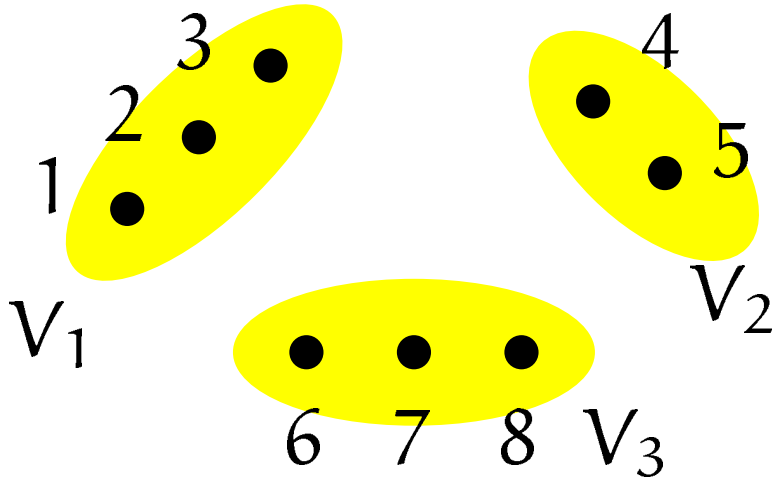
V a nonempty finite set

$\mathcal{P} = \{V_1, V_2, \ldots, V_r\}$ a partition of V

Def.: A partition matroid $\mathcal{I}(\mathcal{P})$ of $\mathcal{P}$ is defined as

$$\mathcal{I}(\mathcal{P}) = \{X \subseteq V : |X \cap V_i| \leq 1 \text{ for all } i = 1, \ldots, r\}.$$

Example: $\mathcal{P} = \{V_1, V_2, V_3\}$



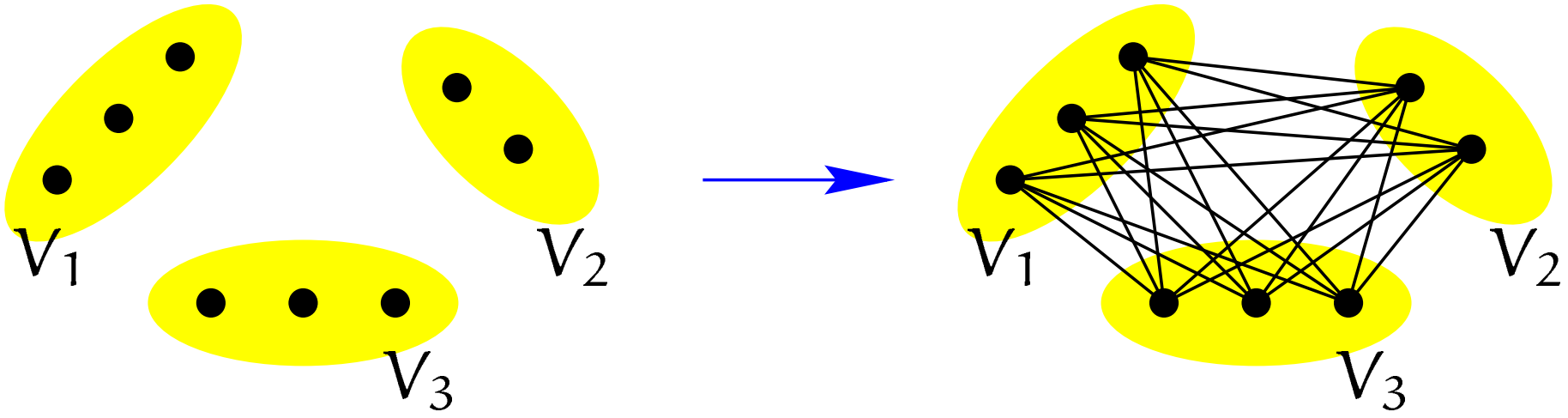
$\mathcal{I}(\mathcal{P}) =$
$\{146, 147, 148, 156, 157, 158,$
$246, 247, 248, 256, 257, 258,$
$346, 347, 348, 356, 357, 358,$
and their subsets$\}$

Notice: $\mathcal{I}(\mathcal{P})$ is really a matroid.

Partition matroids are clique complexes

Lemma 0: Partition matroids are clique complexes.

Proof. Given a partition $\mathcal{P} = \{V_1, \ldots, V_r\}$ of V.



Construct a complete r-partite graph $G_{\mathcal{P}}$ s.t. each part of $G_{\mathcal{P}}$ corresponds to V_i.

Then, $\mathrm{cliq}(G_{\mathcal{P}}) = \mathcal{I}(\mathcal{P})$. [qed]

Remark: G is complete r-partite $\Longrightarrow$ cliq(G) is a matroid.

Theorem 1: G a graph

$$G \text{ is complete } r\text{-partite} \iff \mathrm{cliq}(G) \text{ is a matroid.}$$

Proof.

[$\Longrightarrow$] ... is the previous Lemma 0.

[$\Longleftarrow$] ... needs two more lemmas.

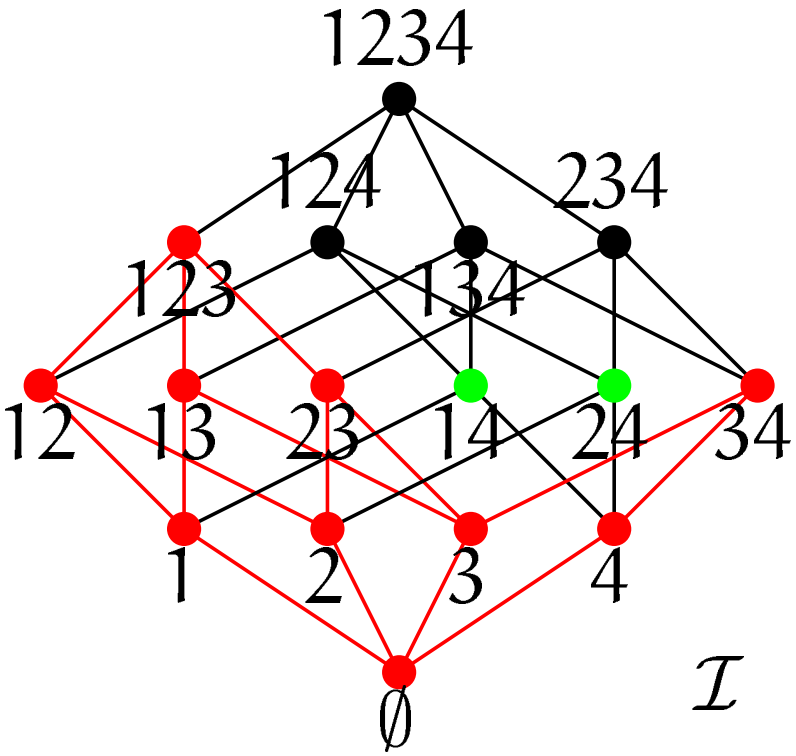
A characterization of clique complexes

Lemma 1a: $\mathcal{I}$ an independence system

$\mathcal{I}$ is a clique complex $\Longleftrightarrow$ every circuit of $\mathcal{I}$ is of size 2.

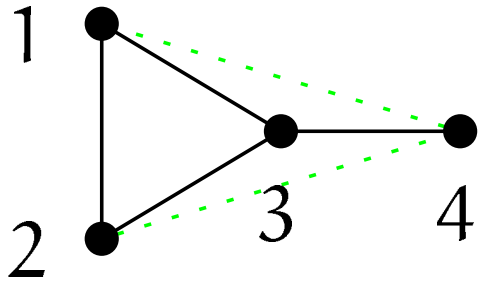
Particularly: every circuit of cliq(G) is a nonedge of G.

Proof.



$\mathcal{I}$

$\mathcal{I} = \text{cliq}(G)$



G

[qed]

Two-element circuits of matroids

$\mathcal{I}$ a matroid on V, without a loop
(meaning that $\{v\} \in \mathcal{I}$ for every $v \in V$)

Lemma 1b: $x, y, z \in V$

$\{x, y\}$ is a circuit of $\mathcal{I}$
$\{y, z\}$ is a circuit of $\mathcal{I}$ $\Longrightarrow$ $\{x, z\}$ is also a circuit of $\mathcal{I}$.

Proof. Suppose $\{x, z\} \in \mathcal{I}$.

$X = \{x, z\}$
$Y = \{y\}$ $\Rightarrow$(augmentation axiom)$\Rightarrow$ $\{x, y\} \in \mathcal{I}$ or $\{y, z\} \in \mathcal{I}$

Contradiction.

$\Rightarrow \{x, z\} \notin \mathcal{I} \Rightarrow \{x, z\}$ is a circuit. [qed]

We have two lemmas

Lemma 1a:

every circuit of $\mathrm{cliq}(G)$ is a nonedge of G.

Lemma 1b: $\mathcal{I}$ a matroid on V s.t. $\{v\} \in \mathcal{I}$ for every $v \in V$

$x, y, z \in V$

$\{x, y\}$ is a circuit of $\mathcal{I}$
$\{y, z\}$ is a circuit of $\mathcal{I}$ $\Longrightarrow$ $\{x, z\}$ is also a circuit of $\mathcal{I}$.

In particular: Two-element circuits give a partition of V.

Back to the proof of Theorem 1

Want to show:

$\mathrm{cliq}(G)$ is a matroid $\Longrightarrow$ G is complete r-partite.

Proof (cont'd). Assume: $\mathrm{cliq}(G)$ is a matroid.

From Lemma 1a:

Every circuit of $\mathrm{cliq}(G)$ is a nonedge of G.

From Lemma 1b and our assumption:

These circuits give a partition $\mathcal{P} = \{V_1, \ldots, V_r\}$ of V s.t. u, v is not adjacent in G iff $\{u, v\} \subseteq V_i$ for some i.

Therefore:

G is a complete r-partite graph. [qed]

The talk plan

In the rest of my talk

(1) clique complex which is a matroid

- Key concept: partition matroid

(2) clique complex which is the intersection of k matroids

- Key concept: stable-set partition

Def.: Stable-set partitions

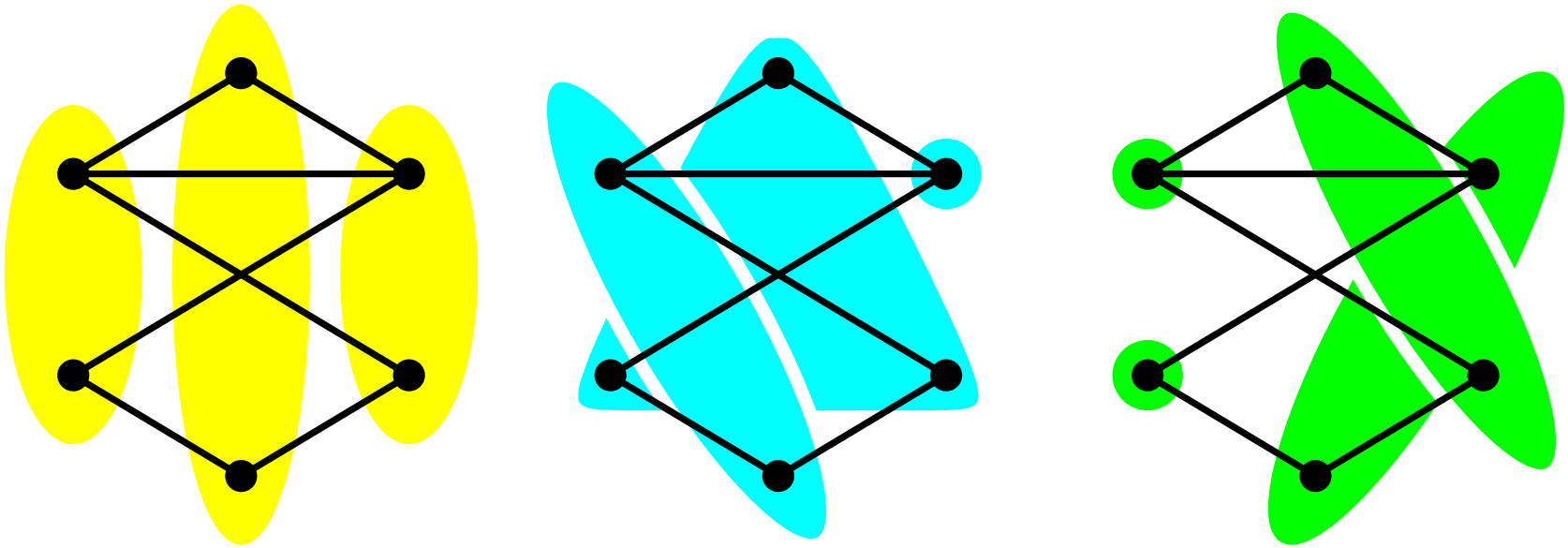
$G = (V, E)$ a graph

Def.: A stable-set partition of G

is a partition $\mathcal{P} = \{V_1, \ldots, V_r\}$ of V

s.t. V_i is a stable set of G for all $i = 1, \ldots, r$.

Example:



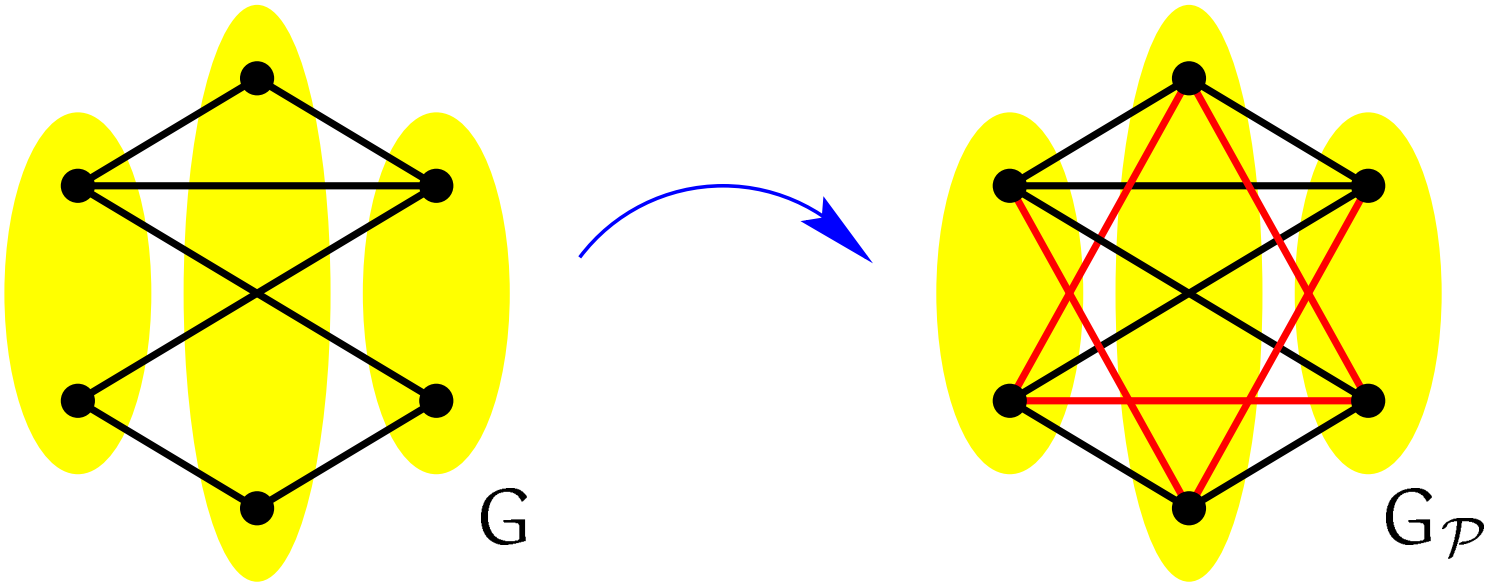
Construct a graph from a stable-set partition

G a graph
$\mathcal{P}$ a stable-set partition of G

Construct a complete r-partite graph $G_{\mathcal{P}}$ from $\mathcal{P}$.

Observation: An edge of G is also an edge of $G_{\mathcal{P}}$.

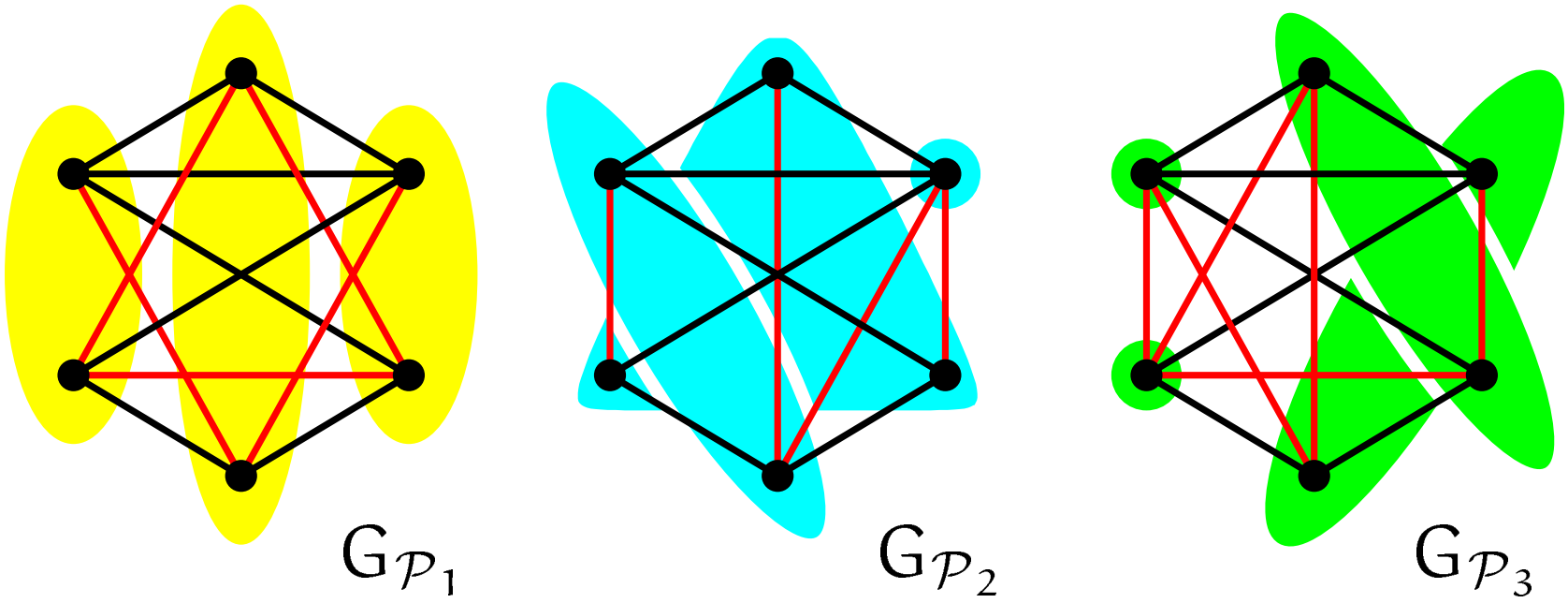
This means: $\mathrm{cliq}(G) \subseteq \mathrm{cliq}(G_{\mathcal{P}}) = \mathcal{I}(\mathcal{P})$.



Three stable-set partitions

Generally, $\mathrm{cliq}(G) \subseteq \mathrm{cliq}(G_{\mathcal{P}_1}) \cap \mathrm{cliq}(G_{\mathcal{P}_2}) \cap \mathrm{cliq}(G_{\mathcal{P}_3})$.

However, in this example,



$G_{\mathcal{P}_1}$ $G_{\mathcal{P}_2}$ $G_{\mathcal{P}_3}$

$$\begin{aligned}\mathrm{cliq}(G) &= \mathrm{cliq}(G_{\mathcal{P}_1}) \cap \mathrm{cliq}(G_{\mathcal{P}_2}) \cap \mathrm{cliq}(G_{\mathcal{P}_3}) \\ &= \mathcal{I}(\mathcal{P}_1) \cap \mathcal{I}(\mathcal{P}_2) \cap \mathcal{I}(\mathcal{P}_3).\end{aligned}$$

That is because

There is no red edge contained in all of these graphs.

To summarize

$G = (V, E)$ a graph

$\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k$ stable-set partitions of G

$$\text{cliq}(G) = \bigcap \{\mathcal{I}(\mathcal{P}_i) : i = 1, \ldots, k\}$$

$$\Updownarrow$$

There is no red edge contained in all $G_{\mathcal{P}_i}$'s.

$$\left(\begin{array}{l} \text{For every nonedge } \{u, v\} \notin E \\ \exists\, S \in \bigcup \{\mathcal{P}_i : i = 1, \ldots, k\} \text{ s.t. } \{u, v\} \subseteq S. \end{array} \right)$$

Theorem 2: $G = (V, E)$ a graph

$\mathrm{cliq}(G)$ is represented as the intersection of k matroids

$$\Updownarrow$$

$\exists$ k stable-set partitions $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k$ of G s.t. $\mathrm{cliq}(G) = \bigcap\{\mathcal{I}(\mathcal{P}_i) : i = 1, \ldots, k\}$

$$\Updownarrow$$

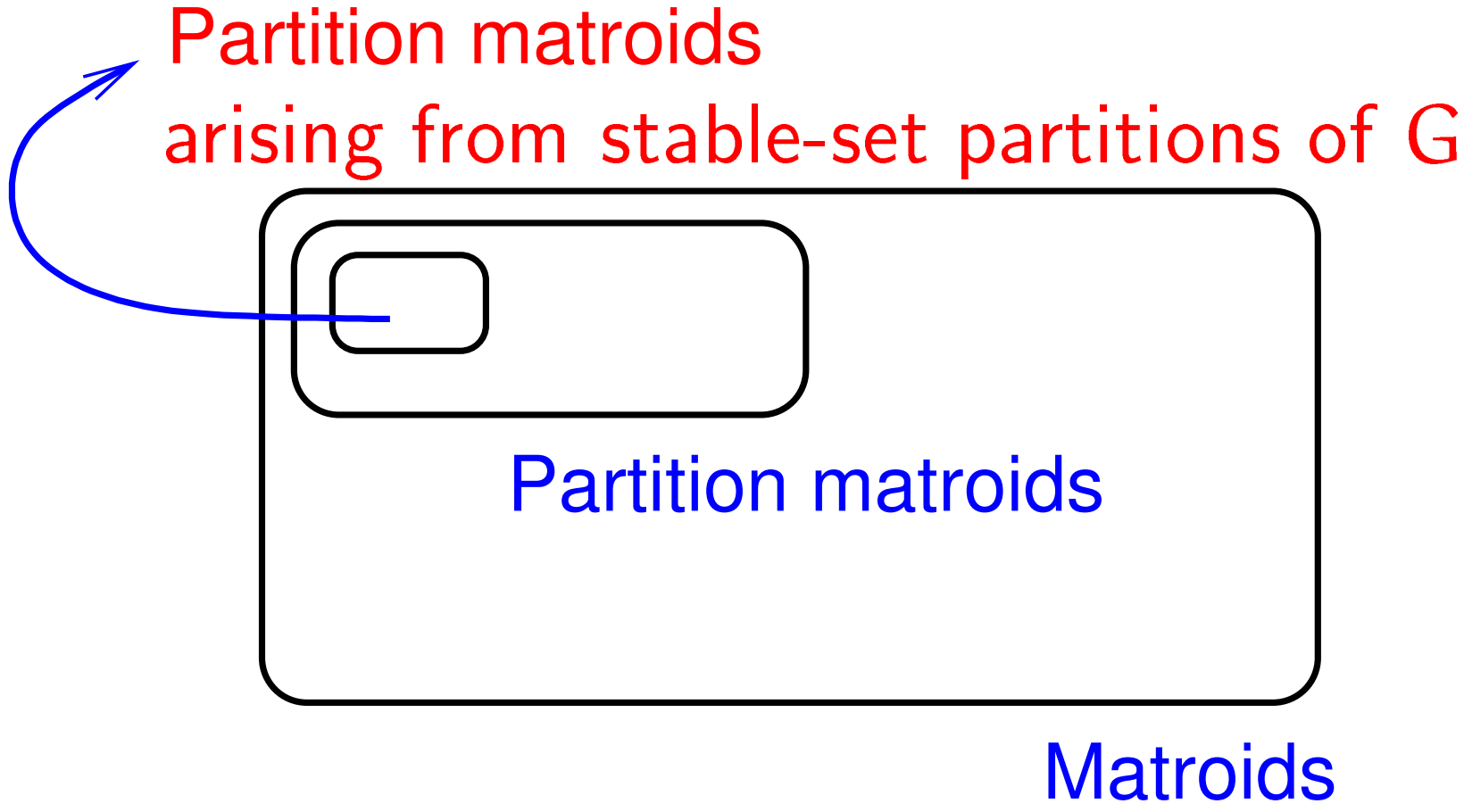
$\exists$ k stable-set partitions $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k$ of G s.t. there is no red edge contained in all $G_{\mathcal{P}_i}$'s.

$$\left(\begin{array}{l} \text{For every nonedge } \{u, v\} \notin E \\ \exists\ S \in \bigcup\{\mathcal{P}_i : i = 1, \ldots, k\} \text{ s.t. } \{u, v\} \subseteq S \end{array} \right)$$

Proof. Omitted.

What does this theorem mean? (1)

Given a graph G.



When we want a representation of $\mathrm{cliq}(G)$ by k matroids, just find k partition matroids from stable-set partitions of G.

What does this theorem mean? (2)

Consider the following decision problem.

Instance: a graph G and a positive integer k

Question: ? $\exists$ k matroids $\mathcal{I}_1, \ldots, \mathcal{I}_k$

s.t. $\mathrm{cliq}(G) = \mathcal{I}_1 \cap \cdots \cap \mathcal{I}_k$?

Theorem 2 implies: This problem $\in \mathcal{NP}$.

Summary

Question

- How many matroids do we need to represent a given clique complex as their intersection??

Result

- $\mathrm{cliq}(G)$ is the intersection of k matroids

$$\Updownarrow$$

$\mathrm{cliq}(G)$ is the intersection of k partition matroids arising from stable-set partitions of G

[End of the talk]

Here are extra slides for questions from the audience.

Complexity issue

Consider the following decision problem.

Fix: a positive integer k

Instance: a graph G

Question: ? $\exists$ k matroids $\mathcal{I}_1, \ldots, \mathcal{I}_k$
s.t. $\mathrm{cliq}(G) = \mathcal{I}_1 \cap \cdots \cap \mathcal{I}_k$?

What is the complexity of this problem for each k?

k	1	2	3	4	$\cdots$
	$\mathcal{P}$	$\mathcal{P}$	?	?	$\ldots$

Conj.: The problem for $k = 3$ is $\mathcal{NP}$-complete.

Clique complexes represented by two matroids

Theorem 3: G a graph

$\mathrm{cliq}(G)$ is the intersection of two matroids

$\Updownarrow$

The stable-set graph of G is bipartite.

Proof. Omitted.

Note: Combining $\begin{cases} \text{Theorem 3 and} \\ \text{a result by Protti–Szwarcfiter ('02),} \end{cases}$

we can tell whether

the clique complex of a given graph
is the intersection of two matroids or not

in polynomial time.

Graphs as independence systems

A graph can be seen as an independence system like

$$X \in G \quad \Leftrightarrow \quad X = \emptyset \text{ or } \{v\} \text{ or } \{u, v\} \text{ if it's an edge.}$$

Theorem 4: G a graph

$$G \text{ is the intersection of } k \text{ matroids}$$

$$\Updownarrow$$

$$\text{cliq}(G) \text{ is the intersection of } k \text{ matroids.}$$

Proof. Omitted.

Number of matroids we need

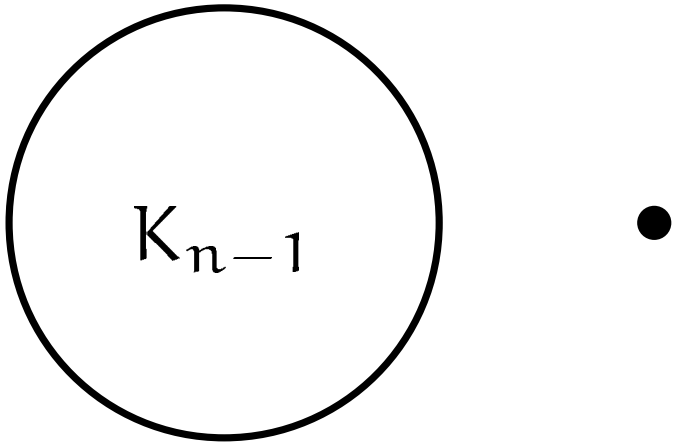
Let

$$k(G) = \min\{k : \text{cliq}(G) \text{ is the intersection of } k \text{ matroids }\}.$$

$$k(n) = \max\{k(G) : G \text{ has } n \text{ vertices }\}.$$

Theorem 5: $k(n) = n - 1$.

Proof. "$k(n) \geq n - 1$" follows from the example below.



"$k(n) \leq n - 1$" ... a proof omitted.