The affine representation theorem for abstract convex geometries

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ETH Zürich

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Joint work with
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Combinatorics
Geometry
Computation

Combinatorial abstract models of geometric concepts

- **Matroids** ..... abstraction of dependence

  Application: { Finite geometry; Coding theory; Combinatorial optimization }

- **Oriented Matroids** ..... abstraction of dependence

  Application: { Convex polytopes; Computational geometry; Discrete geometry; Optimization }

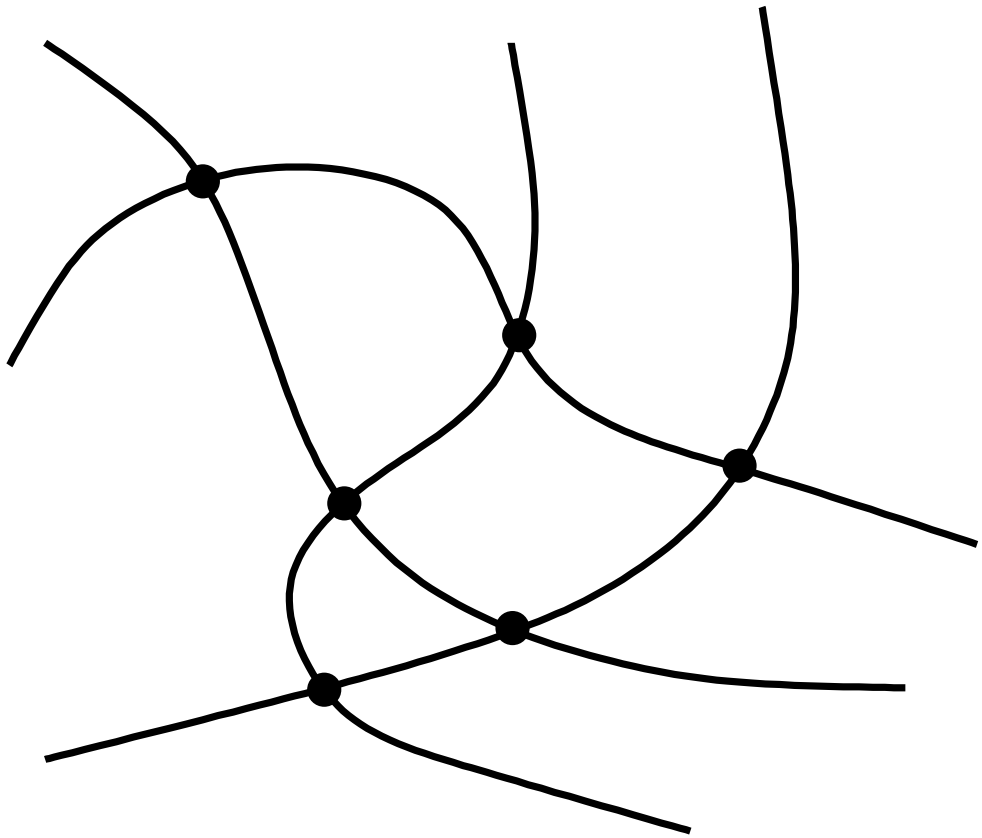
- **Convex geometries** ..... abstraction of convexity

  Application: { Discrete geometry; Social choice theory; Mathematical psychology }

$\sqrt[2]{}$ The topological representation theorem for oriented matroids

One of the most important theorem for oriented matroids
(Folkman–Lawrence, '78)

Any (simple) oriented matroid is isomorphic to some pseudohyperplane arrangement.

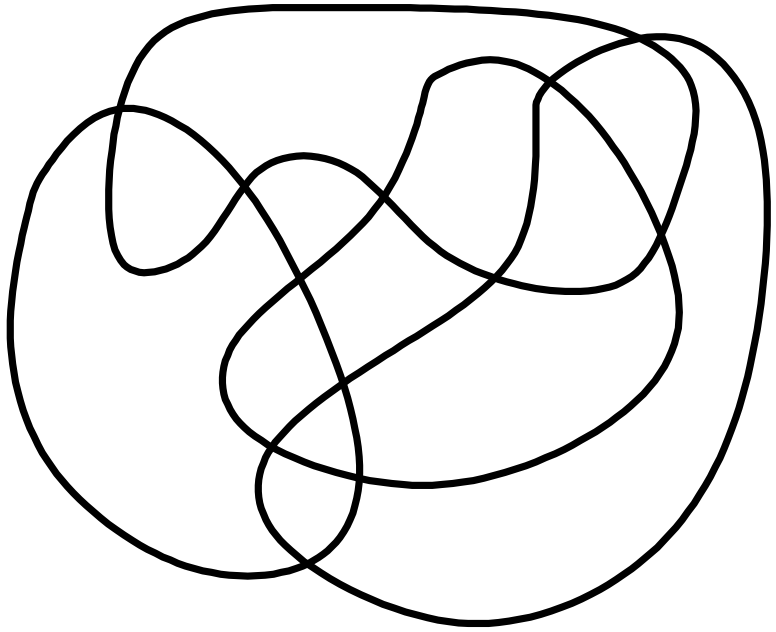


The topological representation theorem for matroids

A quite recent progress

(Swartz, '02)

Any (simple) matroid is isomorphic to some homotopy-sphere arrangement.



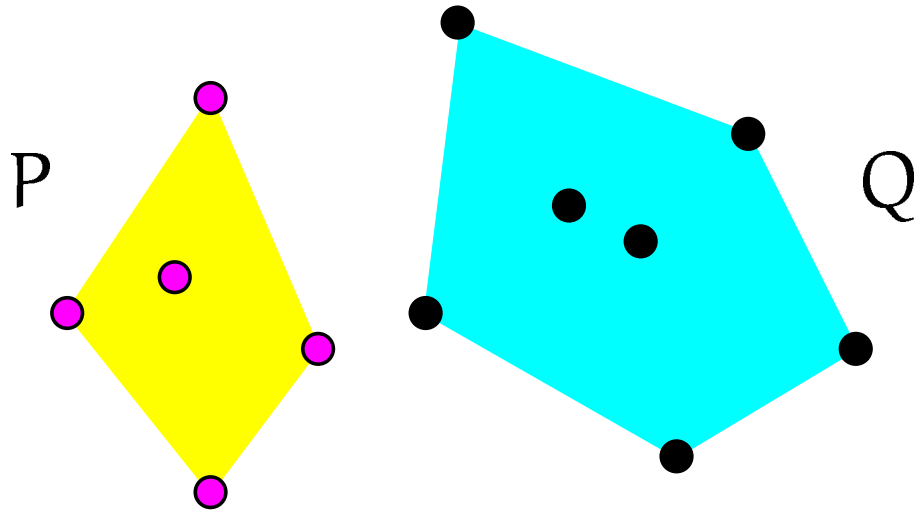
Question:

What is a

representation theorem for convex geometries?

Our Theorem:

Any convex geometry is isomorphic to
some generalized convex shelling,



determined by two point sets P and Q
satisfying that $\mathrm{conv}(\mathrm{P}) \cap \mathrm{conv}(\mathrm{Q}) = \emptyset$.

This gives an affine representation of a convex geometry.

Our Theorem:

Any convex geometry is isomorphic to
some generalized convex shelling.

In the rest of my talk

- Definition of a convex geometry
- Examples of a convex geometry
- Definition of a generalized convex shelling
- Our theorem
- Outline of the proof

Convex geometries

(Edelman–Jamison '85)

E a nonempty finite set

$\mathcal{L}$ a nonempty family of subsets of E

Def. $\mathcal{L} \subseteq 2^E$ is called a convex geometry on E if $\mathcal{L}$ satisfies the following three conditions.

(1) $\emptyset \in \mathcal{L}$, $E \in \mathcal{L}$.

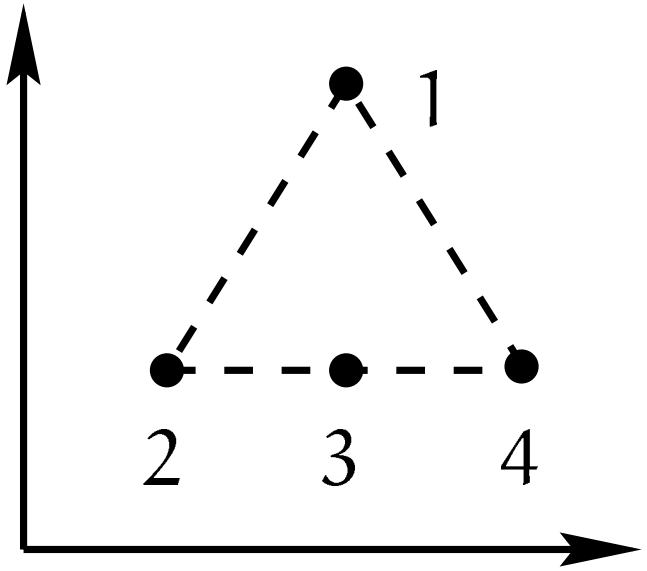
(2) $X, Y \in \mathcal{L} \Longrightarrow X \cap Y \in \mathcal{L}$.

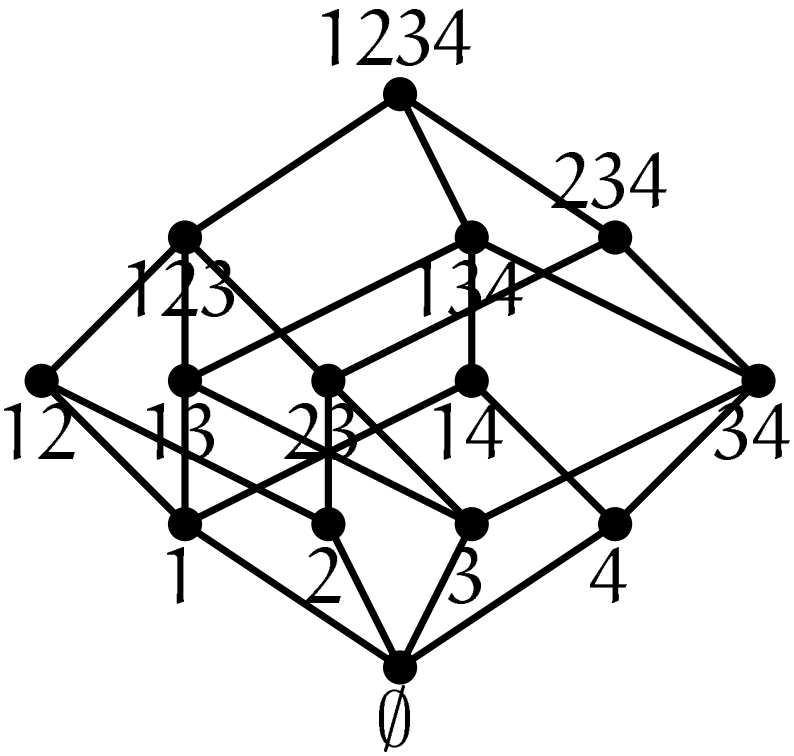
(3) $X \in \mathcal{L} \setminus \{E\} \Longrightarrow \exists\, e \in E \setminus X$ s.t. $X \cup \{e\} \in \mathcal{L}$.

Example 1: convex shelling

Q a finite point set in $\mathbb{R}^d$

Define: $\mathcal{L} = \{X \subseteq Q : \mathrm{conv}(X) \cap (Q \setminus X) = \emptyset\}$.





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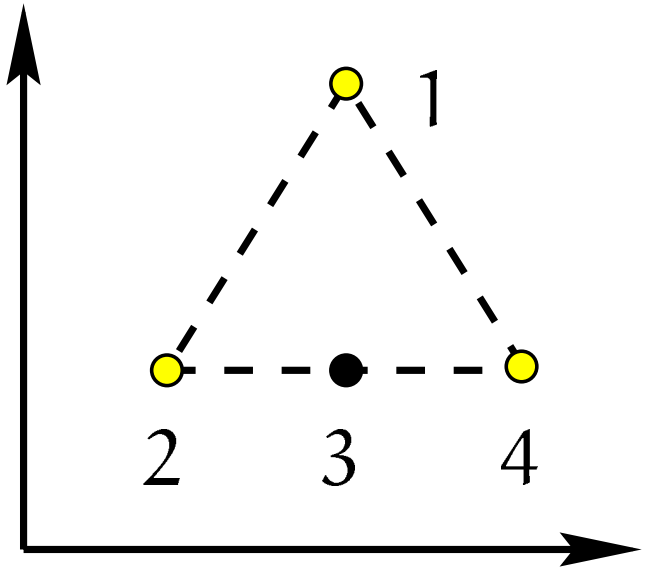

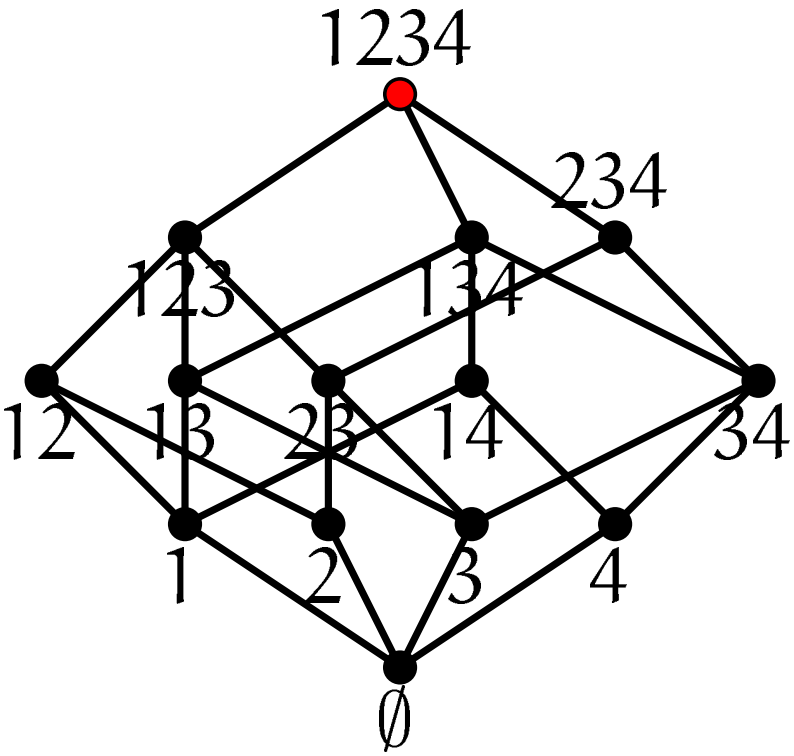

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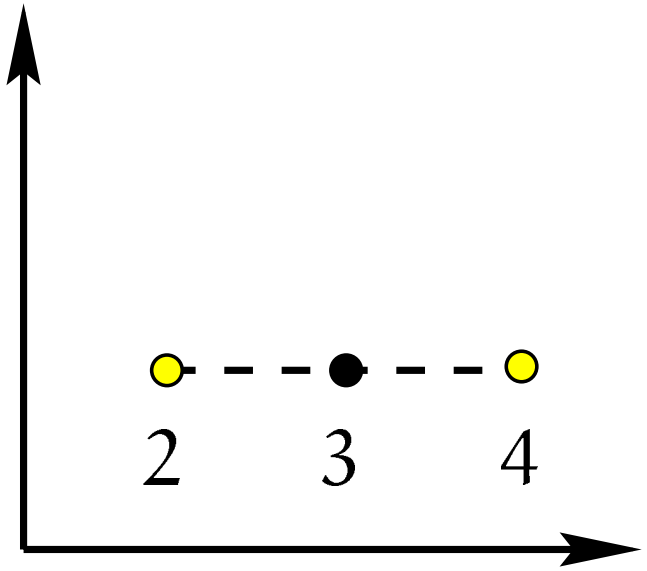

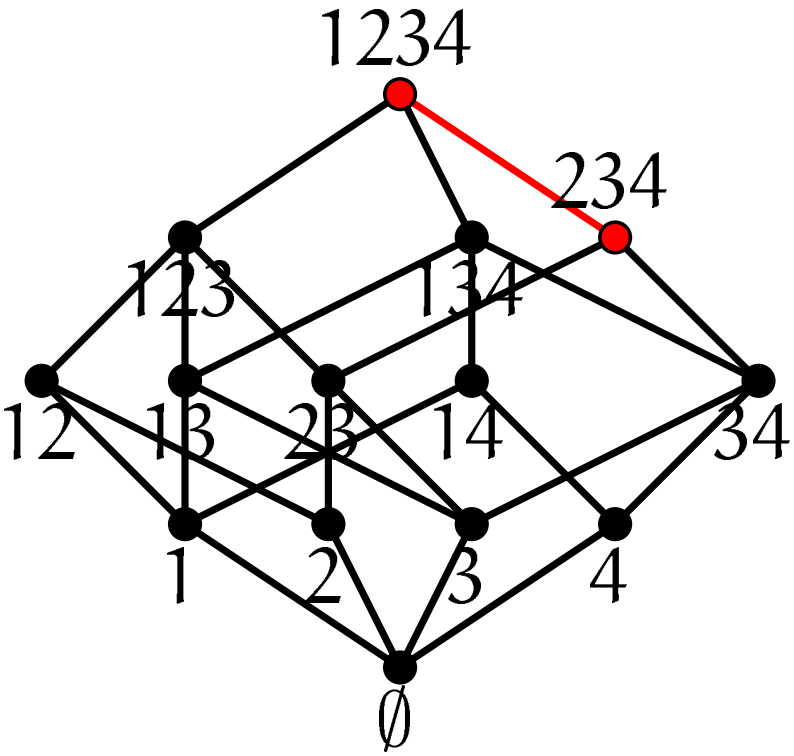

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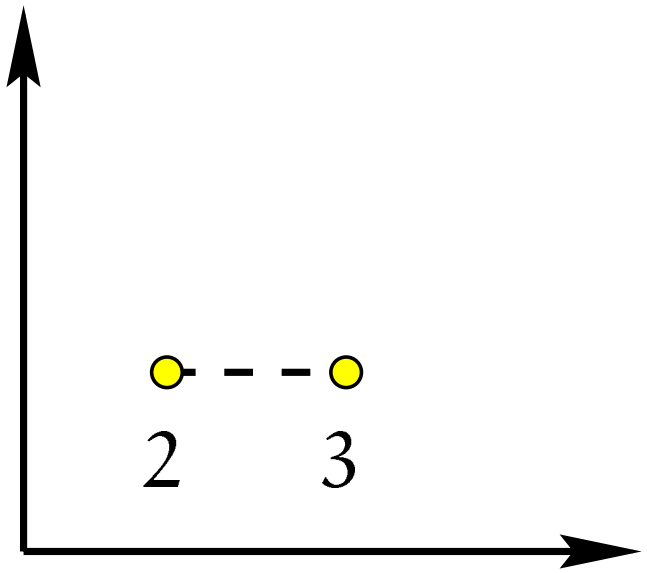

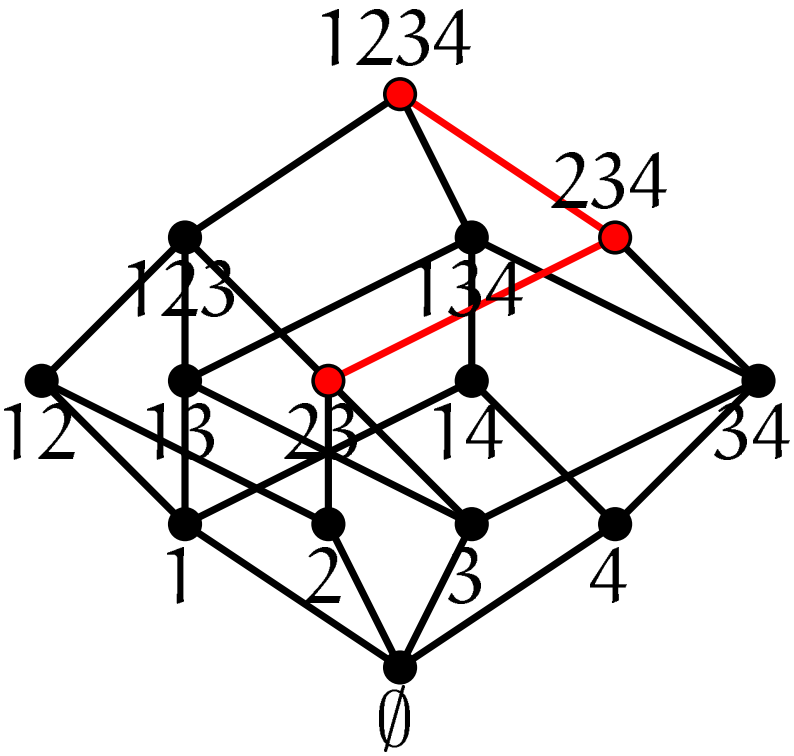

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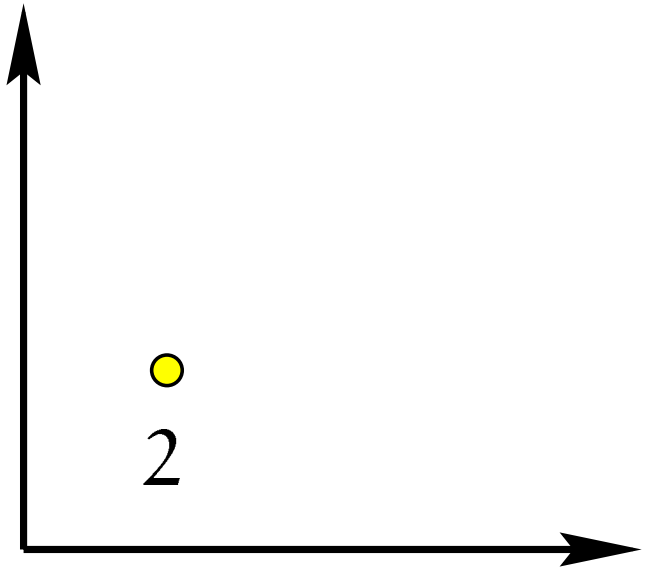

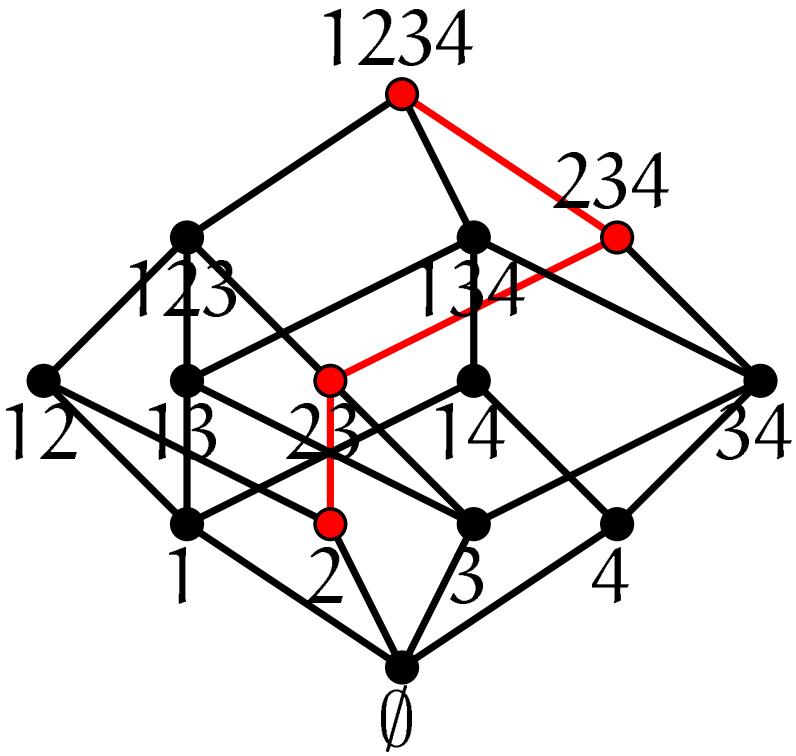

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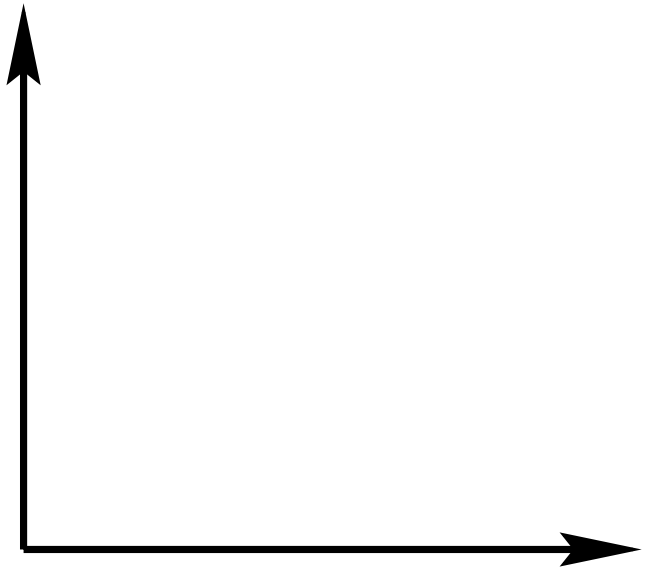

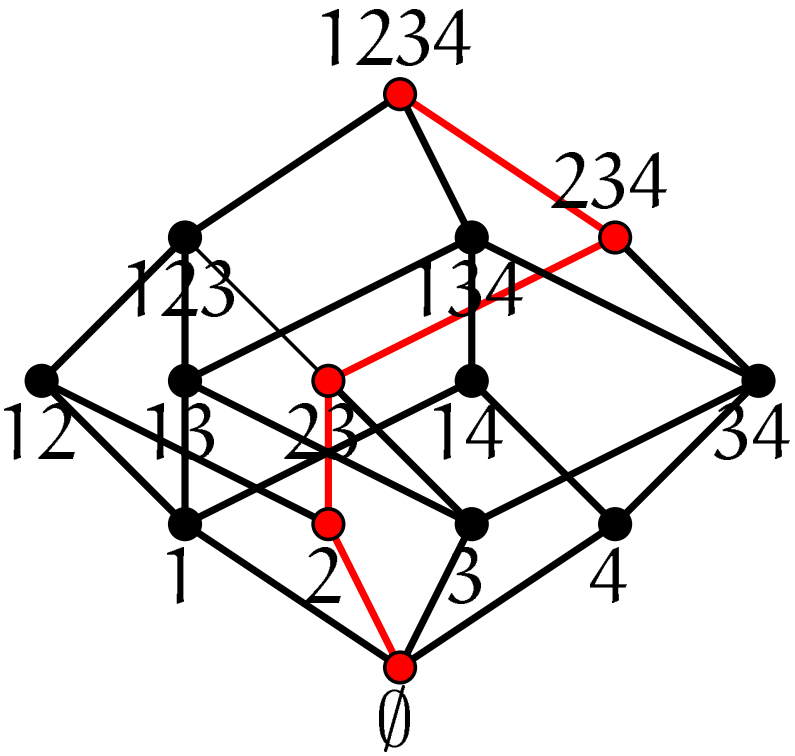

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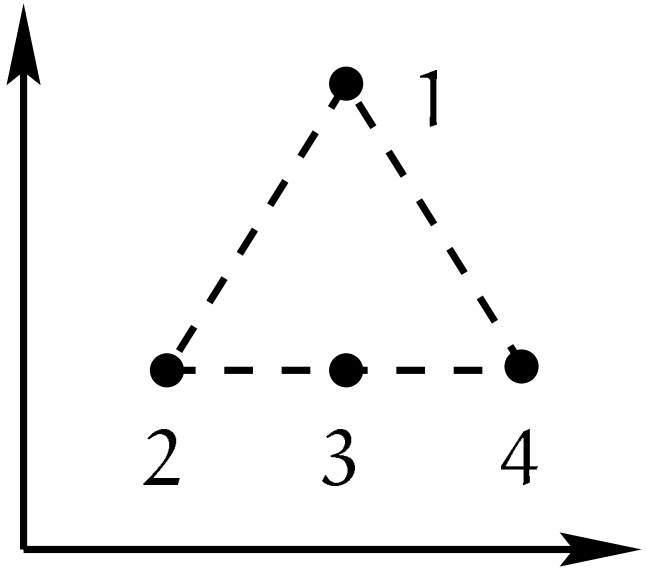

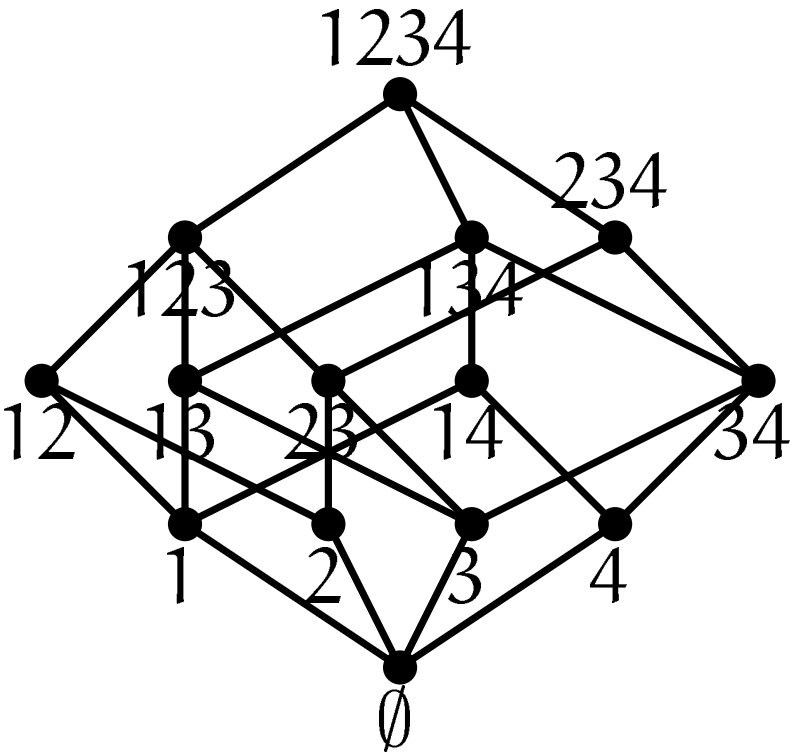

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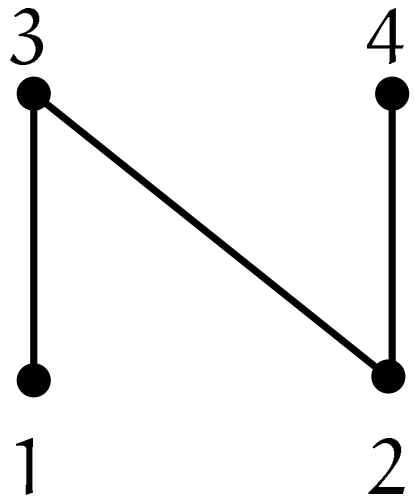
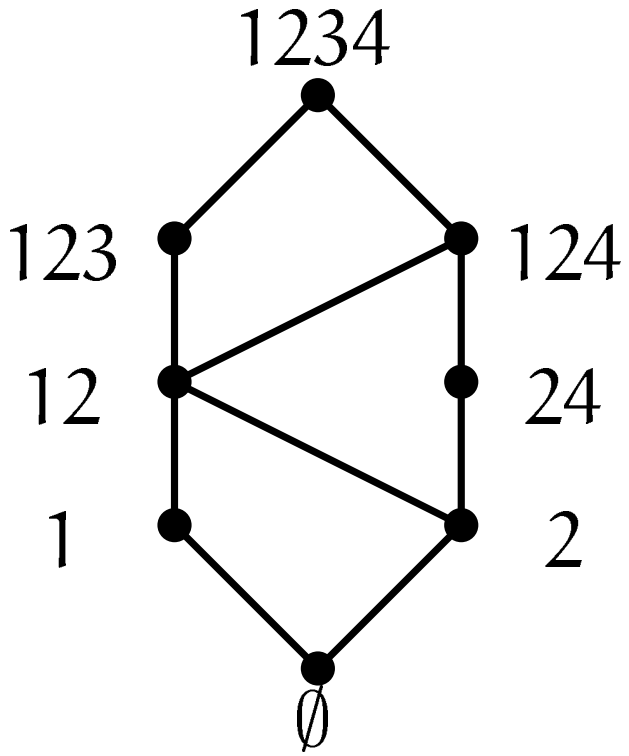
Example 2: poset shelling

$\mathcal{P} = (E, \leq)$ a partially ordered set

Define: $\mathcal{L} = \{X \subseteq E : e \in X, f \leq e \Rightarrow f \in X\}$.





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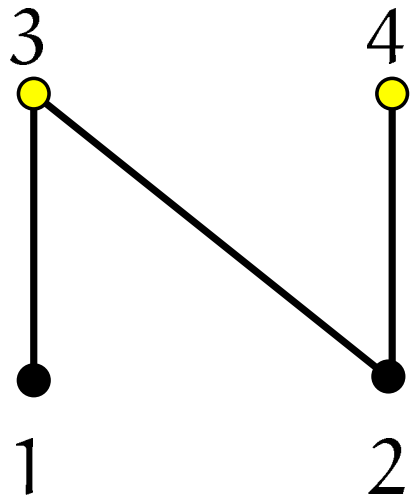

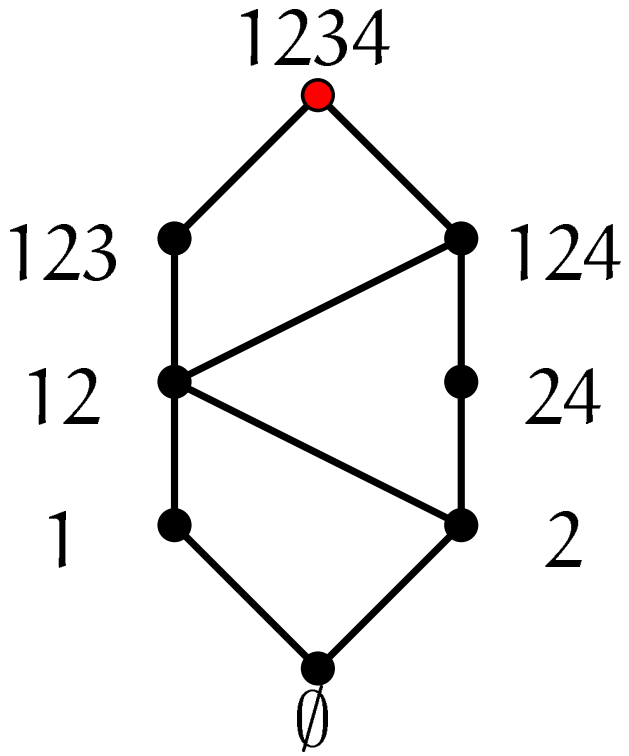

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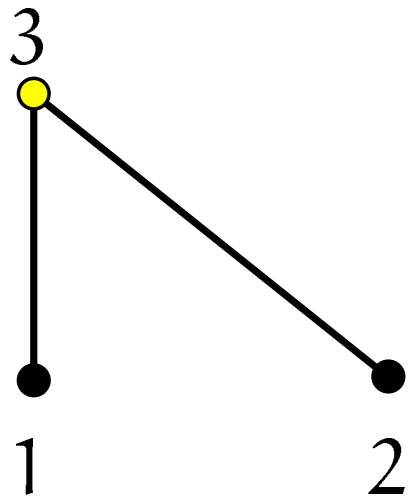

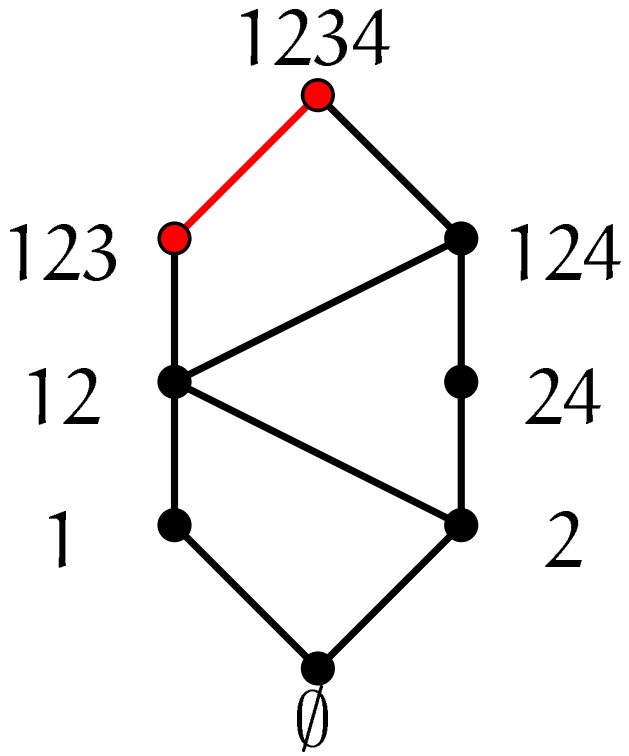

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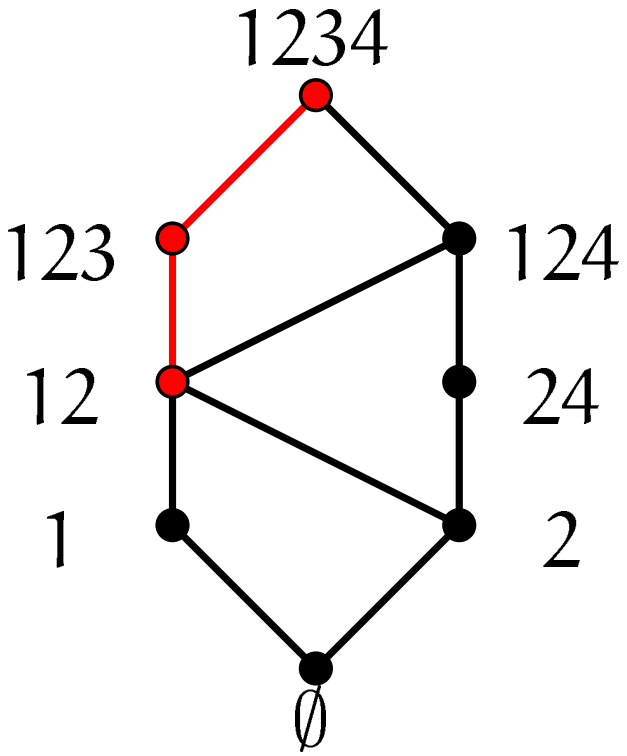

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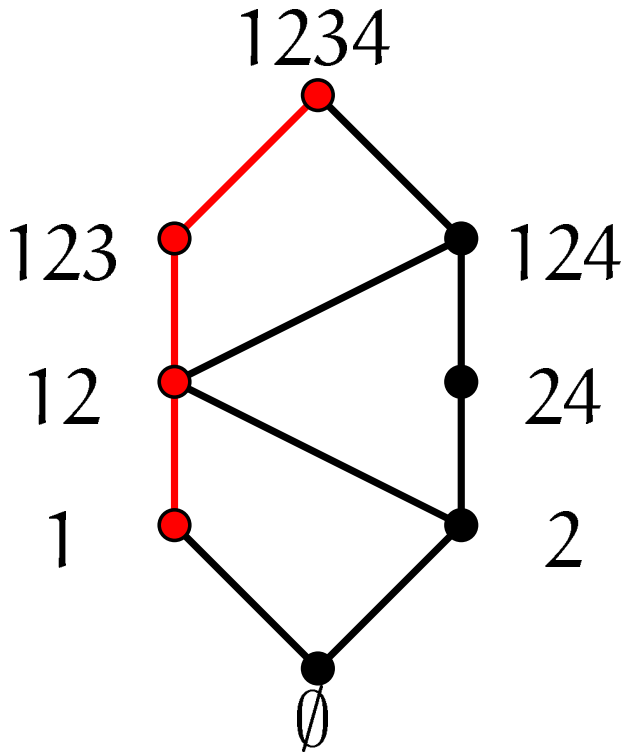

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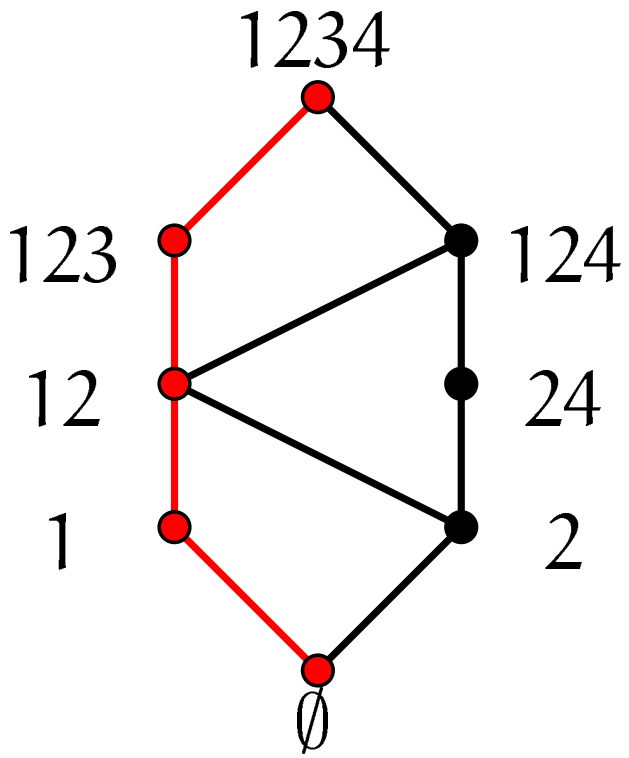

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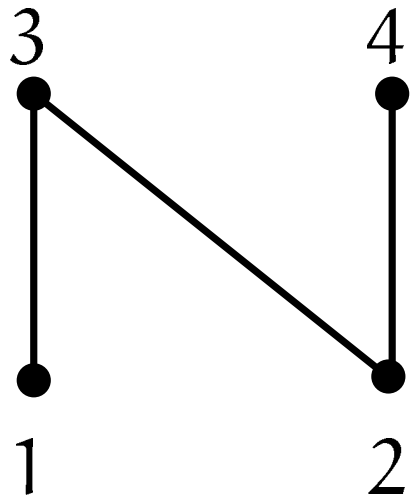

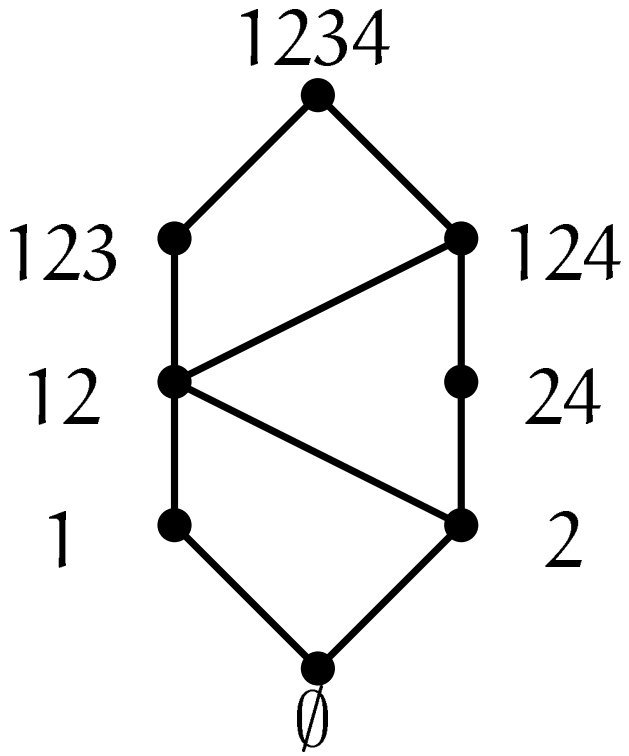

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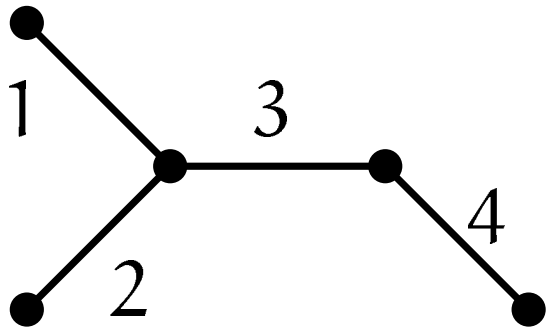
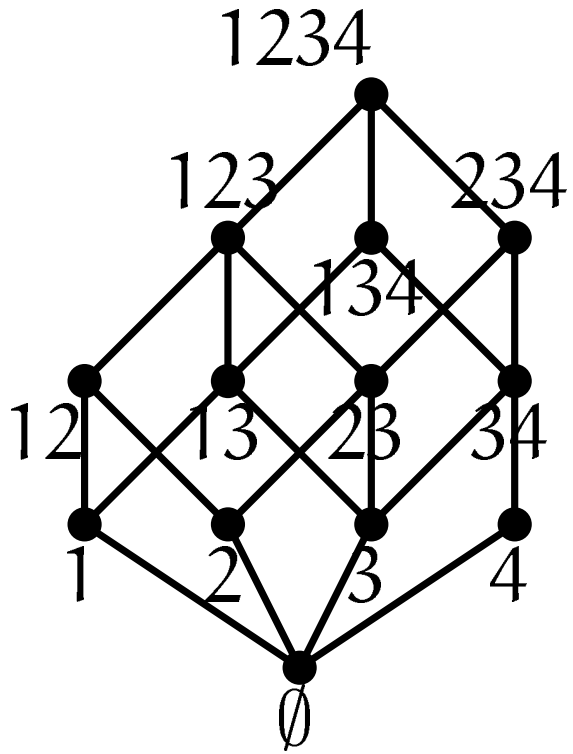
Example 3: tree shelling

$T = (V, E)$ a tree

Define:

$$\mathcal{L} = \{X \subseteq E : X \text{ forms a subtree of } T\}.$$

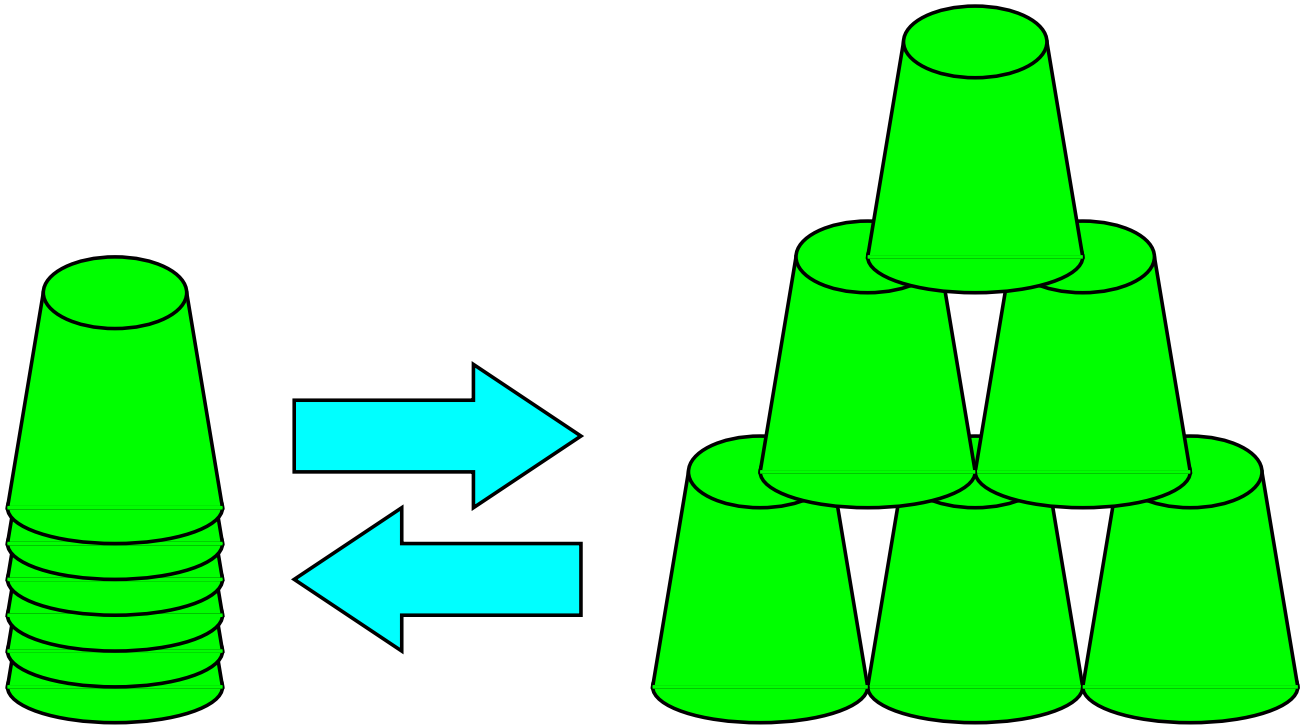




$\mathcal{L}$ is a convex geometry on E and called the tree shelling of T

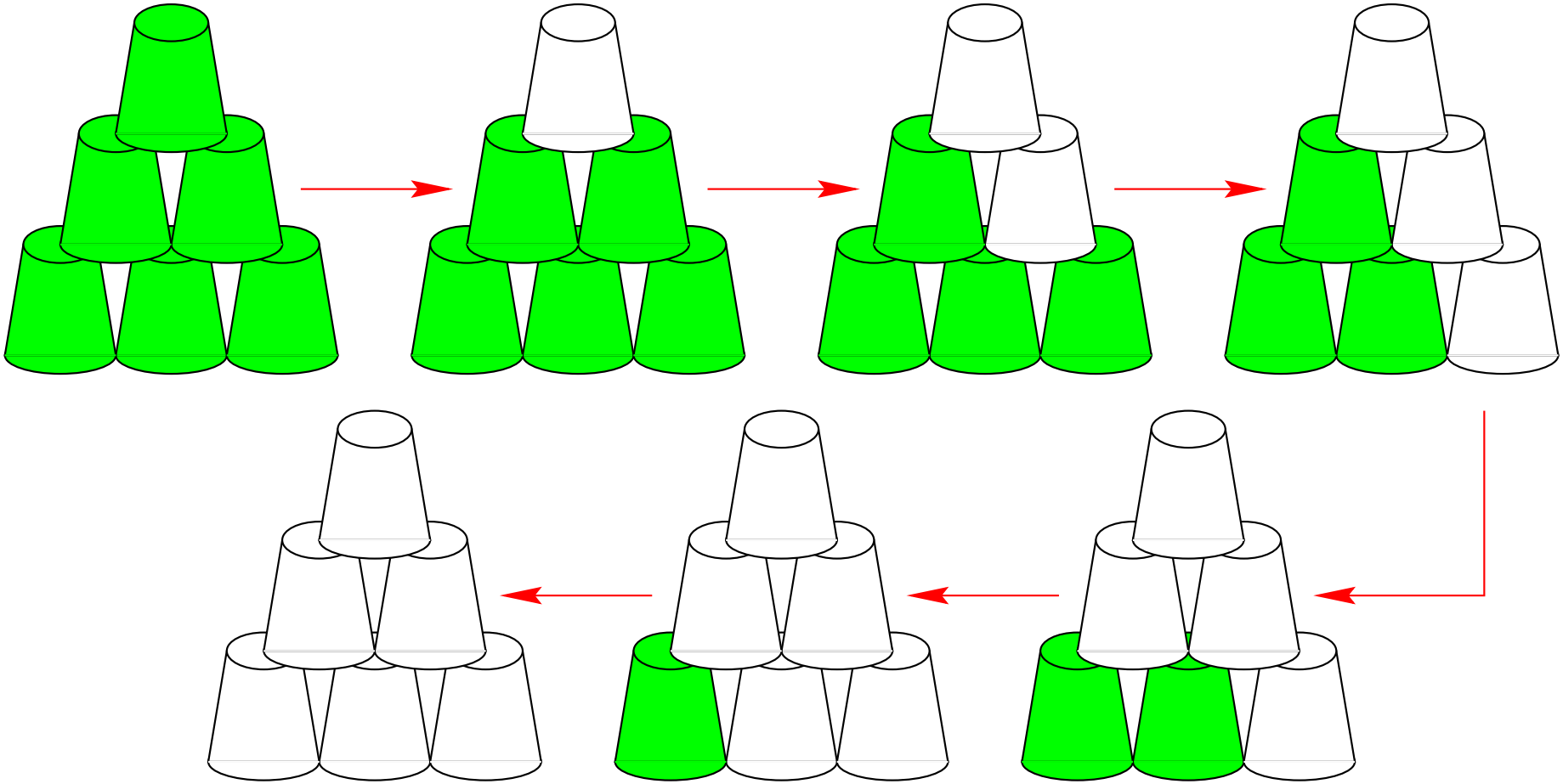
What is "cupstacks"?

Construct the tower from the pile and get it back as quickly as possible.



Example 4: cupstacks

A sequence in collapsing



Other examples

Various objects yield convex geometries.

- From graphs
  - Tree shellings on vertices
  - Graph search
  - Simplicial elimination of chordal graphs
- From partially ordered sets
  - Poset double shellings
  - k-chains in partially ordered sets
- From finite point sets in $\mathbb{R}^d$
  - lower convex shellings on point sets
- From oriented matroids
  - Convex shellings of acyclic oriented matroids
- ...

Yet another example: generalized convex shelling

P,Q finite point sets in $\mathbb{R}^d$ satisfying $\mathrm{conv}(P) \cap Q = \emptyset$

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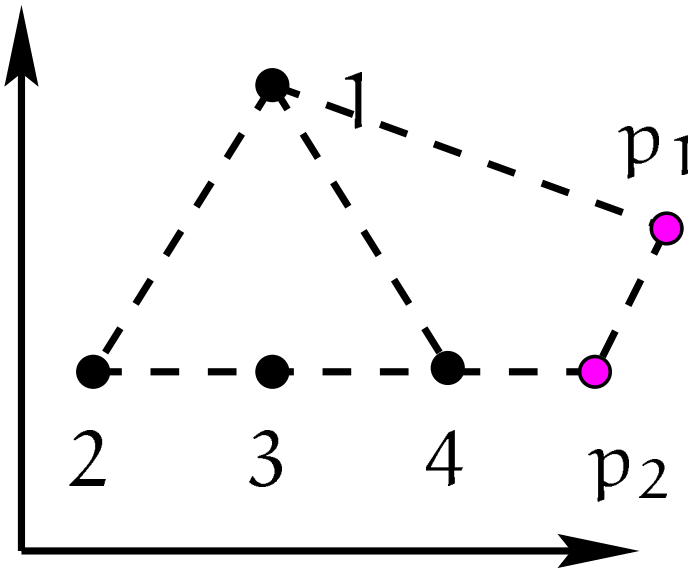

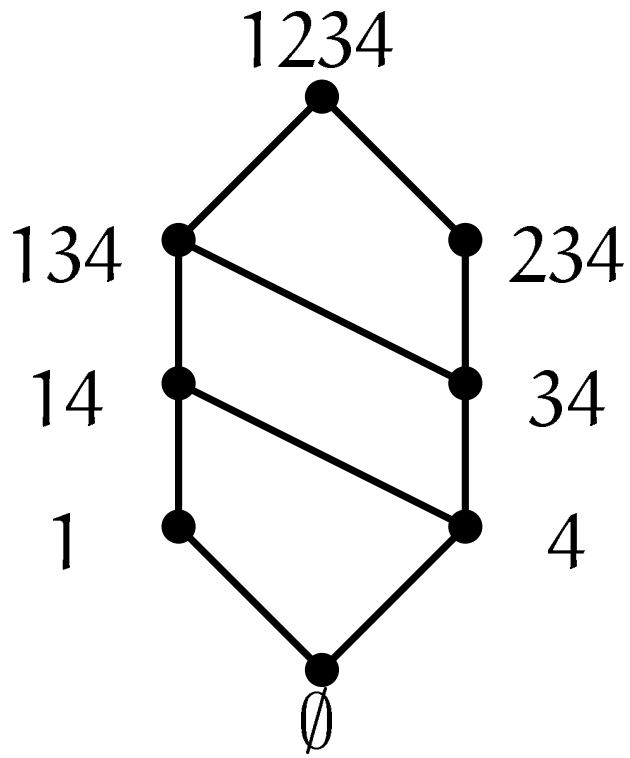

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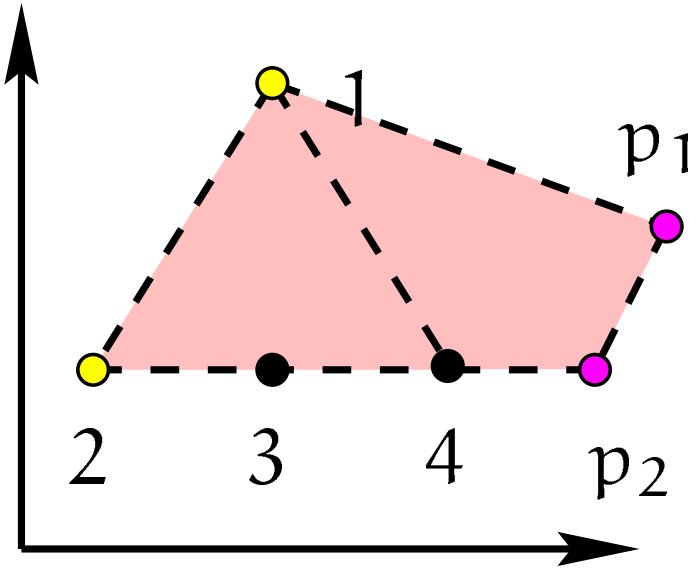

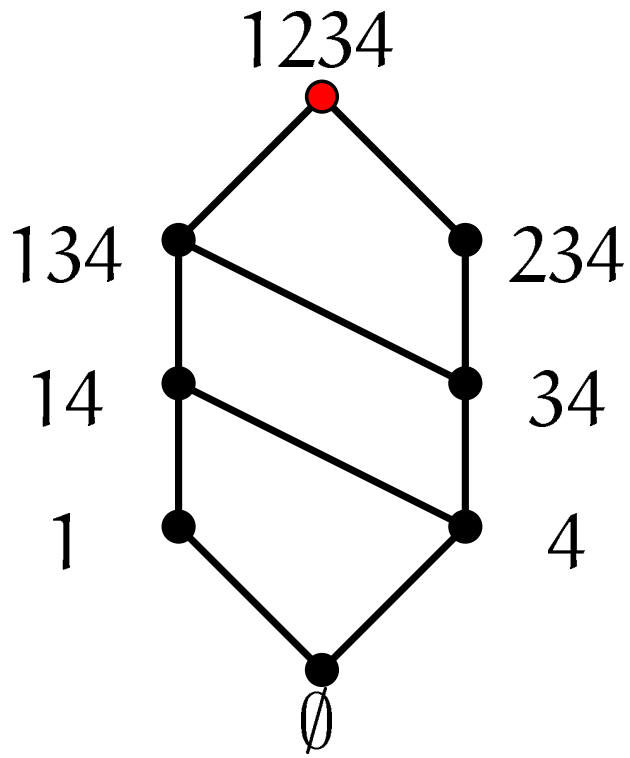

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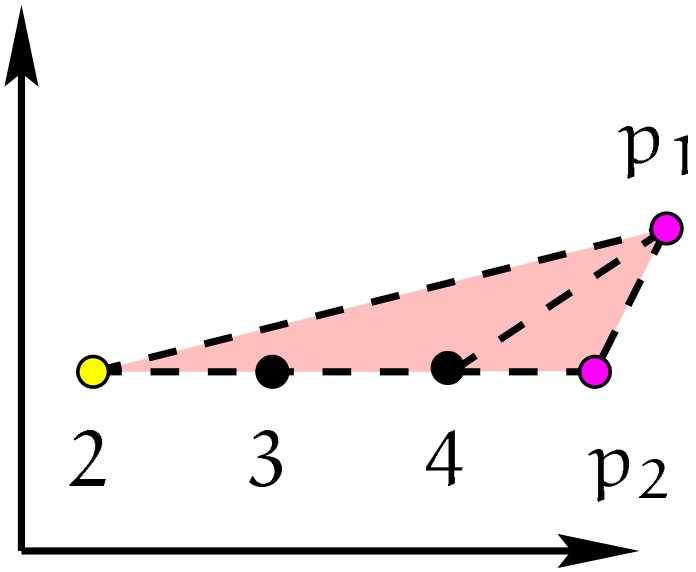

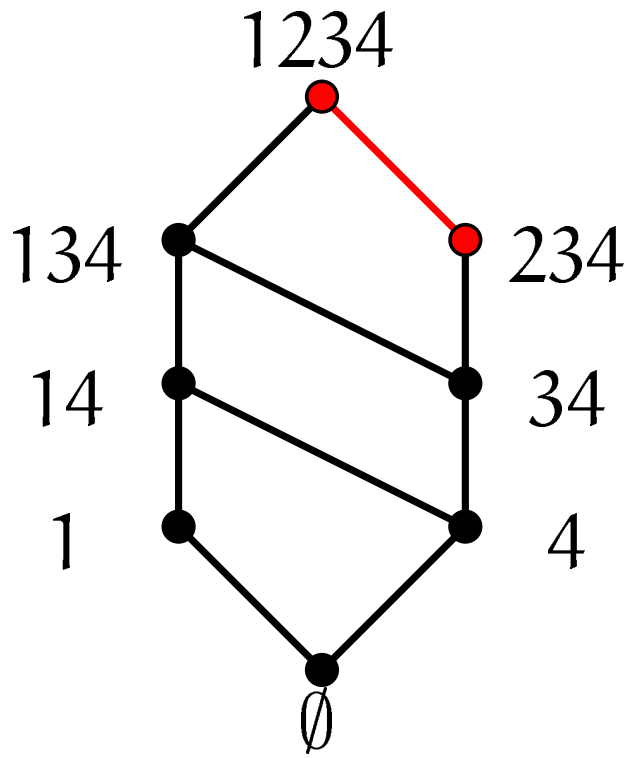

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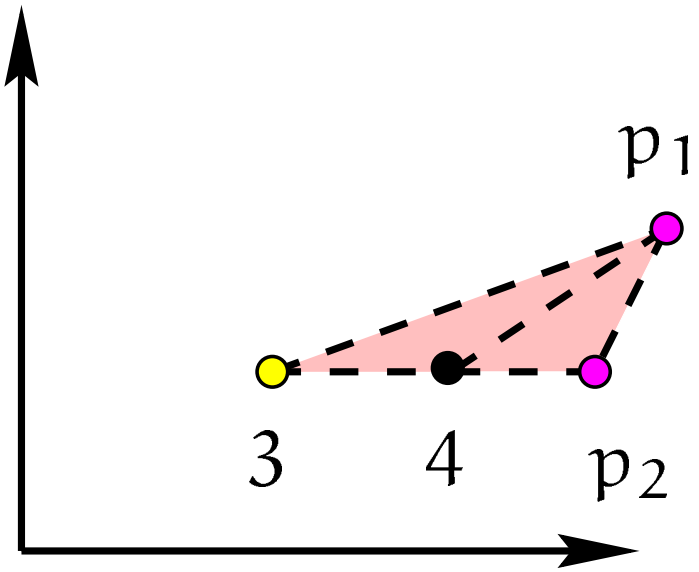

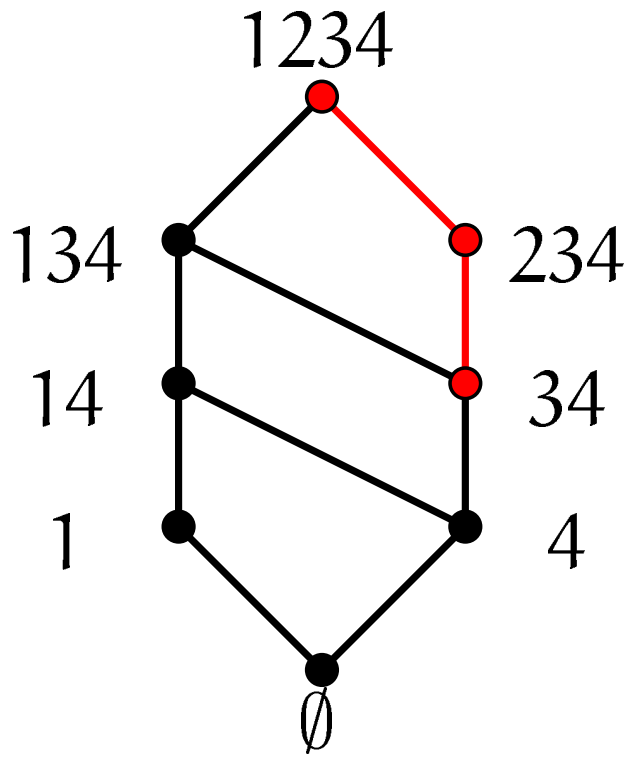

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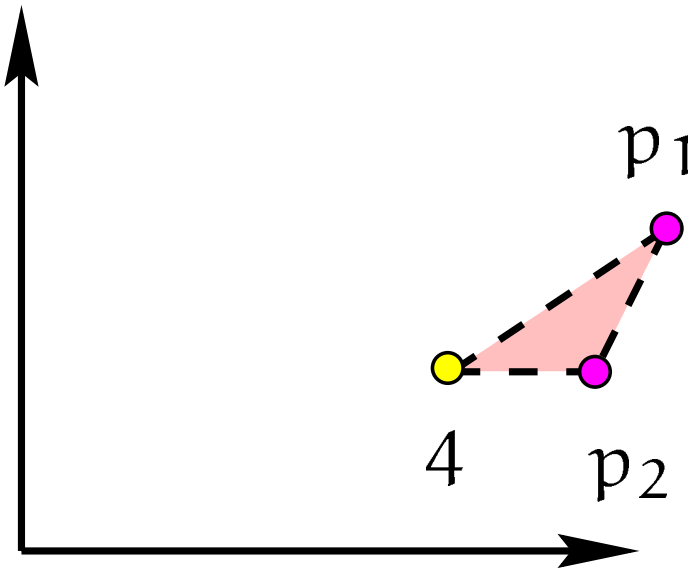
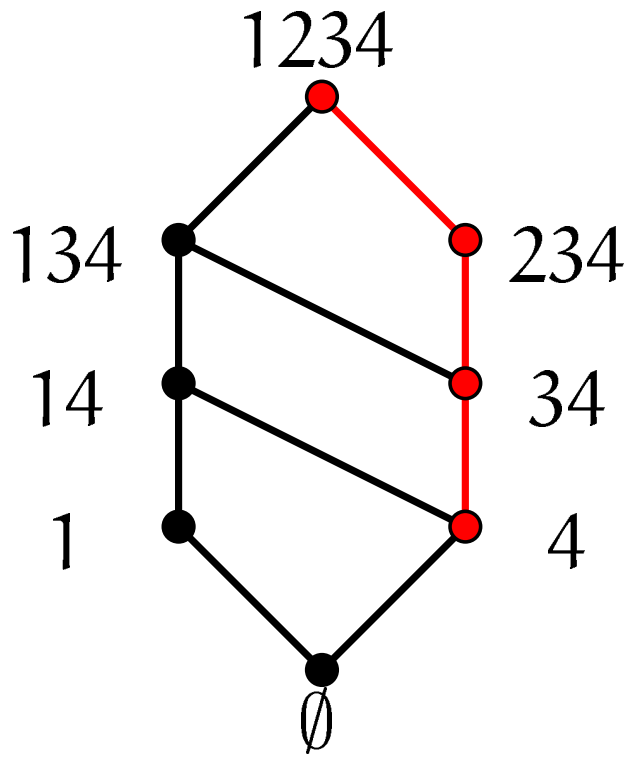


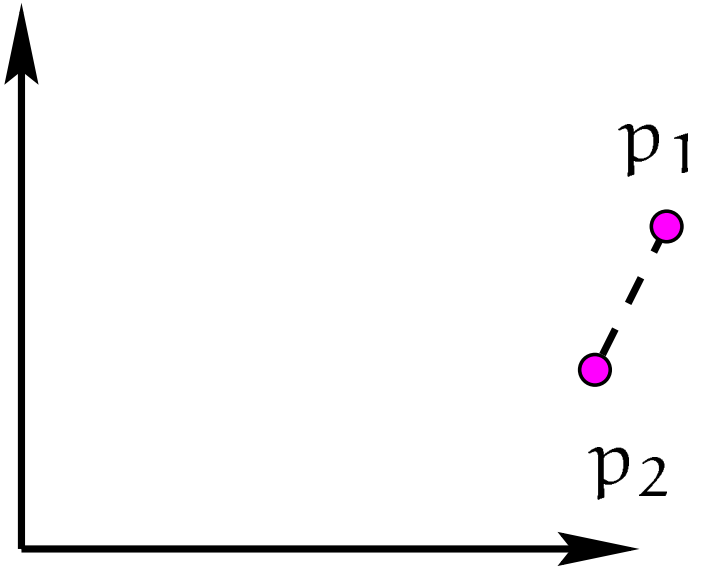


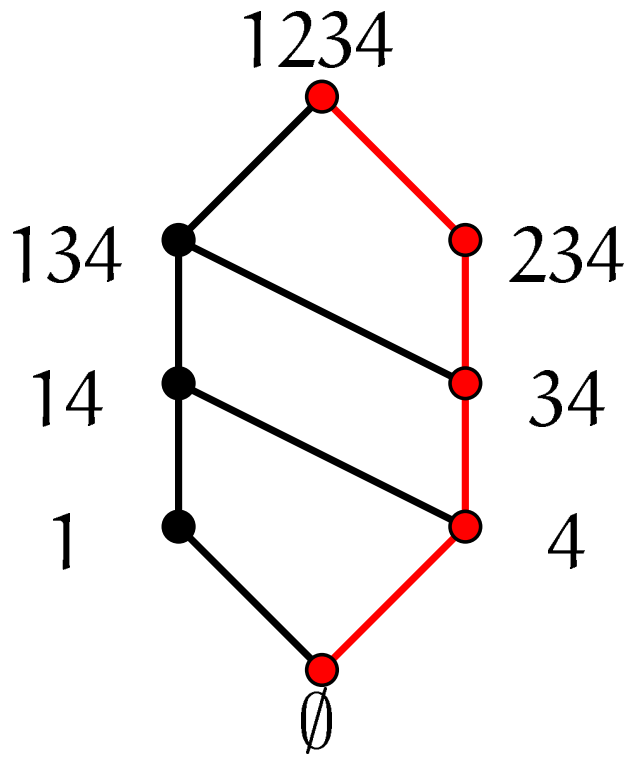
$\mathcal{L}$ is a convex geometry on Q and called
the generalized convex shelling on Q with respect to P.

Yet another example: generalized convex shelling

P,Q finite point sets in $\mathbb{R}^d$ satisfying $\mathrm{conv}(P) \cap Q = \emptyset$

Define: $\mathcal{L} = \{X \subseteq Q : \mathrm{conv}(X \cup P) \cap (Q \setminus X) = \emptyset\}$.





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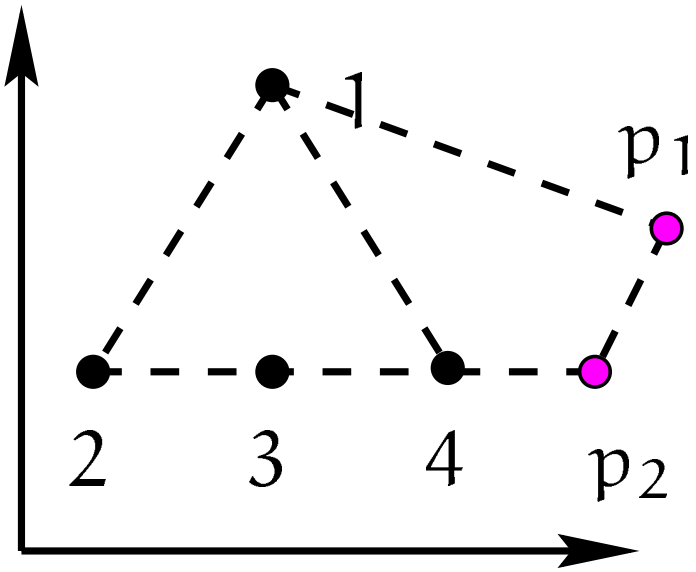
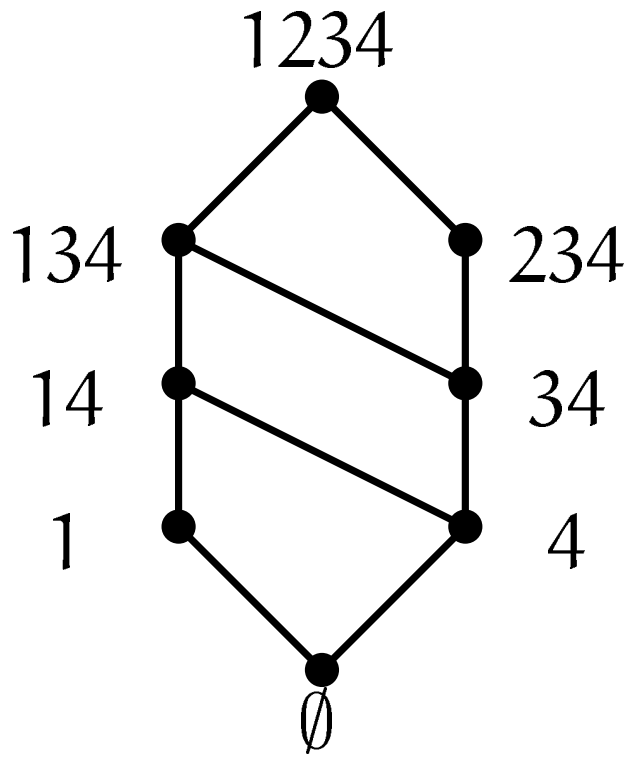




$\mathcal{L}$ is a convex geometry on Q and called
the generalized convex shelling on Q with respect to P.

Our theorem (again)

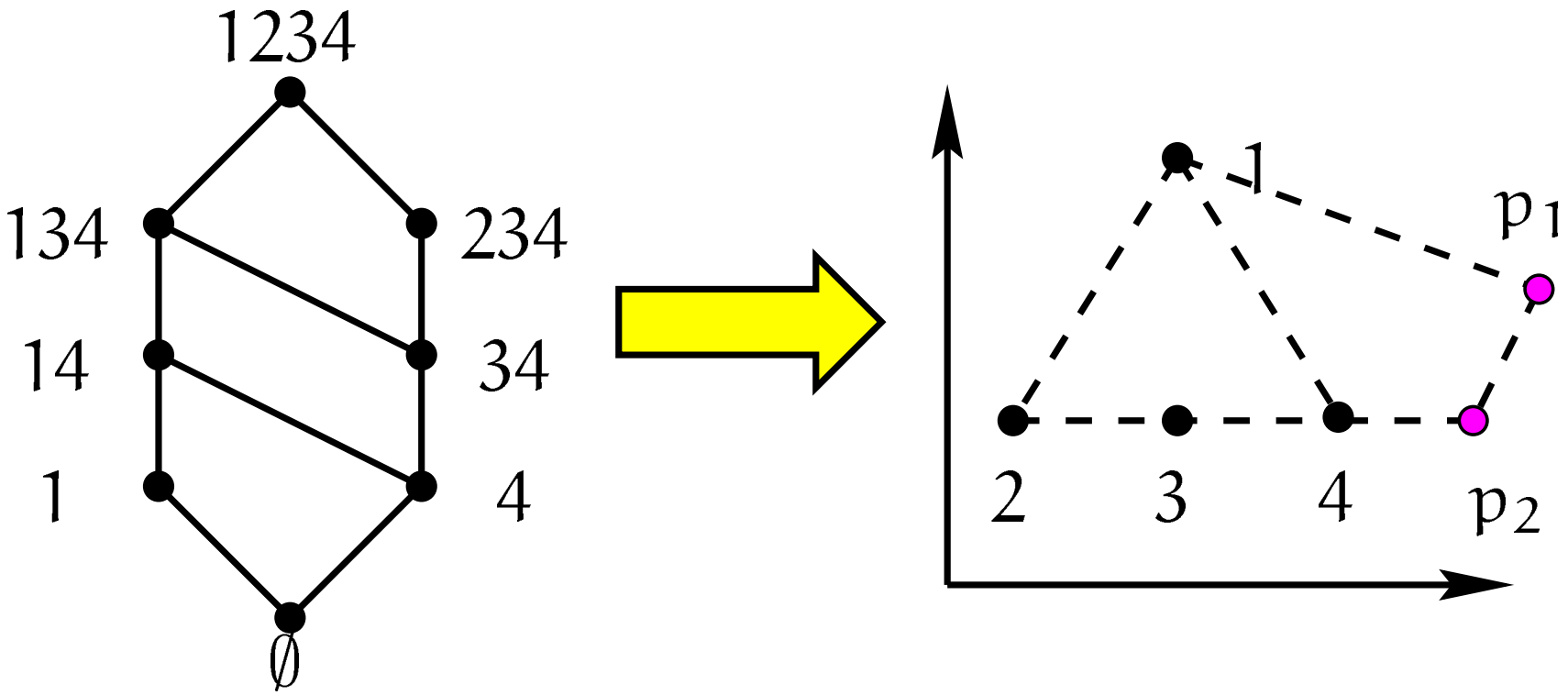
Our Theorem

Any convex geometry is isomorphic to
some generalized convex shelling.

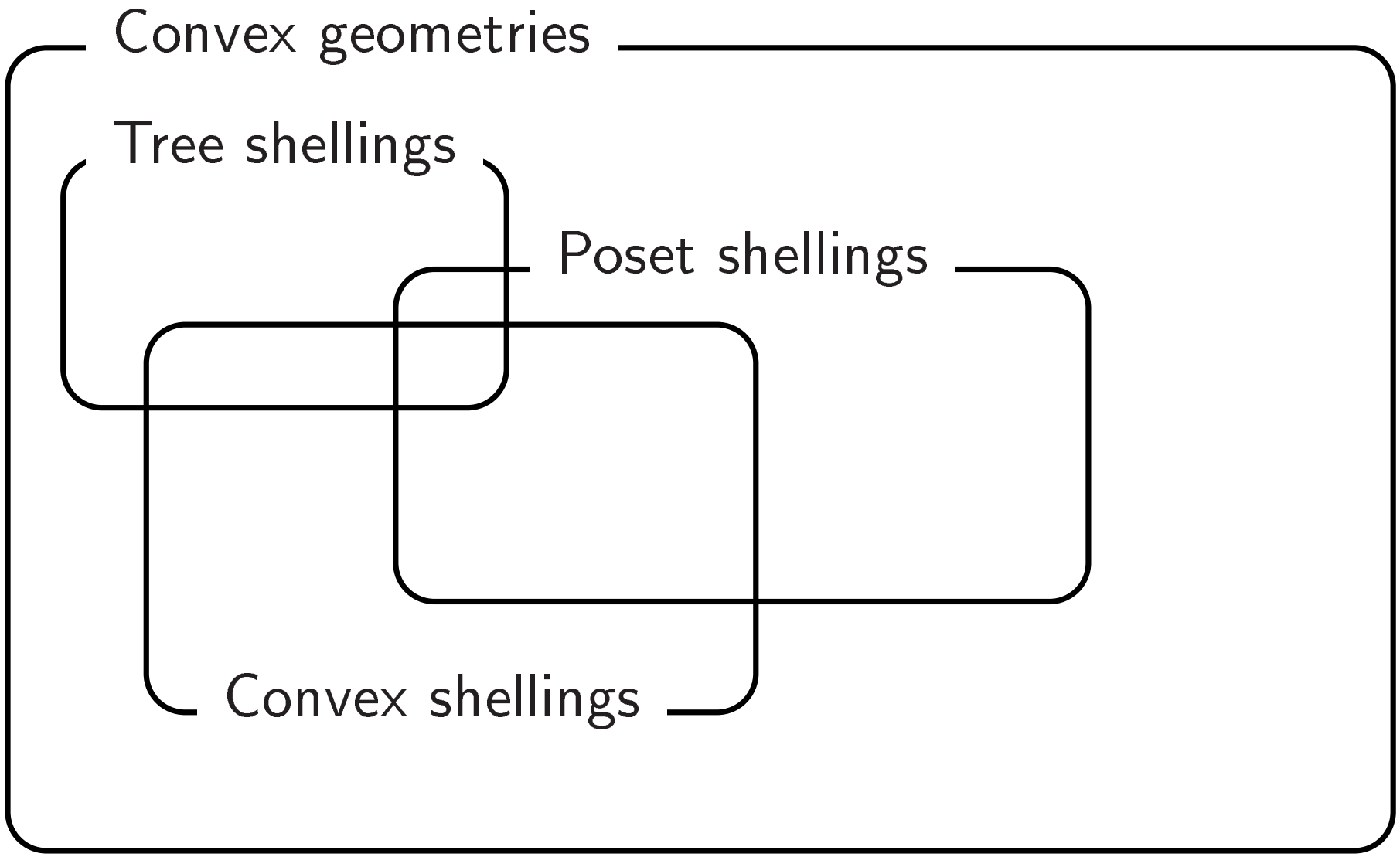
In other words,

For any convex geometry $\mathcal{L}$,
there exist finite point sets P and Q such that
$\mathcal{L}$ is isomorphic to
the generalized convex shelling on Q w.r.t. P.

An illustration

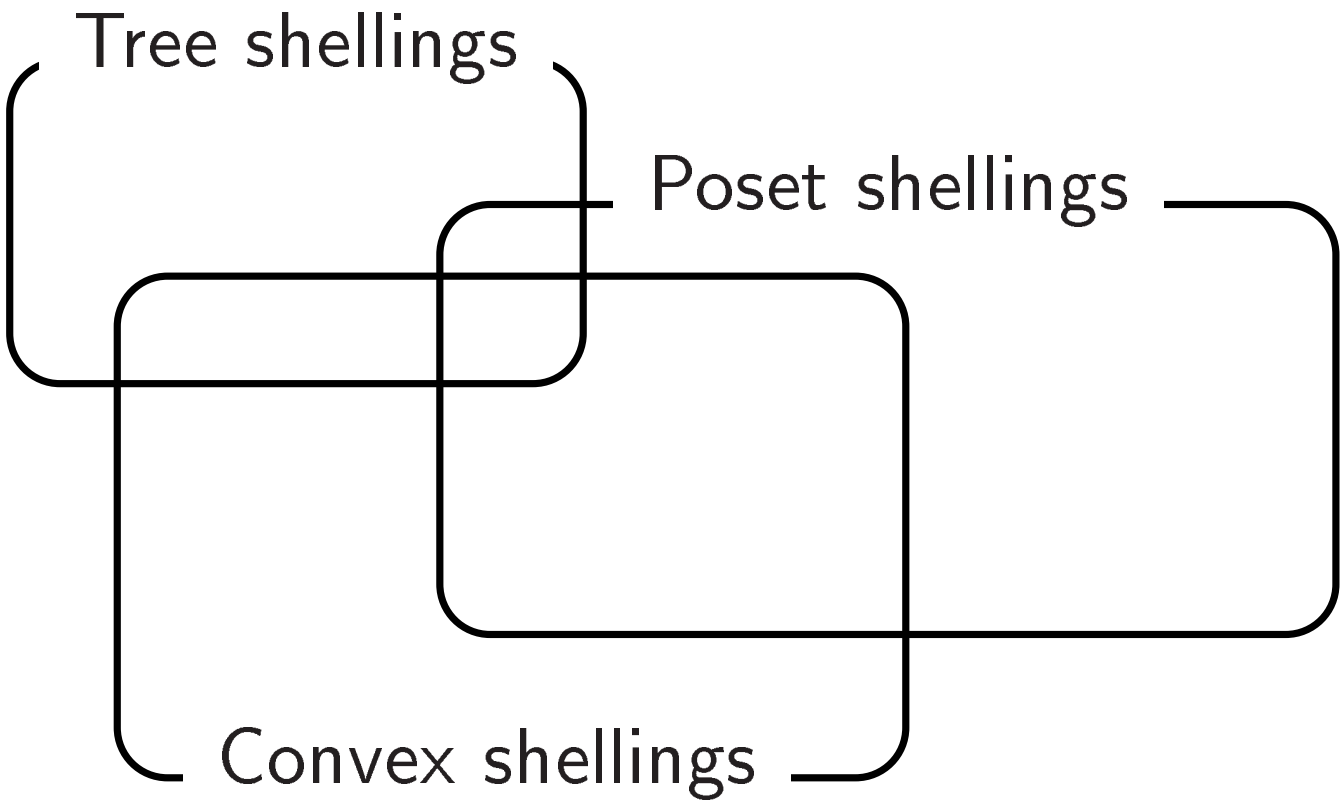


What does the theorem mean? (1)



What does the theorem mean? (1)

Convex geometries = Generalized convex shellings



What does the theorem mean? (2)

- For oriented matroids and matroids, we have

  Topological representation theorems.

- For convex geometries, we have

  Affine representation theorem.

$\Longrightarrow$ An intrinsic simplicity of convex geometries

Outline of the proof

The proof goes along the following line.

We are given a convex geometry $\mathcal{L}$.

(1) Construct:
point sets P and Q from $\mathcal{L}$.

(2) Show:
$\mathcal{L} \cong$ the generalized convex shelling on Q w.r.t. P.

Proof for a special case

To illustrate the proof, we will show a much weaker version.

What we will show

For any poset shelling $\mathcal{L}$

there exist point sets P and Q such that

$\mathcal{L}$ is isomorphic to

the generalized convex shelling on Q w.r.t. P.

Construction of a point set Q

Given a partially ordered set $\mathcal{P} = (\mathrm{E}, \leq)$. Let $\mathrm{n} := |\mathrm{E}|$.

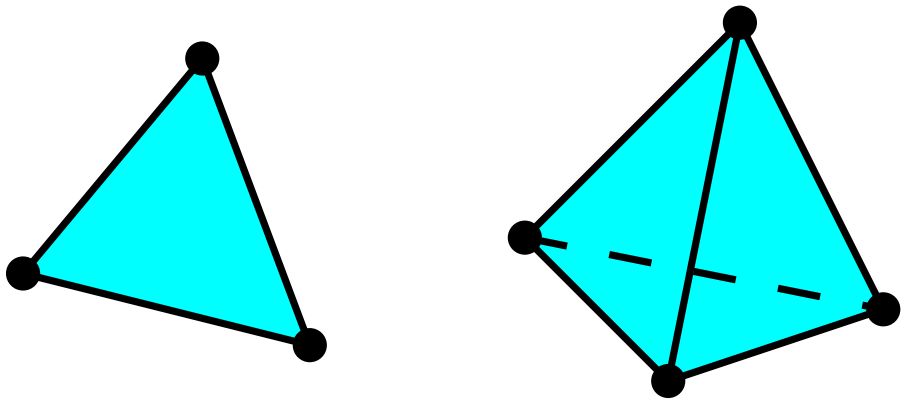
Construction of Q

We use the $(\mathrm{n}-1)$-dimensional space $\mathbb{R}^{\mathrm{n}-1}$.

For each $e \in \mathrm{E}$, put a point $\mathbf{q}(e)$ such that

$\{\mathbf{q}(e) : e \in \mathrm{E}\}$ is affinely independent,

$(\mathrm{conv}(\{\mathbf{q}(e) : e \in \mathrm{E}\})$ is an $(\mathrm{n}-1)$-simplex$)$.



Let $\mathrm{Q} = \{\mathbf{q}(e) : e \in \mathrm{E}\}$. (If $\mathcal{P}$ is an antichain, we are done.)

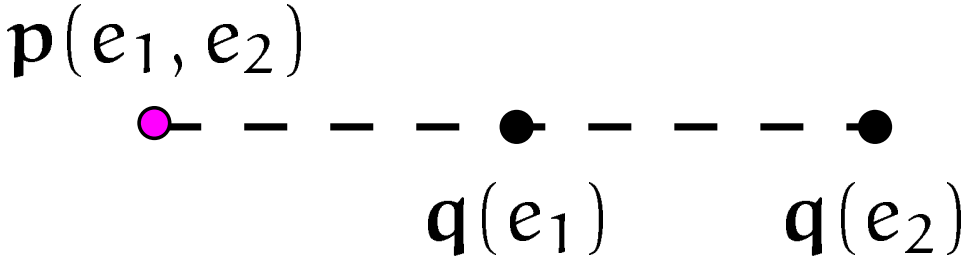
Construction of a point set P

Given a partially ordered set $\mathcal{P} = (E, \leq)$. Let $n := |E|$.

Construction of P

For each $e_1, e_2 \in E$ such that $e_1 < e_2$,

Put a point $\mathbf{p}(e_1, e_2)$ such that $\mathbf{q}(e_1) = \dfrac{\mathbf{p}(e_1, e_2) + \mathbf{q}(e_2)}{2}$.



Let $P = \{\mathbf{p}(e_1, e_2) : e_1, e_2 \in E, e_1 < e_2\}$.

Construction of a point set P

Given a partially ordered set $\mathcal{P} = (E, \leq)$. Let $n := |E|$.

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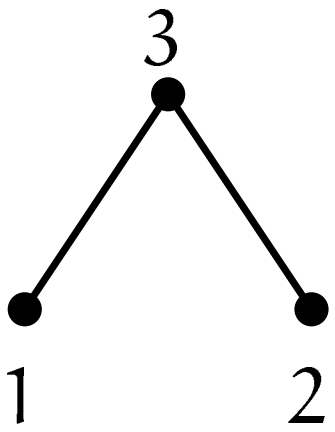

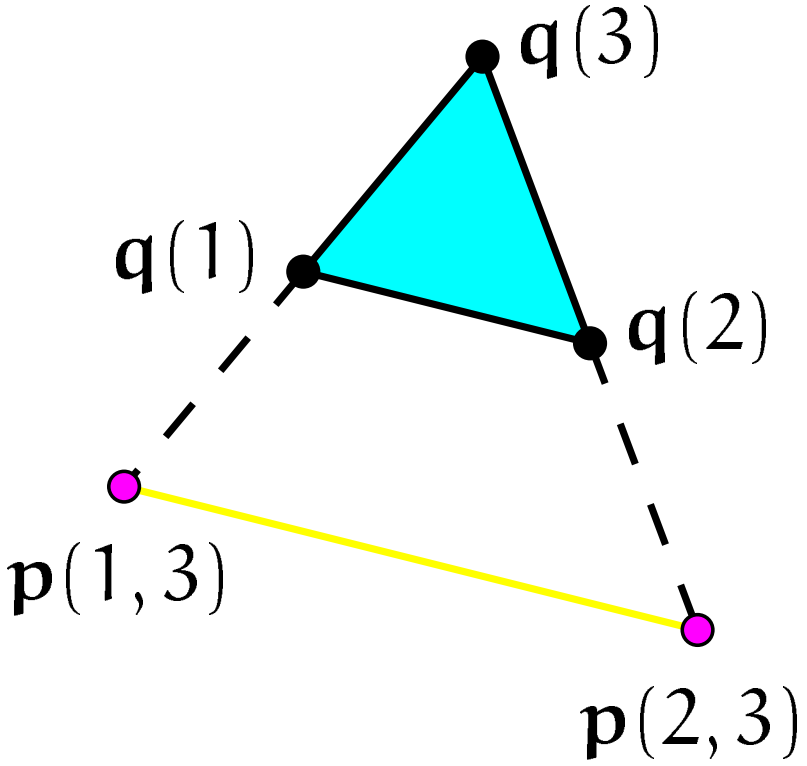

Construction of a point set P

Given a partially ordered set $\mathcal{P} = (\mathrm{E}, \leq)$. Let $n := |\mathrm{E}|$.

Construction of P

For each $e_1, e_2 \in \mathrm{E}$ such that $e_1 < e_2$,

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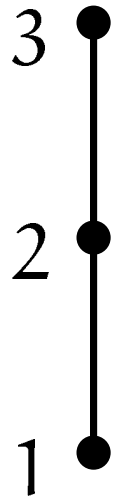

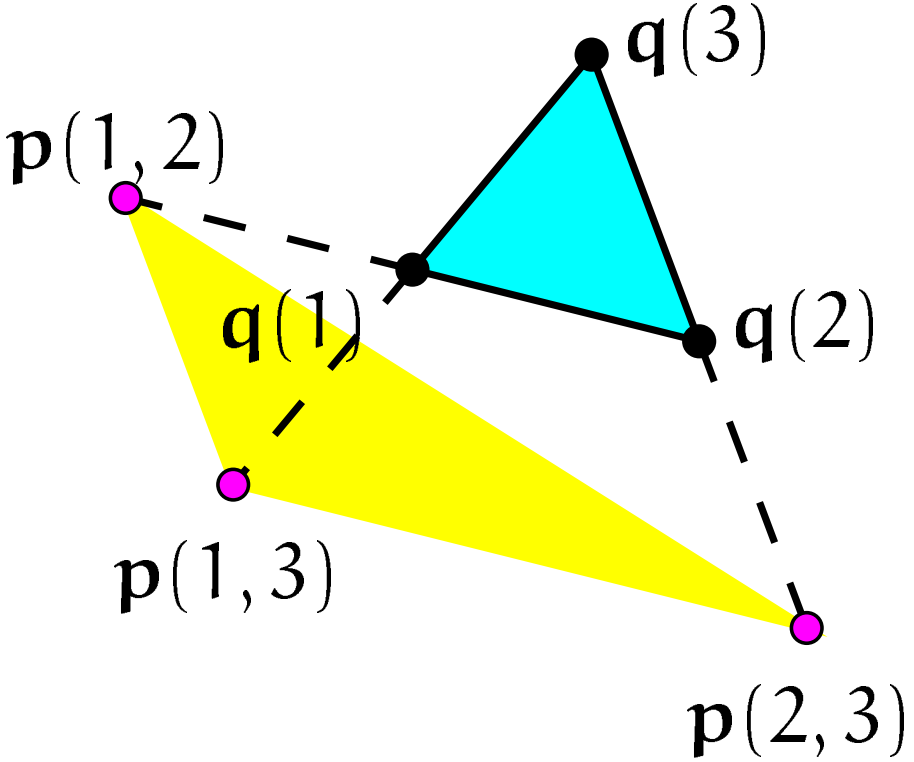

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Construction of P

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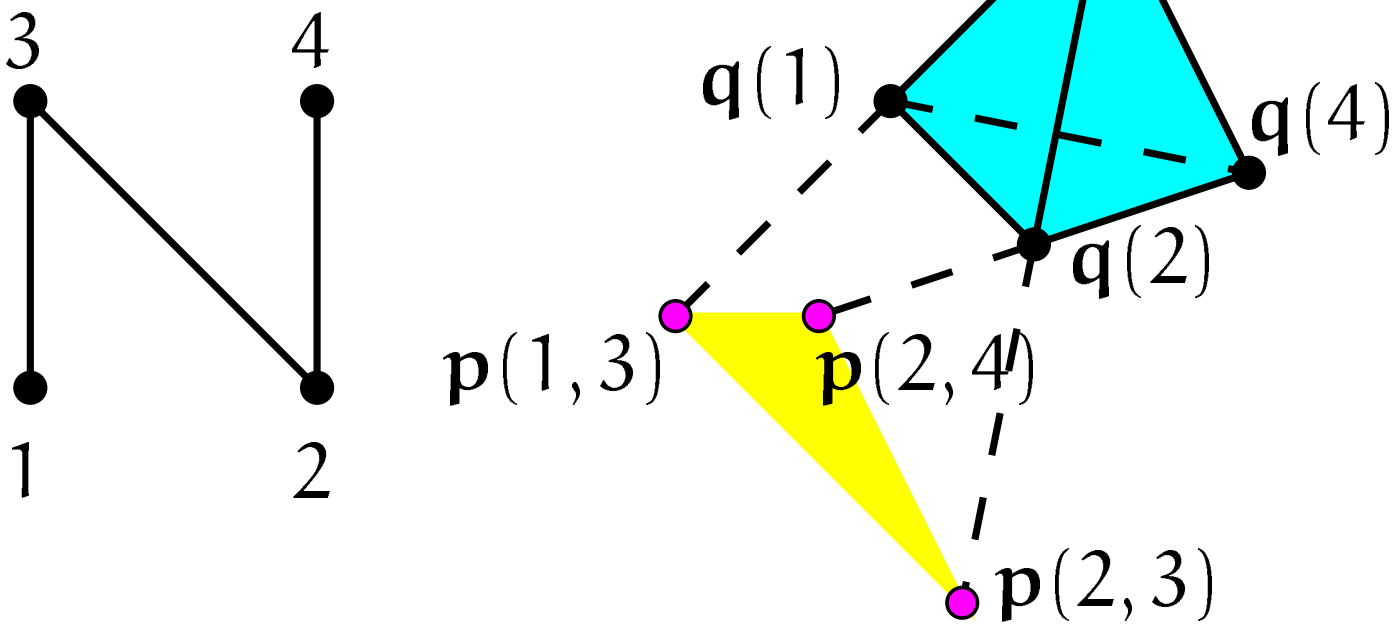


Summary: construction of point sets

Given a partially ordered set $\mathcal{P} = (E, \leq)$. Let $n := |E|$.

We use the $(n-1)$-dimensional space $\mathbb{R}^{n-1}$.

Construction of Q $\quad Q = \{\mathbf{q}(e) : e \in E\}$.

For each $e \in E$, put a point $\mathbf{q}(e)$ such that

$\{\mathbf{q}(e) : e \in E\}$ is affinely independent,

$(\mathrm{conv}(\{\mathbf{q}(e) : e \in E\})$ is an $(n-1)$-simplex$)$.

Construction of P $\quad P = \{\mathbf{p}(e_1, e_2) : e_1, e_2 \in E, e_1 < e_2\}$.

For each $e_1, e_2 \in E$ such that $e_1 < e_2$,

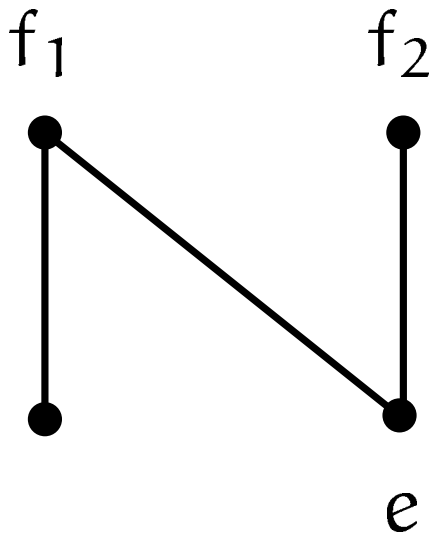
Put a point $\mathbf{p}(e_1, e_2)$ such that $\mathbf{q}(e_1) = \dfrac{\mathbf{p}(e_1, e_2) + \mathbf{q}(e_2)}{2}$.

Claim the poset shelling of $\mathcal{P} = (E, \leq)$
$\cong$ the generalized convex shelling on Q w.r.t. P.

Proof sketch.

(1) When is $e \in E$ allowed to be removed?

e is allowed to be removed $\iff$ all f's such that $e < f$ have been already removed.

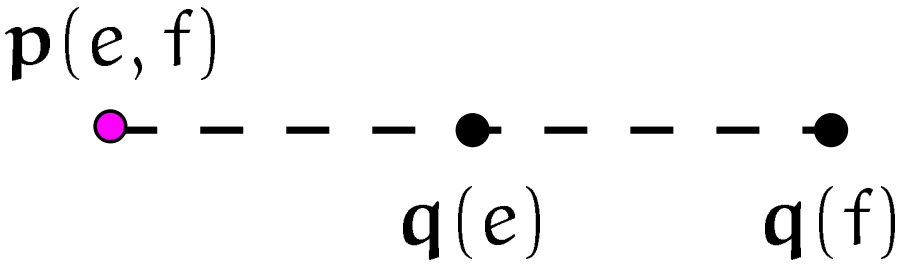


Claim the poset shelling of $\mathcal{P} = (\mathrm{E}, \leq)$
$\cong$ the generalized convex shelling on Q w.r.t. P.

Proof sketch.

(2) When is $\mathbf{q}(e) \in \mathrm{Q}$ allowed to be removed?

$\mathbf{q}(e)$ is allowed to be removed $\iff$ all $\mathbf{q}(\mathrm{f})$'s s.t. $\mathbf{p}(e, \mathrm{f}) \in \mathrm{P}$ have been already removed.



Claim the poset shelling of $\mathcal{P} = (\mathrm{E}, \leq)$
$\cong$ the generalized convex shelling on Q w.r.t. P.

Proof sketch.

(1) When is $e \in \mathrm{E}$ allowed to be removed?

e is allowed to be removed $\iff$ all f's such that $e < \mathrm{f}$ have been already removed.

(2) When is $\mathbf{q}(e) \in \mathrm{Q}$ allowed to be removed?

$\mathbf{q}(e)$ is allowed to be removed $\iff$ all $\mathbf{q}(\mathrm{f})$'s s.t. $\mathbf{p}(e, \mathrm{f}) \in \mathrm{P}$ have been already removed.

(3) $e < \mathrm{f} \iff \mathbf{p}(e, \mathrm{f}) \in \mathrm{P}$.

Claim the poset shelling of $\mathcal{P} = (\mathrm{E}, \leq)$
$\cong$ the generalized convex shelling on Q w.r.t. P.

Proof sketch.

(1) When is $e \in \mathrm{E}$ allowed to be removed?

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$\mathbf{q}(e)$ is allowed to be removed $\iff$ all $\mathbf{q}(\mathrm{f})$'s s.t. $\mathbf{p}(e, \mathrm{f}) \in \mathrm{P}$ have been already removed.

$\Longrightarrow$ The mapping "$e \longmapsto \mathbf{q}(e)$" is an isomorphism. [qed]

The final slide

What was our theorem??

Our Theorem

Any convex geometry is isomorphic to
some generalized convex shelling.

This theorem is expected to be useful
for a lot of problems in convex geometries.

Further Work How useful can it be?

[End of the talk]