♠♢♣♡♠♢♣♡

Traveling Salesman Games with the Monge Property

Yoshio Okamoto

ETH Zürich

okamotoy@inf.ethz.ch

June 4, 2002

♠♢♣♡♠♢♣♡

Traveling salesman problem (TSP)

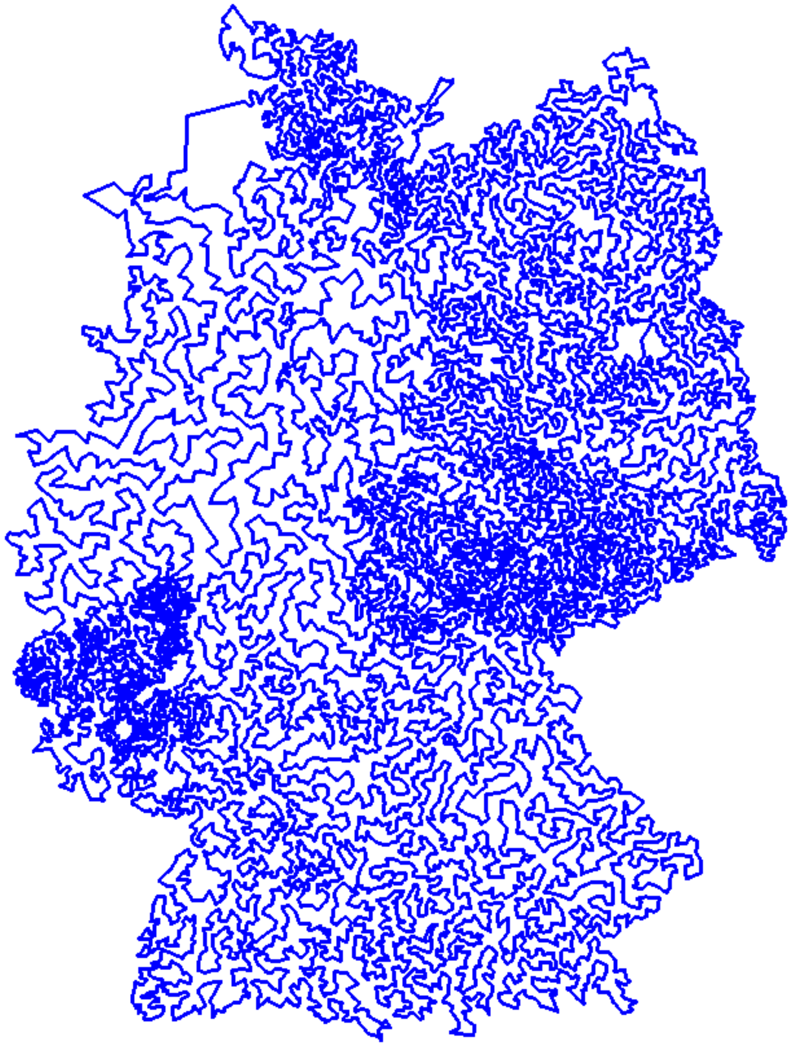
One of the most famous $\mathcal{NP}$-hard problems.

A salesman wants to travel along a tour of minimum cost.

$\Downarrow$

Find such a tour!

15,112 cities in Germany $\rightarrow$



An optimal tour by Applegate–Bixby–Chvátal–Cook ('01)

Traveling salesman game (TSG)

TSG is a kind of cost allocation problems

- Asian countries invite a football team from Europe.
- The invited team visits each country.

$\Rightarrow$ How do these countries pay for the travel cost?

We consider how to allocate the total cost to each country.

Contributions to traveling salesman games (TSG)

- It is $\mathcal{NP}$-hard to test the core non-emptiness of a TSG.
- The core of a TSG with the Monge property is always non-empty.

Outline

1. Definitions of TSG and the core
2. Description of the first result
3. The Monge property
4. Description of the second result
5. Proof
6. Remarks

Def: Traveling salesman game

Potters–Curiel–Tijs ('92)

$N = \{0, \ldots, n\}$	Countries
$h \in N$	fixed (Home of an invited team)
C	$N \times N$ cost matrix
$N_h := N \setminus \{h\}$	players (Asian countries)
$S \subseteq N_h$	coalition
$v_C : 2^{N_h} \to \mathbb{R}$	characteristic function

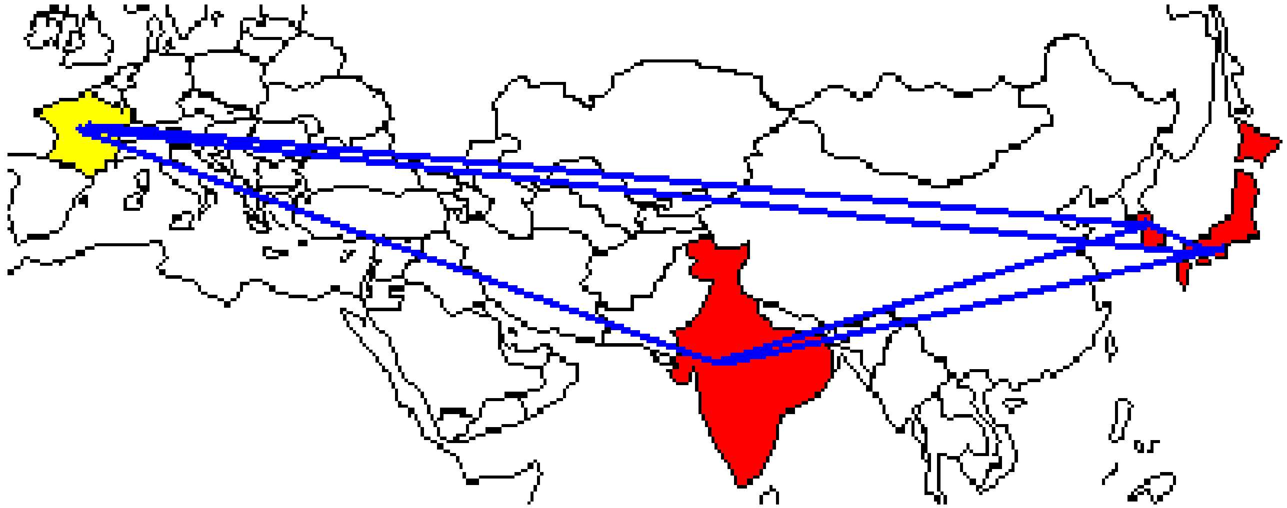
$v_C(S)$ = the cost of a shortest tour of $S \cup \{h\}$
= the optimal value of TSP on $S \cup \{h\}$
(determined by C)

(N_h, v_C)	traveling salesman game (TSG)

Goal: To allocate "fairly" the cost to each player

Example: Traveling salesman game

$N = \{$France, India, Japan, Korea$\}$, $h =$ France



$$
\begin{array}{llll}
v_C(\emptyset) = & 0 & v_C(\{\mathsf{I},\mathsf{J}\}) = & 23 \\
v_C(\{\mathsf{I}\}) = & 14 & v_C(\{\mathsf{I},\mathsf{K}\}) = & 21 \\
v_C(\{\mathsf{J}\}) = & 20 & v_C(\{\mathsf{J},\mathsf{K}\}) = & 20 \\
v_C(\{\mathsf{K}\}) = & 18 & v_C(\{\mathsf{I},\mathsf{J},\mathsf{K}\}) = & 23 \ (\times 10^3 \mathrm{km})
\end{array}
$$

We want to allocate fairly $v_C(\{\mathsf{I},\mathsf{J},\mathsf{K}\}) = 23$ to $\{\mathsf{I},\mathsf{J},\mathsf{K}\}$.

Assumptions on the matrix

Some assumptions on the cost matrix C

- The diagonal entries of C are all 0.
- The non-diagonal entries of C are all positive.
- C is symmetric.

Remark that

★ C may not satisfy the triangle inequality.

$x \in \mathbb{R}^{N_h}$ is a core allocation of (N_h, v_C) if

$$\sum \{x[i] : i \in N_h\} = v_C(N_h),$$

(x is an allocation,)

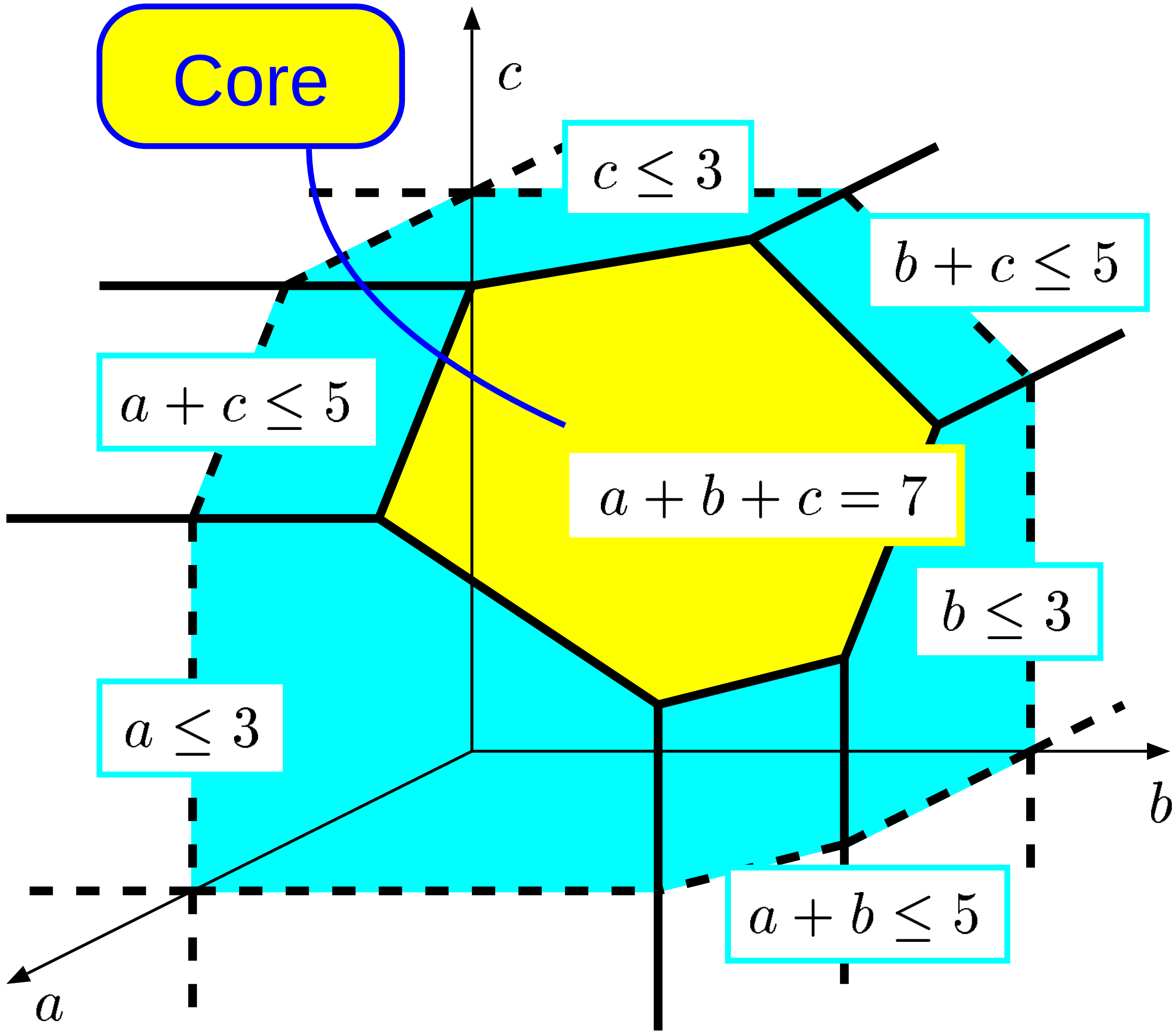
$$\sum \{x[i] : i \in S\} \leq v_C(S) \quad \text{for all } S \subseteq N_h,$$

(each coalition $S \subseteq N_h$ is satisfied with x.)

- A core allocation represents a “fair” allocation.

The core of (N_h, v_C) is the set of all core allocations.

Example: Core



$N_h = \{a, b, c\}$	
$v_C(\emptyset)$	0
$v_C(\{a\})$	3
$v_C(\{b\})$	3
$v_C(\{c\})$	3
$v_C(\{a, b\})$	5
$v_C(\{a, c\})$	5
$v_C(\{b, c\})$	5
$v_C(\{a, b, c\})$	7

[Remark] The core is possibly empty.

$\Longrightarrow$ There may not exist a core allocation!!

It's important to test the core non-emptiness!!

Is it easy?

$\Longrightarrow$ <u>We consider the complexity of this problem.</u>

The formal description of the problem:

Problem:

CORE NON-EMPTINESS OF TSG

Instance:

N	countries
$h \in N$	home
C	$N \times N$ symm. cost matrix

Question:

Is the core of the TSG (N_h, v_C) non-empty?

Theorem

CORE NON-EMPTINESS OF TSG is $\mathcal{NP}$-hard.

Idea of Proof

Reduction from HAMILTONIAN PATH PROBLEM.

This result implies that

- we're unlikely to have a good characterization of core non-emptiness of TSG.

In the literature

The core is non-empty

when C satisfies the triangle inequality and

- TSG = a routing game (Potters, '89)
- $n \leq 4$ (Tamir, '89)
- $n = 5$ (Kuipers, '93)

We show

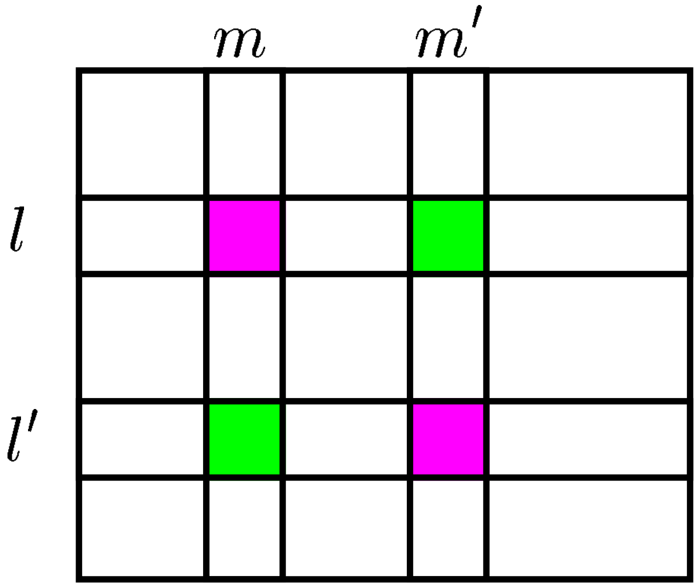
another condition based on the Monge property.

Def: Monge matrix

C is a Monge matrix if C satisfies

$$\boxed{c[l,m] + c[l',m']} \leq \boxed{c[l,m'] + c[l',m]}$$

for all $l < l'$ and $m < m'$.



Theorem

TSG with a symmetric Monge cost matrix has the non-empty core.

. . . .

This result is important since

- it's independent from # of the players,
- it suggests the relationship between
  - the core non-emptiness of games
  - well-solved cases for optimization.

Monge matrices and TSP

[Proposition] (Klyaus, '76; Gilmore–Lawler–Shmoys, '85)

TSP with a Monge matrix is solved in $\mathrm{O}(n^2)$.

[Proposition] (Supnick, '57)

TSP with a symmetric Monge matrix has an optimal tour as

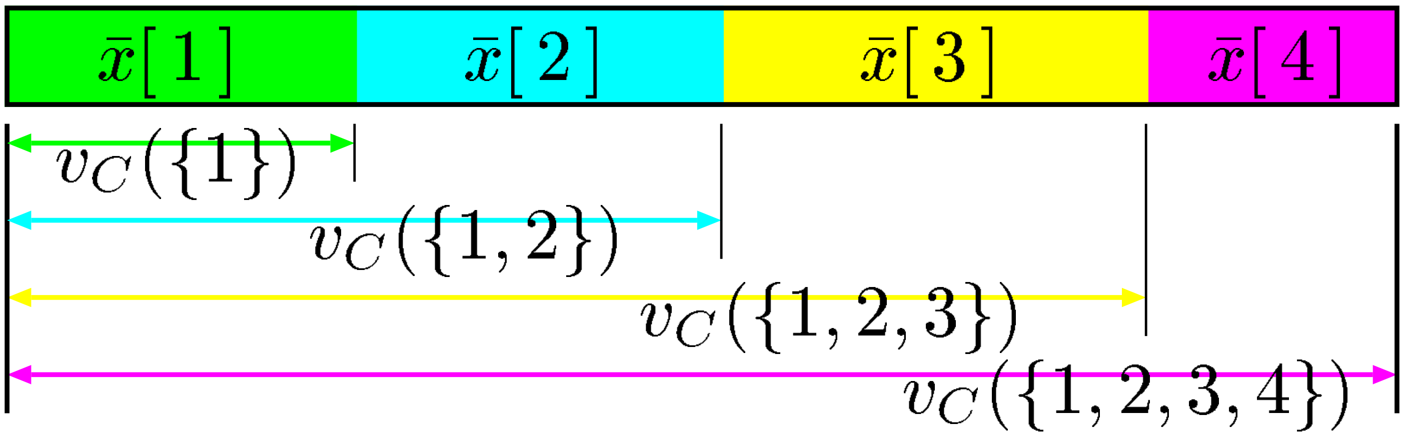
Proof (1/11): Construction

We will look at a proof for $h = 0$ only.
(the general case is more involved.)

Construct a core allocation $\bar{x} \in \mathbb{R}^{N_0}$:

$$\bar{x}[\,i\,] = v_C(\{1, 2, \dots, i\}) - v_C(\{1, 2, \dots, i-1\}).$$

(For a case of $N = \{0, 1, 2, 3, 4\}$, $N_0 = \{1, 2, 3, 4\}$)



To show: $\bar{x}$ is indeed a core allocation.

Proof (2/11): Induction

- $\sum\{\bar{x}[i] : i \in N_0\} = v_C(N_0)$ — easily shown
- $\sum\{\bar{x}[i] : i \in S\} \leq v_C(S)$ for all $S \subseteq N_0$

  — to be shown

$\Rightarrow$ Induction: suppose that

$$\sum\{\bar{x}[i] : i \in S\} \leq v_C(S) \quad \text{for all } S \subseteq N_0 \setminus \{n\}$$

Proof (3/11): Calculation

We want to show for all $S \subseteq N_0 \setminus \{n\}$

$$\sum\{\bar{x}[i] : i \in S \cup \{n\}\} \leq v_C(S \cup \{n\})$$

$$\begin{aligned}
&\sum\{\bar{x}[i] : i \in S \cup \{n\}\} \\
&\quad = \bar{x}[n] + \sum\{\bar{x}[i] : i \in S\} \\
&\quad \leq \bar{x}[n] + v_C(S) \qquad \text{(the induction hypothesis)} \\
&\quad = v_C(N_0) - v_C(N_0 \setminus \{n\}) + v_C(S) \\
&\qquad\quad \text{(construction of } \bar{x}\text{)}
\end{aligned}$$

Then, it suffices to show

$$\boxed{v_C(N_0) - v_C(N_0 \setminus \{n\})} \le \boxed{v_C(S \cup \{n\}) - v_C(S)}.$$

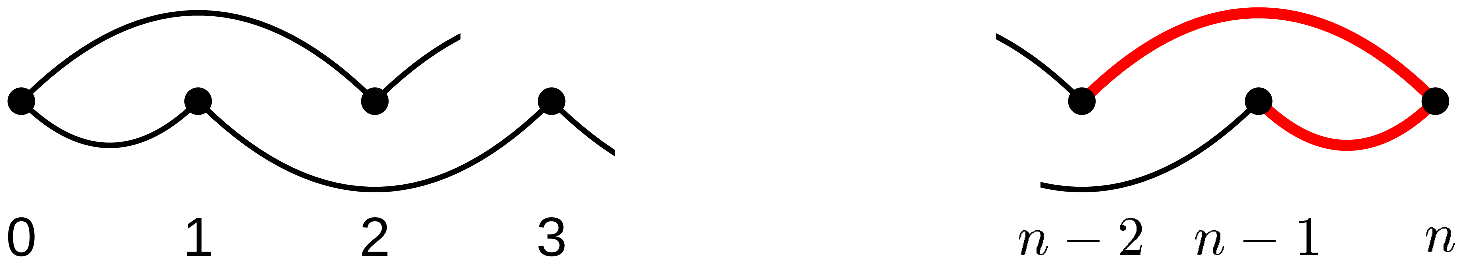
Let $S = \{i_1, i_2, \ldots, i_s\} \subseteq N_0 \setminus \{n\}$ with $i_1 < \cdots < i_s$,

$$\begin{aligned}
\boxed{\text{LHS}} &= v_C(N_0) - v_C(N_0 \setminus \{n\}) \\
&= c[\,n-2, n\,] + c[\,n-1, n\,] - c[\,n-2, n-1\,], \\
\boxed{\text{RHS}} &= v_C(S \cup \{n\}) - v_C(S) \\
&= c[\,i_{s-1}, n\,] + c[\,i_s, n\,] - c[\,i_{s-1}, i_s\,].
\end{aligned}$$

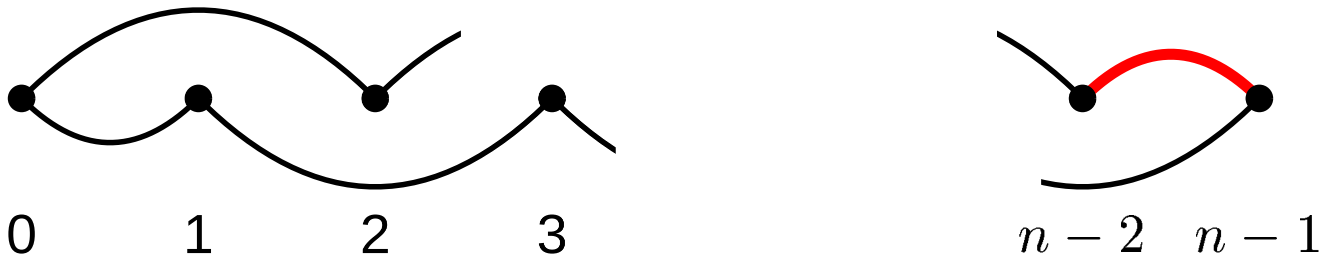
Proof (5/11): Why is it?

Remember
the form of an optimal solution for symm. Monge matrices:

$v_C(N)$



$v_C(N \setminus \{n\})$



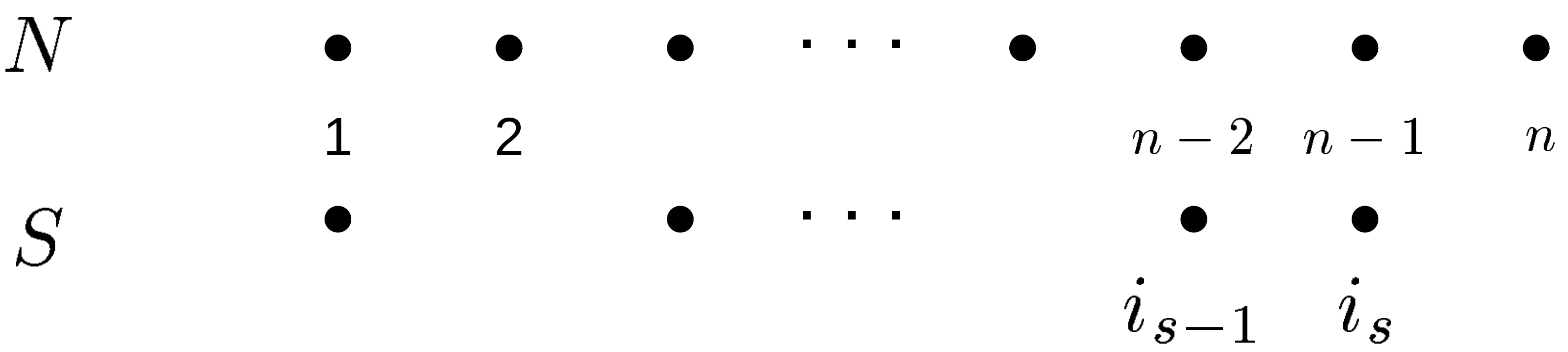
We have four cases:

- $\boxed{\text{Case 1}}$: $i_{s-1} = n-2$, $i_s = n-1$
- $\boxed{\text{Case 2}}$: $i_{s-1} < n-2$, $i_s = n-1$
- $\boxed{\text{Case 3}}$: $i_{s-1} < n-2$, $i_s = n-2$
- $\boxed{\text{Case 4}}$: $i_{s-1} < n-2$, $i_s < n-2$

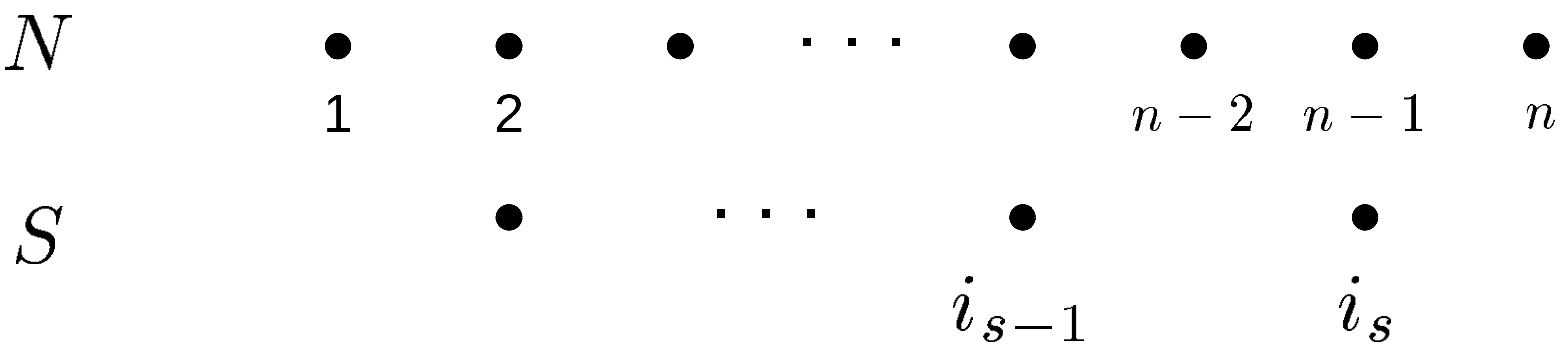
[Note] Always, $i_{s-1} \leq n-2$, $i_s \leq n-1$.

Proof (7/11): Cases 1 and 2

- $\boxed{\text{Case 1}}$: $i_{s-1} = n-2,\ i_s = n-1$



- $\boxed{\text{Case 2}}$: $i_{s-1} < n-2,\ i_s = n-1$



- $\boxed{\text{Case 3}}$: $i_{s-1} < n-2,\ i_s = n-2$

N	$\bullet$	$\bullet$	$\bullet$	$\cdots$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
	1	2				$n-2$	$n-1$	n
S		$\bullet$	$\cdots$	$\bullet$		$\bullet$		
				i_{s-1}		i_s		

- $\boxed{\text{Case 4}}$: $i_{s-1} < n-2,\ i_s < n-2$

N	$\bullet$	$\bullet$	$\bullet$	$\cdots$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
	1	2				$n-2$	$n-1$	n
S	$\bullet$	$\bullet$	$\cdots$	$\bullet$	$\bullet$			
				i_{s-1}	i_s			

$\boxed{\text{Case 1}}$: $i_{s-1} = n-2,\ i_s = n-1$

$$\boxed{\text{LHS}} = c[\,n-2, n\,] + c[\,n-1, n\,] - c[\,n-2, n-1\,],$$

$$\boxed{\text{RHS}} = c[\,i_{s-1}, n\,] + c[\,i_s, n\,] - c[\,i_{s-1}, i_s\,]$$

$$= c[\,n-2, n\,] + c[\,n-1, n\,] - c[\,n-2, n-1\,].$$

Hence, $\boxed{\text{LHS}} = \boxed{\text{RHS}}$ OK.

$\boxed{\text{Case 2}}$: $i_{s-1} < n-2,\ i_s = n-1$

$$\boxed{\text{LHS}} = c[\,n-2, n\,] + c[\,n-1, n\,] - c[\,n-2, n-1\,],$$

$$\begin{aligned}\boxed{\text{RHS}} &= c[\,i_{s-1}, n\,] + c[\,i_s, n\,] - c[\,i_{s-1}, i_s\,] \\ &= c[\,i_{s-1}, n\,] + c[\,n-1, n\,] - c[\,i_{s-1}, n-1\,].\end{aligned}$$

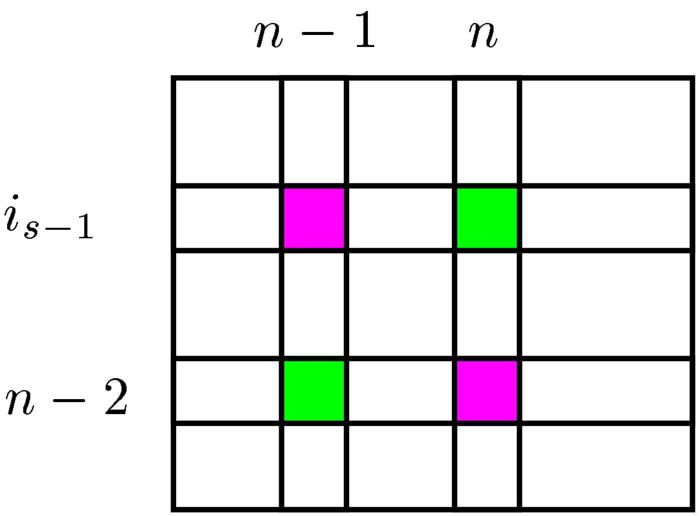
Hence,

$$\begin{aligned}\text{LHS} - \text{RHS} = {} & c[\,n-2, n\,] - c[\,n-2, n-1\,] \\ & - c[\,i_{s-1}, n\,] + c[\,i_{s-1}, n-1\,]\end{aligned}$$

Proof (11/11): Case 2 (cont'd)

$\boxed{\text{Case 2}}$: $i_{s-1} < n-2,\ i_s = n-1$

$$\boxed{\text{LHS}} - \boxed{\text{RHS}} = \boxed{c[n-2, n]} - \boxed{c[n-2, n-1]}$$
$$- \boxed{c[i_{s-1}, n]} + \boxed{c[i_{s-1}, n-1]}$$
$$\leq 0 \qquad \text{(Monge property)}$$



Hence,

$\boxed{\text{LHS}} \leq \boxed{\text{RHS}}$ OK.

$\boxed{\text{Cases 3,4}}$ can be shown similarly. [QED]

How about other well solvable cases??

C: a symmetric matrix

- symm. Monge matrices

  $$c[l,m] + c[l',m'] \le c[l,m'] + c[l',m] \quad (l < l',\ m < m')$$

- Kalmanson matrices

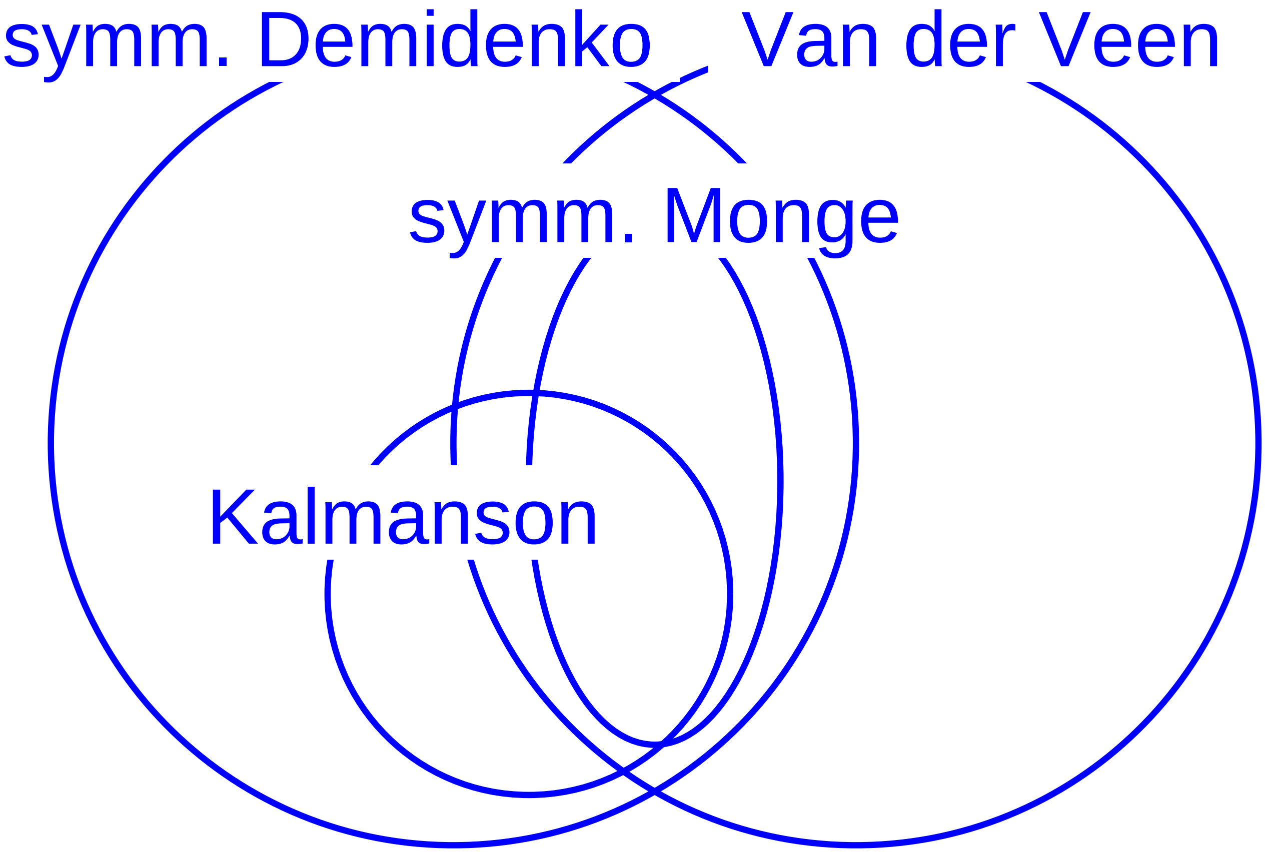
  $$\begin{aligned} &c[i,j] + c[k,l] \le c[i,k] + c[j,l], \\ &c[i,l] + c[j,k] \le c[i,k] + c[j,l] \qquad (i < j < k < l) \end{aligned}$$

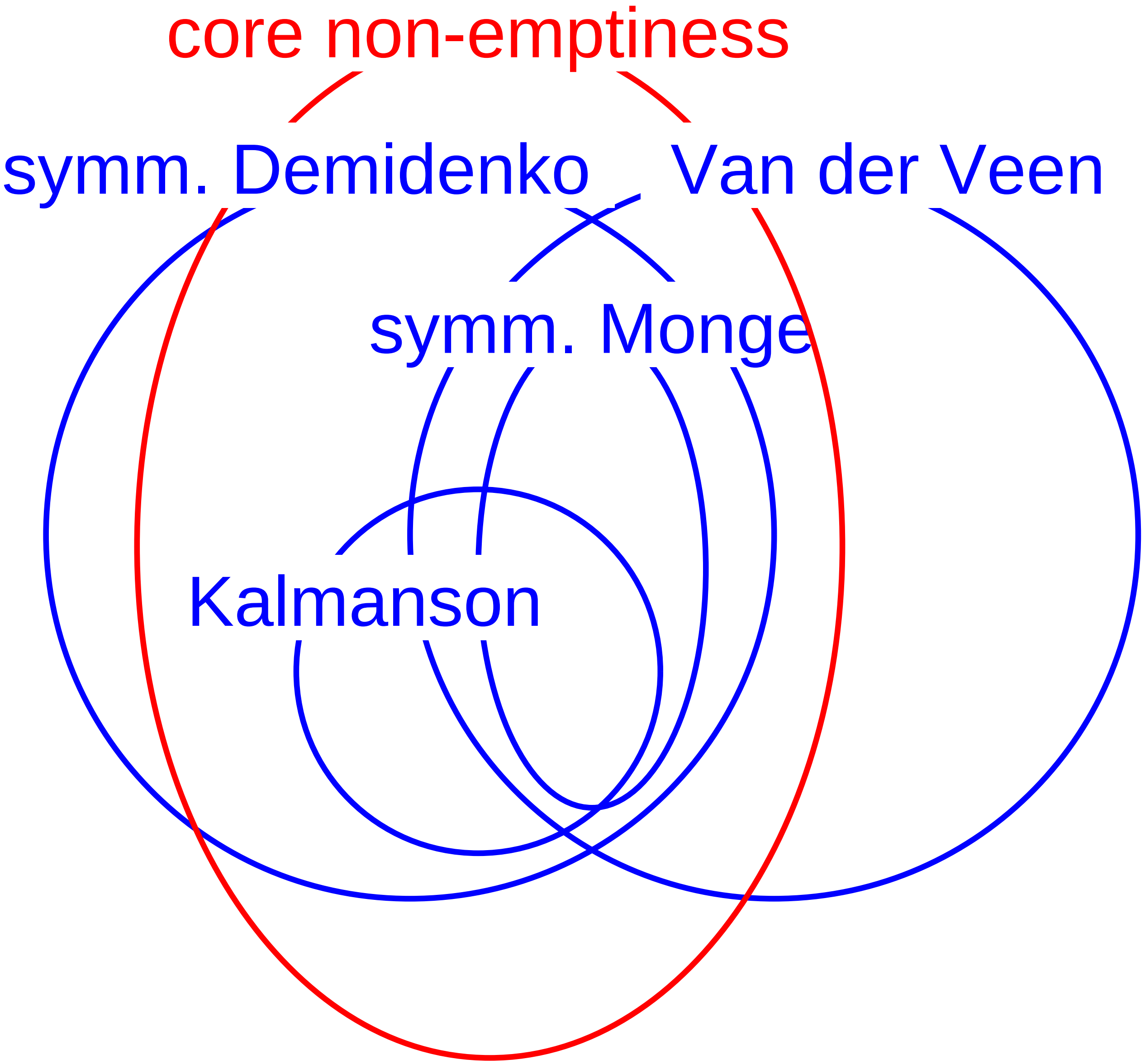
- symm. Demidenko matrices

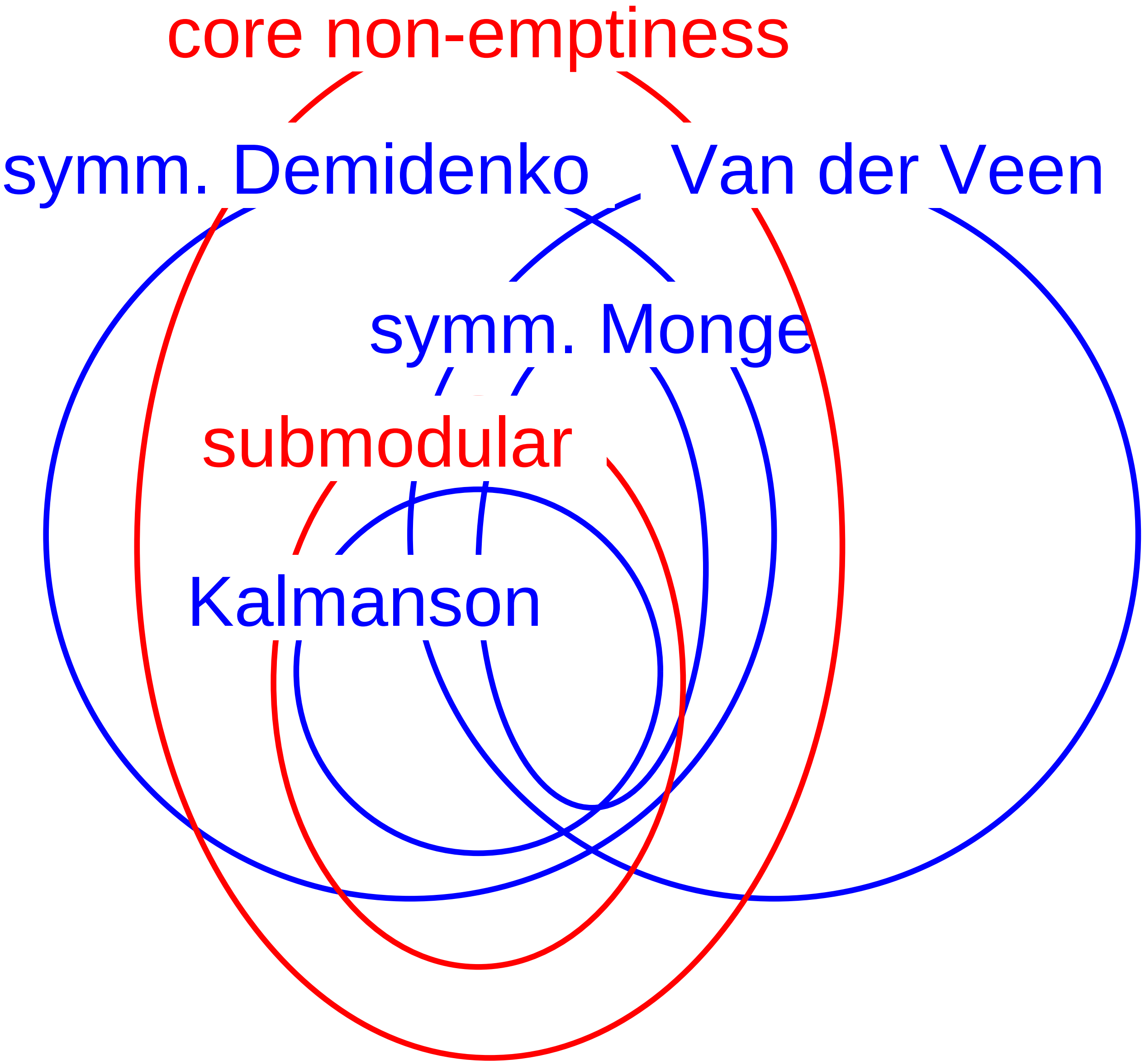
  $$c[i,j] + c[k,l] \le c[i,k] + c[j,l] \qquad (i < j < k < l)$$

- Van der Veen matrices

  $$c[i,j] + c[k,l] \le c[i,l] + c[j,k] \qquad (i < j < k < l)$$


symm. Demidenko
Van der Veen
symm. Monge
Kalmanson


core non-emptiness
symm. Demidenko
Van der Veen
symm. Monge
Kalmanson


core non-emptiness
symm. Demidenko
Van der Veen
symm. Monge
submodular
Kalmanson

(N_h, v_C) is submodular (or concave) if for all $S, T \subseteq N_h$

$$v_C(S) + v_C(T) \geq v_C(S \cup T) + v_C(S \cap T).$$

Basic property:

A game is submodular $\Rightarrow$ it has the non-empty core.
(Shapley,'71)

Theorem

TSG with a Kalmanson cost matrix is submodular.

Questions

- How about other solution concepts?

  (solution concepts $\approx$ criteria of fairness)
- How about asymmetric TSP?
- Can we obtain new well solvable cases
  from the core non-emptiness results for TSG?

  e.g.: facility location games (Goemans–Skutella, '00)
- How about other optimization problems?

Combinatorial optimization games
for which testing the core non-emptiness is intractable:

- Min coloring games (Deng–Nagamochi–Ibaraki, '99)
- Facility location games (Goemans–Skutella, '00)
- Traveling salesman games (Okamoto)

Combinatorial optimization games
with the non-empty cores:

- Assignment games (Shapley–Shubik, '72)
- Min-cost spanning tree games (Granot–Huberman, '81)
- Simple flow games (Kalai–Zemel, '82)

Combinatorial optimization games
with good characterizations of core non-emptiness:

- Min-cost base games on matroids (Nagamochi–Zeng–Kabutoya–Ibaraki, '97)
- Max matching games (Deng–Nagamochi–Ibaraki, '99)
- Min vertex cover games (Deng–Nagamochi–Ibaraki, '99)
- Min edge cover games (Deng–Nagamochi–Ibaraki, '99)
- Max independent set games (Deng–Nagamochi–Ibaraki, '99)
- Delivery games (Granot–Hamers–Tijs, '99)

Contribution to traveling salesman games

- It is $\mathcal{NP}$-hard to test the core non-emptiness of a traveling salesman game.
- The core of a traveling salesman game with the Monge property is always non-empty.

What I want to tell you

- When you have an optimization problem,
  go by one step further,
  then you will find a game-theoretic problem.

Thanks to

- http://www.math.princeton.edu/tsp/
(World record of TSP, and more information on TSP)
- http://www.geographic.org/
(Map of Asia)

You can get my slides from

http://www.inf.ethz.ch/personal/okamotoy/.

A cooperative game (or a game) is a pair (X, γ) of

- a finite set X
  (called the set of players)
- a function $\gamma : 2^X \to \mathbb{R}$ with $\gamma(\emptyset) = 0$
  (called the characteristic function).

For $S \subseteq X$,

$\gamma(S)$ represents $\left\{ \begin{array}{c} \text{the min. possible cost} \\ \text{the max. possible profit} \end{array} \right\}$

owed by the players of S when they cooperate.

In TSG, $X = N_h$, $\gamma = v_C$.