Several Aspects of Antimatroids and Convex Geometries

Master's Thesis

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Abstract

Convexity is important in several fields, and we have some theories on it. In this thesis, we discuss a kind of combinatorial convexity, in particular, antimatroids and convex geometries.

An antimatroid is a combinatorial abstraction of convexity. It has some different origins; by Dilworth in lattice theory, by Edelman and Jamison in the notions of convexity, by Korte–Lovász who were motivated by scheduling problems. A convex geometry is known as a dual object of an antimatroid.

In this thesis, we have four main topics. The first topic is a characterization result. We characterize line-search antimatroids of rooted digraphs by their forbidden-minors. It implies that the minor theorem for antimatroids does not hold, while it holds for graphs.

The second topic is an antimatroidal analogue of Dilworth's decomposition theorem for partially ordered sets. We show a characterization of coatomic antimatroids via the concept of circuits.

The third topic is related to submodular-type optimization, which discusses what the "good combinatorial structure" for optimization is. We consider a submodular-type optimization problem on the extreme sets of a convex geometry. We introduce a new kind of submodularity, called c-submodularity, and show that the equivalence between the validity of a greedy algorithm and the c-submodularity of a given function. Moreover, we give a new characterization of a poset shelling, related to this result.

The fourth topic is an application of convex geometries to the theory of cooperative games. We show that, if a game on a convex geometry is quasi-convex, then the core is a unique stable set of the game.

Contents

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Yoshio Okamoto

Chapter 1

Introduction

Convexity is important in several fields, and we have some theories on it. In this thesis, we discuss a kind of combinatorial convexity, in particular, antimatroids and convex geometries.

An antimatroid is a combinatorial abstraction of convexity. It has some different origins. Already in 1940, Dilworth [8] considered a special class of semimodular lattices, which would be known as a notion equivalent to antimatroids. Edelman [11] and Jamison [25] introduced a notion of the “anti-exchange” property, which is an “antipodal” concept to the Steinitz-MacLane exchange property of matroids. Indeed, they introduced a convex geometry, which is now known as a “dual” object of an antimatroid. Korte–Lovász [26, 27] introduced an equivalent concept, stimulated by scheduling problems with alternative precedence constraints. They named it a shelling structure or an alternative precedence structure.

Lots of combinatorial objects give rise to antimatroids. In particular, we can obtain many classes of antimatroids by shelling processes or searching processes. By shelling processes, we mean repeated elimination of suitable elements until all the elements are removed. This process yields a convex shelling on the Euclidean space, a shelling of a poset, a double shelling of a poset, a vertex shelling of a tree, a simplicial shelling of a triangulated graph, etc. By search, we mean a gathering process of adjacent elements. This yields a point-search of a rooted (di)graph, a line-search of a rooted (di)graph, etc.

An antimatroid has a good property on some optimization problems. Boyd–Faigle [5] gave an algorithmic characterization of antimatroids. They considered a bottleneck-type optimization problem on them, which generalizes Lawler’s work [31] for a scheduling problem.

We have some surveys and good materials on antimatroids and convex geometries. We can find important properties and results on them which do not appear in the thesis. A book by van de Vel [46] is a comprehensive survey of combinatorial convexity, however it contains little on convex geometries and antimatroids. Duchet [10] is a compact survey of combinatorial convexity, which concentrates on the structural properties. Chapter 5 of [10] is devoted to convex geometries. Korte–Lovász–Schrader [28] is a unique book on "greedoids," which includes antimatroids and matroids as the subclasses. The Chapter 3 of the book is devoted to the introduction of antimatroids and convex geometries, and some results on them are scattered over the book. Edelman–Jamison [12] concentrates on convex geometries, and Dietrich [7] is a survey on matroids and antimatroids. Goecke–Korte–Lovász [20] presents lots of examples of greedoids and antimatroids. Moreover, we can see non-simple antimatroids in Björner–Ziegler [4]. Note that non-simple antimatroids are also characterized via a greedy algorithm by Nakamura [34].

Organization of the thesis

In this thesis, we discuss some aspects of antimatroids and convex geometries. The organization of this thesis is as follows.

In Chapter 2, we introduce the definitions of antimatroids and convex geometries, and see some examples and some properties of them.

We have four main topics. The first topic appears in Chapter 3, where we characterize line-search antimatroids of rooted digraphs by their forbidden-minors. Several combinatorial objects are beautifully characterized by their forbidden-minors. We know, for example, Kuratowski's theorem for planar graphs [30] and Tutte's result for binary matroids [45]. For antimatroids, Nakamura [35] gives the forbidden-minor characterizations of poset shellings and point-searches of rooted (di)graphs. Our new result in Chapter 2 is obtained by an application of a consequence of the theory of line graphs [23, 37, 44]. Moreover, it concludes that the Minor Theorem for antimatroids does not hold, while the Graph Minor Theorem is shown to hold by Robertson–Seymour [39].

Secondly, in Chapter 4 we consider an antimatroidal analogue of Dilworth's decomposition theorem for partially ordered sets. In a paper of 1950, Dilworth [9] showed one of the most famous theorems for partially ordered sets, or posets. That is, for any poset, the maximum size of antichains is equal to the minimum number of chains which cover the ground set. In Chapter 4, we consider an extension of this theorem to antimatroids. First, we introduce free sets and linear sets of an antimatroid. While a free set is a well-known concept in antimatroids, a linear set is probably a concept which first appears in this thesis. Via free sets and linear sets, we restate the Dilworth's theorem in terminology of antimatroids: for any poset shelling, the maximum size of free sets is equal to the minimum number of linear sets which cover the ground set. However, this statement does not hold for all antimatroids. We show that it does not hold for coatomic antimatroids. In addition, we show a characterization of coatomic antimatroids via the concept of circuits.

In Chapter 5, we consider a submodular-type optimization problem on the extreme sets of a convex geometry. The theory of submodular-type optimization is a polyhedral extension of matroid theory, in which "good combinatorial structure" for optimization is discussed. It is applied to some optimization problems: networks [17], scheduling [38], cooperative game theory [24, 41], etc. In Chapter 5, we introduce a new kind of submodularity, called c-submodularity, and we show that the equivalence between the validity of a greedy algorithm and the c-submodularity of a given function. Moreover, we give a new characterization of a poset shelling, related to this result.

The last topic is related to a cooperative game on a convex geometry, introduced by Bilbao [2]. In classical cooperative games, every subset of players is a feasible coalition. However, in many social, cultural and geographical situations, this assumption does not apply. Bilbao [2] introduced a model of partial cooperation using convex geometries, that is, a member of a convex geometry is only allowed to form a coalition. In Chapter 6, we show that, if a game on a convex geometry is quasi-convex, then the core is a unique stable set. The core of a game is the set of coalitionally rational imputations, and it is a polyhedral solution concept. A set of imputations of a game is called a stable set if it enjoys the internal stability and the external stability. In general, a stable set is not unique. However, Shapley [41] showed that in the classical situation a convex game has a unique stable set, which coincides with the core. The result of Chapter 6 can be seen as an extension of the work of Shapley [41].

Chapter 2

Antimatroids and Convex Geometries

In this chapter, we review the definitions of antimatroids and convex geometries and some examples of them.

2.1 Definitions

Let E be a non-empty finite set. A family $\mathcal{F} \subseteq 2^E$ of subsets of E is an *antimatroid* on E if $\mathcal{F}$ satisfies the following three properties:

$$\emptyset \in \mathcal{F}, \quad E \in \mathcal{F}, \tag{2.1}$$

$$X \in \mathcal{F} \setminus \{\emptyset\} \text{ implies } X \setminus \{e\} \in \mathcal{F} \text{ for some } e \in X, \tag{2.2}$$

$$X, Y \in \mathcal{F} \text{ imply } X \cup Y \in \mathcal{F}. \tag{2.3}$$

A member of an antimatroid is called a *feasible set*, and E is called the *ground set* of $\mathcal{F}$.

A convex geometry is considered as a "dual" object of an antimatroid. Before the definition of a convex geometry, we must have preliminaries. Let E be a non-empty finite set. Then a family $\mathcal{L} \subseteq 2^E$ is a *closure space* on E if it satisfies the following conditions:

$$\emptyset \in \mathcal{L}, \quad E \in \mathcal{L}, \tag{2.4}$$

$$X, Y \in \mathcal{L} \text{ imply } X \cap Y \in \mathcal{L}. \tag{2.5}$$

A member of a closure space is called a *closed set*.

For a closure space $\mathcal{L}$ on E, we consider the following operator $\tau : 2^E \to 2^E$, for $A \subseteq E$

$$\tau(A) = \bigcap\{X \in \mathcal{L} : A \subseteq X\}. \tag{2.6}$$

That is, $\tau(A)$ is the smallest set in $\mathcal{L}$ containing A. We can easily check that τ has the following four properties:

$$\tau(\emptyset) = \emptyset, \tag{2.7}$$

$$A \subseteq \tau(A), \tag{2.8}$$

$$A \subseteq B \text{ implies } \tau(A) \subseteq \tau(B), \tag{2.9}$$

$$\tau(\tau(A)) = \tau(A). \tag{2.10}$$

Generally, an operator satisfying these four properties is called a *closure operator*. Therefore, τ is a closure operator. Conversely, every closure operator τ defines a closure space $\mathcal{L}$ as follows:

$$\mathcal{L} = \{X \subseteq E : \tau(X) = X\}. \tag{2.11}$$

Now we define a convex geometry. Let $\mathcal{L}$ be a closure space on E with the closure operator $\tau : 2^E \to 2^E$. Then $\mathcal{L}$ is a *convex geometry* if τ satisfies the following axiom, called the *anti-exchange property*, for all $X \subseteq E$,

$$\text{if } a, b \notin \tau(X), a \neq b \text{ and } b \in \tau(X \cup \{a\}), \text{ then } a \notin \tau(X \cup \{b\}). \tag{2.12}$$

A member of a convex geometry is called a *convex set*.

Why is it called a convex geometry? Consider a finite set of points on the Euclidean space, and let τ be a convex hull operator. Then it is straightforward to check that τ satisfies (2.7)–(2.10) and (2.12), as shown in Figure 2.1.

A convex geometry can be also defined without the closure operator, as is shown in the next proposition.

Proposition 2.1. *Let $\mathcal{L}$ be a closure space on E. Then $\mathcal{L}$ is a convex geometry if and only if $X \in \mathcal{L} \setminus \{E\}$ implies $X \cup \{e\} \in \mathcal{L}$ for some $e \in E \setminus X$.*

Proof. See [28, Chapter III, Theorem 1.1]. $\square$

Hence, we show the dual relationship between a convex geometry and an antimatroid:

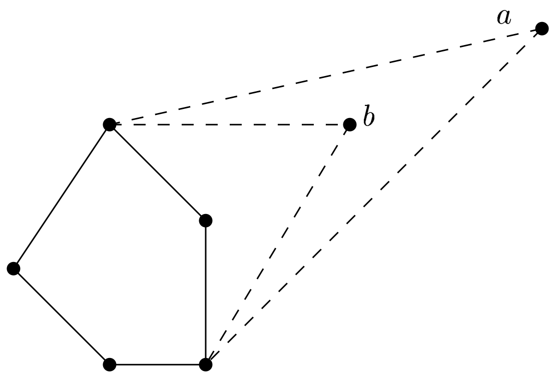

Figure 2.1: The anti-exchange property.

Proposition 2.2. *Let $\mathcal{F}$ be an antimatroid on E, and define $\mathcal{L}$ as $\mathcal{L} = \{E \setminus X : X \in \mathcal{F}\}$. Then $\mathcal{L}$ is a convex geometry on E. Conversely, let $\mathcal{L}$ be a convex geometry on E, and define $\mathcal{F}$ as $\mathcal{F} = \{E \setminus X : X \in \mathcal{L}\}$. Then $\mathcal{F}$ is an antimatroid on E.*

Proof. See [28, Chapter III, Theorem 1.3]. □

2.2 Partially ordered sets and lattices

In this section, we introduce some basic terminology of partially ordered sets and lattices. They appear frequently in the rest of this thesis.

A *partially ordered set* $(E, \leq)$, or a *poset* for short, is a finite set E with a binary relation $\leq$ satisfying:

$$x \leq x, \tag{2.13}$$

$$x \leq y \text{ and } y \leq x \text{ imply } x = y, \tag{2.14}$$

$$x \leq y \text{ and } y \leq z \text{ imply } x \leq z. \tag{2.15}$$

We write $x < y$ if $x \leq y$ and $x \neq y$.

Let $(E, \leq)$ be a poset. An element $x \in E$ is *minimal* if $y \leq x$ implies $y = x$, and *maximal* if $x \leq y$ implies $y = x$. For distinct elements $x, y \in E$, we say that x *covers* y if $y \leq x$ and $y \leq z \leq x$ imply $z = x$ or $z = y$.

Two distinct elements $x, y \in E$ are *comparable* if either $x \leq y$ or $y \leq x$. Otherwise, x and y are *incomparable*. A subset $X \subseteq E$ is a *chain* if any two elements of X are comparable, and an *antichain* if any two elements of X are incomparable. The *length* of the chain X is $|X| - 1$.

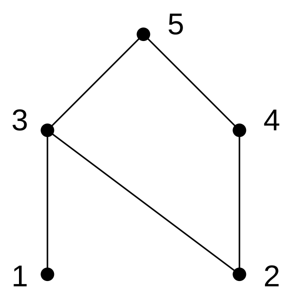

Figure 2.2: An example of the Hasse diagram.

We can visualize a poset $(E, \leq)$ with the *Hasse diagram*. We can construct the Hasse diagram as the following procedure:

1. we choose a standard horizontal/vertical coordinate system in the plane and require that the vertical coordinate of the point corresponding to x be larger than one to y if x covers y.

2. we put all the elements of E so as to satisfy this requirement.

3. Moreover, we add an edge between x and y if and only if x covers y, and we also require that this edge is drawn by a straight line segment which contains no points corresponding to some element of E other than its endpoints x and y.

Then, we have the Hasse diagram of $(E, \leq)$. Figure 2.2 is an example of the Hasse diagram of a poset $(E, \leq)$, where $E = \{1, 2, 3, 4, 5\}$ and

$$x \leq y \text{ if and only if } (x, y) = (1, 3), (1, 5), (2, 3), (2, 4), (2, 5), (3, 5), (4, 5).$$

For a poset $(E, \leq)$ we say that an element $u \in E$ is the *meet* of $x, y \in E$ if it satisfies the following condition: for every $z \in E$ we have that $z \leq u$ precisely when $z \leq x$ and $z \leq y$. Dually, an element $w \in E$ is the *join* of $x, y \in E$ if it satisfies the following condition: for every $z \in E$ we have that $w \leq z$ precisely when $x \leq z$ and $y \leq z$. We denote the meet of x and y by $x \wedge y$, and the join of x and y by $x \vee y$. A poset $(E, \leq)$ is a *lattice* if any two elements $x, y \in E$ have the meet and the join. The *height* of a lattice is the maximal length of its chains.

A lattice $(E, \leq)$ is *semimodular* if, for any $x, y \in E$ such that both cover the meet $x \wedge y$, the join $x \vee y$ covers x and y. It is known that all the maximal chains of a semimodular lattice have the same length. See [43] for a comprehensive survey on semimodular lattices.

A set system is often treated as a poset. Let $\mathcal{F}$ be a family of subset of a finite set E, not necessarily an antimatroid nor a convex geometry. Then, $(\mathcal{F}, \subseteq)$ forms a poset, that is, $\mathcal{F}$ is partially ordered with respect to set-inclusion.

The following shows the relationship between antimatroids and semimodular lattices.

Proposition 2.3. *Let $\mathcal{F} \subseteq 2^E$ be a family of subsets of E with $\emptyset, E \in \mathcal{F}$. Then $\mathcal{F}$ is an antimatroid on E if and only if $(\mathcal{F}, \subseteq)$ is a semimodular lattice of height $|E|$.*

Proof. See [28, Chapter VI, Theorem 1.1]. □

Dually, a lattice $(E, \leq)$ is *lower semimodular* if, for any $x, y \in E$ such that the join $x \vee y$ covers both, the meet $x \wedge y$ is covered by x and y. By Propositions 2.2 and 2.3, a convex geometry is a lower semimodular lattice with respect to set-inclusion.

Let be $\mathcal{F} \subseteq 2^E$ a family of subset of a finite set E. The poset $(\mathcal{F}, \subseteq)$ is a Boolean lattice if $\mathcal{F} = 2^E$. In general, a lattice is *Boolean* if it is isomorphic to $(2^E, \subseteq)$.

2.3 Examples

In this section, we see some examples of antimatroids and convex geometries.

Various combinatorial objects give rise to antimatroids. In particular, we can obtain many classes of antimatroids by shelling processes or searching processes. By *shelling* processes, we mean repeated elimination of suitable elements until all the elements are removed. We see some examples of shelling processes.

Example 2.4 (convex shelling of points on the Euclidean space). Let $E \subseteq \mathbb{R}^n$ be a finite set of points on the Euclidean space, where $\mathbb{R}$ is the set of reals. Then, we consider the following procedure:

1. Set $X \leftarrow \emptyset$ and $\mathcal{C} \leftarrow \{\emptyset\}$.

2. While $X \neq E$ repeat:

   2.1 Choose a vertex of the convex hull of $E \setminus X$, say e.

   2.2 Reset $X \leftarrow X \cup \{e\}$ and $\mathcal{C} \leftarrow \mathcal{C} \cup \{X \cup \{e\}\}$.

   2.3 Return to the head of this repetition.

Here, the *convex hull* of X is the minimal closed set which includes all points of X. For example, we consider the case of Figure 2.3. Figure 2.3 indicates how the procedure executes, and now by this procedure we have

$$\mathcal{C} = \{\emptyset, \{3\}, \{2,3\}, \{2,3,4\}, \{1,2,3,4\}\}.$$

The family $\mathcal{C}$ depends on the order to choose e in each iteration. Here, we enumerate all of the possible cases:

$$\begin{aligned}
\mathcal{C}_1 &= \{\emptyset, \{1\}, \{1,2\}, \{1,2,3\}, \{1,2,3,4\}\}, \\
\mathcal{C}_2 &= \{\emptyset, \{1\}, \{1,2\}, \{1,2,4\}, \{1,2,3,4\}\}, \\
\mathcal{C}_3 &= \{\emptyset, \{1\}, \{1,3\}, \{1,2,3\}, \{1,2,3,4\}\}, \\
\mathcal{C}_4 &= \{\emptyset, \{1\}, \{1,3\}, \{1,3,4\}, \{1,2,3,4\}\}, \\
\mathcal{C}_5 &= \{\emptyset, \{1\}, \{1,4\}, \{1,3,4\}, \{1,2,3,4\}\}, \\
\mathcal{C}_6 &= \{\emptyset, \{1\}, \{1,4\}, \{1,2,4\}, \{1,2,3,4\}\}, \\
\mathcal{C}_7 &= \{\emptyset, \{2\}, \{1,2\}, \{1,2,3\}, \{1,2,3,4\}\}, \\
\mathcal{C}_8 &= \{\emptyset, \{2\}, \{1,2\}, \{1,2,4\}, \{1,2,3,4\}\}, \\
\mathcal{C}_9 &= \{\emptyset, \{2\}, \{2,3\}, \{1,2,3\}, \{1,2,3,4\}\}, \\
\mathcal{C}_{10} &= \{\emptyset, \{2\}, \{2,3\}, \{2,3,4\}, \{1,2,3,4\}\}, \\
\mathcal{C}_{11} &= \{\emptyset, \{2\}, \{2,4\}, \{1,2,4\}, \{1,2,3,4\}\}, \\
\mathcal{C}_{12} &= \{\emptyset, \{2\}, \{2,4\}, \{2,3,4\}, \{1,2,3,4\}\}, \\
\mathcal{C}_{13} &= \{\emptyset, \{3\}, \{1,3\}, \{1,2,3\}, \{1,2,3,4\}\}, \\
\mathcal{C}_{14} &= \{\emptyset, \{3\}, \{1,3\}, \{1,3,4\}, \{1,2,3,4\}\}, \\
\mathcal{C}_{15} &= \{\emptyset, \{3\}, \{2,3\}, \{1,2,3\}, \{1,2,3,4\}\}, \\
\mathcal{C}_{16} &= \{\emptyset, \{3\}, \{2,3\}, \{2,3,4\}, \{1,2,3,4\}\}.
\end{aligned}$$

Then, we define $\mathcal{F}$ as

$$\begin{aligned}
\mathcal{F} &= \bigcup_{i=1}^{16} \mathcal{C}_i \\
&= \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \\
&\qquad \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\}.
\end{aligned}$$

We can easily check that $\mathcal{F}$ is an antimatroid, which is called the *convex shelling* of E.

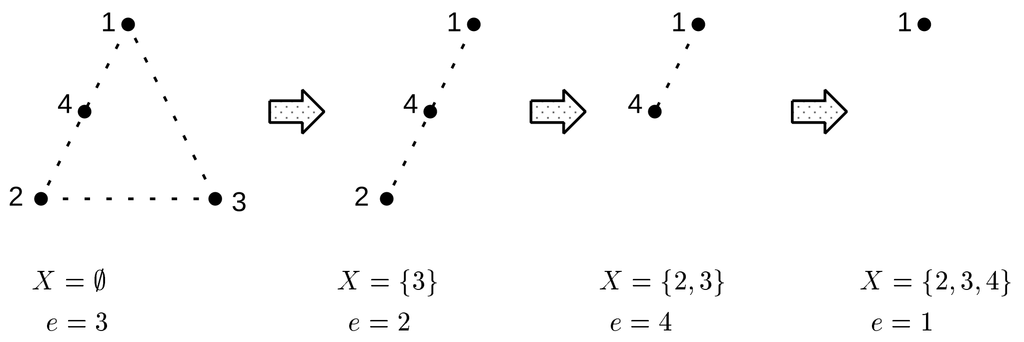

Figure 2.3: A shelling of points on the plane.

In other words, we can represent this antimatroid $\mathcal{F}$ as:

$$\mathcal{F} = \{X \subseteq E : \text{ conv.hull}(E \setminus X) \cap X = \emptyset\}, \tag{2.16}$$

where conv.hull$(E \setminus X)$ is the convex hull of $E \setminus X$. In general, if a given antimatroid is isomorphic to some convex shelling, then it is also called a convex shelling.

Often, we represent an antimatroid by the Hasse diagram as Figure 2.4.

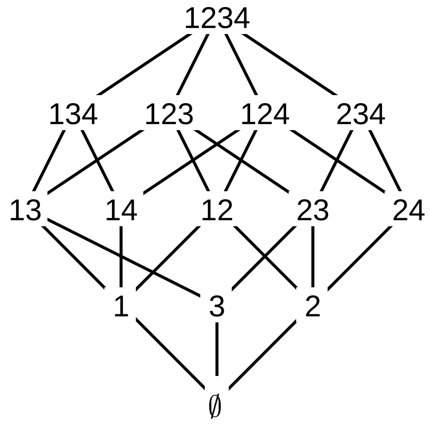

Figure 2.4: the Hasse diagram of the convex shelling.

Dually, a convex geometry $\mathcal{L}$ defined as

$$\mathcal{L} = \{X \subseteq E : \text{ conv.hull}(X) \cap E = X\}. \tag{2.17}$$

is also called a convex shelling. We use the same words “convex shelling” for antimatroids and convex geometries. When it is confusing, we use the words “convex shelling antimatroids,” etc. We use this convention for the following examples.

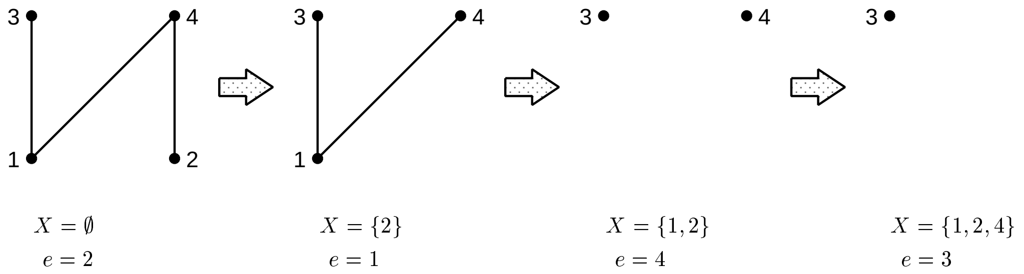

Figure 2.5: A shelling of a poset.

Now, we see another example of shelling processes.

Example 2.5 (poset shelling). Let $(E, \leq)$ be a poset. Then we consider the following procedure:

1. Set $X \leftarrow \emptyset$ and $\mathcal{C} \leftarrow \{\emptyset\}$.

2. While $X \neq E$ repeat:

   2.1 Choose a minimal element of $E \setminus X$, say e.

   2.2 Reset $X \leftarrow X \cup \{e\}$ and $\mathcal{C} \leftarrow \mathcal{C} \cup \{X \cup \{e\}\}$.

   2.3 Return to the head of this repetition.

For example, we consider the case of Figure 2.5. Figure 2.5 indicates how the procedure executes, and now by this procedure we have

$$\mathcal{C} = \{\emptyset, \{2\}, \{1,2\}, \{1,2,4\}, \{1,2,3,4\}\}.$$

The family $\mathcal{C}$ depends on the order to choose e in each iteration. Then,

$$\begin{aligned} \mathcal{F} &= \bigcup \mathcal{C} \\ &= \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{1,3\}, \{1,2,3\}, \{1,2,4\}, \{1,2,3,4\}\} \end{aligned}$$

forms an antimatroid, which is called the *poset shelling* of $(E, \leq)$.

An *order ideal* of the poset $(E, \leq)$ is a subset $X \subseteq E$ such that $b \in X, a \leq b$ imply $a \in X$. We also define a poset shelling $\mathcal{F}$ by order ideals as follows:

$$\mathcal{F} = \{X \subseteq E : X \text{ is an order ideal of } (E, \leq)\}. \tag{2.18}$$

Figure 2.5 shows the representation by the Hasse diagram of this antimatroid.

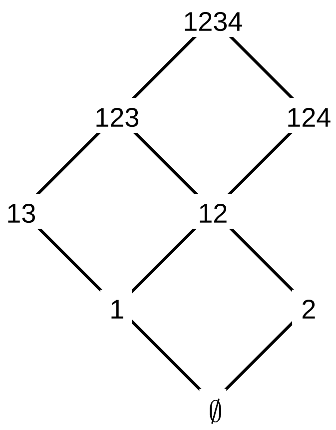

Figure 2.6: The Hasse diagram of the poset shelling.

The only difference of the procedures in Example 2.4 and Example 2.5 is Step 2.1. In Example 2.4, Step 2.1 of the procedure lets us choose a vertex of the convex hull, while in Example 2.5, Step 2.1 of the procedure lets us choose a minimal element. Thus, by modifying Step 2.1, we can generate some antimatroids in similar ways.

Example 2.6 (vertex shelling of a tree / edge shelling of a tree). Let $G = (V, E)$ be a tree. A *leaf* of G is a vertex with degree 1. If we delete the leaves step by step, then we obtain an antimatroid on V, which is called the *vertex shelling* of the tree G. Analogously, we define the *edge shelling* of a tree.

Example 2.7 (double shelling of a poset). Let $(E, \leq)$ be a poset. Now, we consider deleting minimal elements or maximal elements repeatedly. Then, we obtain an antimatroid on E, which is called the *double shelling* of the poset $(E, \leq)$.

Example 2.8 (simplicial shelling of a triangulated graph). A graph $G = (V, E)$ is *triangulated*, or *chordal*, if it contains no induced cycles other than triangles. A vertex of a graph is *simplicial* if its neighbors form a complete subgraph.

Let $G = (V, E)$ be a triangulated graph. If we eliminate a simplicial vertex in each iteration, then we obtain an antimatroid on V, which is called the *simplicial shelling* of the triangulated graph G.

A *search* is another process to obtain antimatroids, which means a gathering process of adjacent elements. Let us see an example.

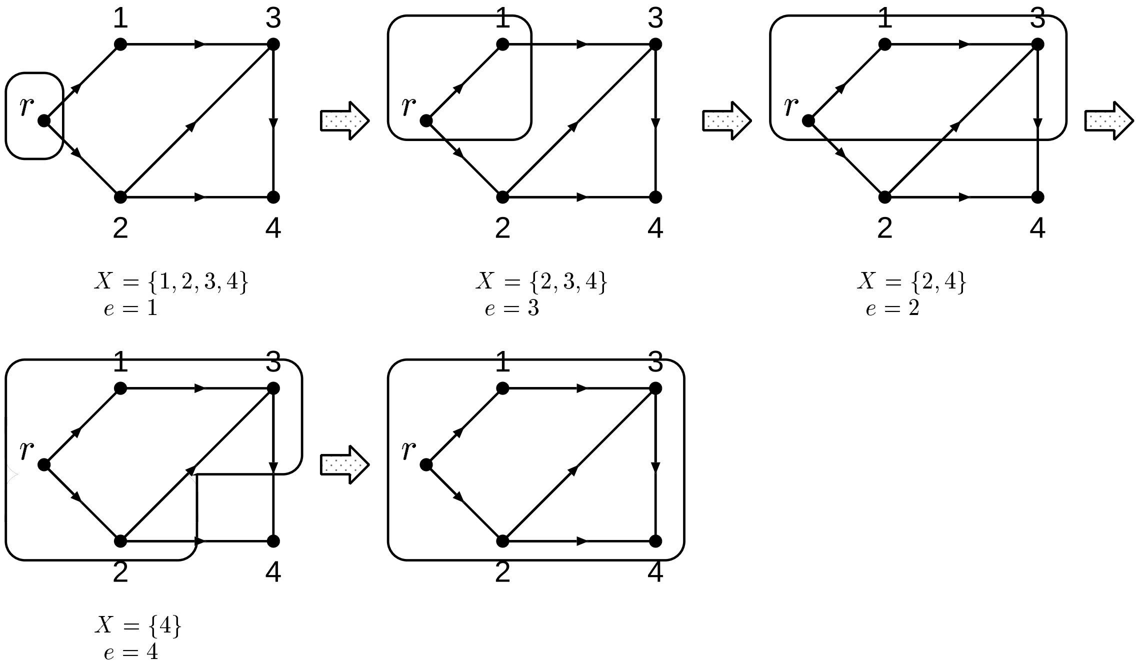

Figure 2.7: A searching process of a graph.

Example 2.9 (point-search of a rooted directed graph). Let $G = (V \cup \{r\}, E)$ be a directed graph with a specified vertex r called the *root*. Then we consider the following procedure:

1. Set $X \leftarrow \emptyset$ and $\mathcal{C} \leftarrow \{\emptyset\}$.

2. While $X \neq V$ repeat:

   2.1 Choose a vertex of $V \setminus X$ adjacent to a vertex of $X \cup \{r\}$, say e.

   2.2 Reset $X \leftarrow X \cup \{e\}$ and $\mathcal{C} \leftarrow \mathcal{C} \cup \{X \cup \{e\}\}$.

   2.3 Return to the head of this repetition.

For example, we consider a rooted graph G with the vertices $\{r, 1, 2, 3, 4\}$ and edges $\{(r, 1), (r, 2), (1, 3), (2, 3), (2, 4), (3, 4)\}$ and the search process as Figure 2.7. By the procedure in Figure 2.7 we have $\mathcal{C} = \{\emptyset, \{1\}, \{1, 3\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}$. The family $\mathcal{C}$ depends on the order that we choose e in each iteration. Then,

$$\begin{aligned}
\mathcal{F} &= \bigcup \mathcal{C} \\
&= \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \\
&\qquad \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}
\end{aligned}$$

forms an antimatroid on V, which is called the *point-search* of the rooted directed graph G.

Analogously, we also consider point-searches of rooted undirected graphs, line-searches of rooted directed graphs, and line-searches of rooted undirected graphs.

In the proceeding chapters, we introduce some other concepts on antimatroids and convex geometries; in Chapter 3, minors of an antimatroid; in Chapter 4, traces, free sets, linear sets and circuits of an antimatroid; in Chapter 5, extreme points and extreme sets of a convex geometry.

Chapter 3

The Forbidden Minor Characterization of Line-Search Antimatroids of Rooted Digraphs

3.1 Introduction to this chapter

In this chapter, we give the forbidden minor characterization for line-search antimatroids of rooted digraphs.

For an antimatroid $\mathcal{F}$, a *minor* $\mathcal{F}[A, B]$ is defined as follows:

$$\mathcal{F}[A, B] = \{X \setminus A : X \in \mathcal{F}, A \subseteq X \subseteq B\}, \tag{3.1}$$

where $A, B \in \mathcal{F}$ and $A \subseteq B$. We can easily check that each minor of an antimatroid is also an antimatroid.

In greedoid theory, some classes are characterized by their forbidden minors: local poset greedoids [27], undirected branching greedoids [21, 40], poset shelling antimatroids and point-search antimatroids of rooted (di)graphs [35].

Note that there are still other antimatroids whose forbidden minor characterizations have not known yet; for example, line-search antimatroids of rooted undirected graphs.

3.2 Basics for this chapter

3.2.1 Point-search antimatroids of rooted digraphs

In Chapter 2, we briefly see point-search antimatroids of rooted digraphs. Now we introduce them more formally. A *digraph* G is a pair (V, E) such that V is a non-empty finite set of *vertices*, and E is a subset of $\{(x, y) : x, y \in V, x \neq y\}$ called a set of *edges*. For simplicity, we write xy instead of (x, y). For an edge $xy \in E$, x is called the *tail*, and y is called the *head*.

A *path* P in $G = (V, E)$ is a sequence of vertices $x_1x_2 \cdots x_m$ with $x_ix_{i+1} \in E$ for $i = 1, \ldots, m-1$. A path $P = x_1 \cdots x_m$ is also called a path from x_1 to x_m. For a path $P = x_1 \cdots x_m$, if there exists an edge $x_ix_j \in E$ $(i + 1 < j)$, then the edge x_ix_j is called a *short-cut* of the path P. A path without repeated vertices is called *elementary*. An elementary path without any short-cuts is called *straight*.

A digraph G is called a *rooted digraph*, or an *r-digraph*, if it has an distinguished vertex r called the *root* and there exists a path from r to every other vertex of G. A path from the root r is called a *rooted path*, or an *r-path*. A vertex v is called an *atom* if $rv \in E$.

For an r-digraph $G = (V \cup \{r\}, E)$, we consider the following procedure: first we choose one of the atoms, say v; next we shrink v to the root. If we repeat this procedure until all vertices are shrunk to the root, then we will obtain a sequence of vertices selected by the above procedure of shrinking. If we gather all of these sequences, then they form an antimatroid. Formally, for an r-digraph $G = (V \cup \{r\}, E)$, we define the *point-search antimatroid* $\mathfrak{PS}_D(G)$ as follows:

$$\begin{aligned}\mathfrak{PS}_D(G) = \{X \subseteq V :\ & \text{every vertex } v \in X \text{ can be reached by an} \\ & \text{r-path in the subgraph induced by } X \cup \{r\}\}.\end{aligned} \tag{3.2}$$

Note that the class of point-search antimatroids is closed under taking minors.

In an r-digraph $G = (V, E)$, let $e \in E$ be an edge of G. Suppose $P = ru_1u_2 \cdots u_m$ to be a straight r-path such that $u_{m-1}u_m = e$. Then we say that *e is supported by P*, or *P supports e*. If there is no path supporting e, then e is called a *redundant* edge. If an r-digraph contains no redundant edge, then it is called an *irredundant* r-digraph. Note that redundant edges have no use for defining point-search antimatroids. In particular, irredundant r-digraphs have no edge whose head is the root r or an atom. For an r-digraph G, define G_0 as the r-digraph such that the redundant edges of G

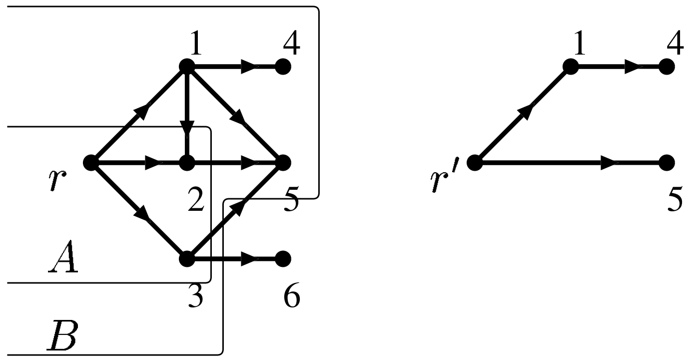

Figure 3.1: A rooted digraph and the r-minor.

are deleted, then point-search antimatroids of G and G_0 are the same. Therefore, without loss of generality, when we consider point-search antimatroids of r-digraphs, we only have to handle irredundant r-digraphs.

Let $G = (V \cup \{r\}, E)$ be an r-digraph, and $\mathfrak{PS}_D(G)$ be the point-search antimatroid of G. For $A, B \in \mathfrak{PS}_D(G)$ with $A \subseteq B$, remove $V \setminus B$ and the edges incident to $V \setminus B$ from G, shrink the vertices A to r. Then delete all the redundant edges from the resultant graph. This procedure gives us an irredundant r-digraph, which we call an *r-minor* and denote by $G[A, B]$. Figure 3.1 shows an example of r-minors. Note that every r-minor of an irredundant r-digraph is also irredundant. Clearly, the point-search antimatroid of $G[A, B]$ is equal to the minor $\mathfrak{PS}_D(G)[A, B]$, namely $\mathfrak{PS}_D(G[A, B]) = \mathfrak{PS}_D(G)[A, B]$. Furthermore, suppose G' to be another irredundant r-digraph. Then $\mathfrak{PS}_D(G)$ contains a minor isomorphic to $\mathfrak{PS}_D(G')$ if and only if there exists an r-minor of G which is isomorphic to G'.

3.2.2 Line graphs

A *multi-digraph* H is a quadruple $(N, A; h, t)$ where N is a non-empty finite set of *nodes*, A is a finite set of *arcs*, and h, t are maps from A to N. For $a \in A$, $h(a) \in N$ is the *head* of a, and $t(a) \in N$ is the *tail* of a. A digraph is a special case of multi-digraphs in a natural way. A *path* in H is a sequence of arcs $a_1 \cdots a_k$ such that $h(a_i) = t(a_{i+1})$ for $i = 1, \ldots, k-1$. If a path has no repeated arcs, it is called *simple*.

A multi-digraph $H = (N, A; h, t)$ defines a digraph $G = (A, E)$ by $E = \{(a, b) : a, b \in A, a \neq b, h(a) = t(b)\}$, which is called the *line graph* of H. A digraph G is a *line graph* if there exists some multi-digraph of which G is the line graph. Syslo

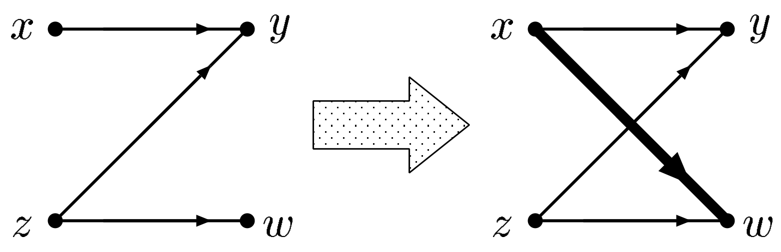

Figure 3.2: The Heuchenne condition.

[44] gives a polynomial-time algorithm which decides whether the given digraph is a line graph or not. The algorithm is based on the following characterization of line graphs [23, 37]:

Proposition 3.1. *Let $G = (V, E)$ be a digraph. G is a line graph if and only if for every $x, y, z, w \in V$, $(x, y), (z, y), (z, w) \in E$ imply $(x, w) \in E$, as shown in Figure 3.2.*

The condition of this proposition is called the *Heuchenne condition*, or the *H-condition*, for short.

A *rooted multi-digraph*, or an *r-multi-digraph*, is a multi-digraph with a distinguished node r' called a *root* such that for every arc there exists a simple path from r' which contains it. An r-multi-digraph $H = (N \cup \{r'\}, A, h, t)$ also gives its *rooted line graph* as follows: add a new node r'' and insert an arc $r''r'$ to H, and construct the line graph of this resultant multi-digraph, then we have a digraph G whose vertices are $A \cup \{r\}$ where r is a vertex corresponding to the arc $r''r'$. By assumption, it is obvious that there exists an r-path to every vertex in G. Hence G is an r-digraph.

3.3 The forbidden minor characterization of line-search antimatroids

As is analogous to point-search antimatroids, we define the *line-search antimatroid* $\mathfrak{LS}_D(H)$ of an r-multi-digraph $H = (N \cup \{r'\}, A, h, t)$ as follows:

$$\begin{aligned} \mathfrak{LS}_D(H) = \{ X \subseteq A : & \text{ every arc } a \in A \text{ is contained in a simple} \\ & \text{path from } r' \text{ on the subgraph induced by } X \cup \{r'\}\}. \end{aligned} \tag{3.3}$$

Note that line-search antimatroids of rooted multi-digraphs are also closed under taking their minors.

Let G be the rooted line graph of an r-multi-digraph H. Then the line-search antimatroid of H coincides with the point-search antimatroid of G. Therefore, the class of point-search antimatroids of rooted digraphs includes that of line-search antimatroids of rooted multi-digraphs. It is easily checked that there is a one-to-one correspondence between line-search antimatroids of rooted multi-digraphs and irredundant r-digraphs which satisfy the H-condition.

Point-search antimatroids of rooted digraphs are characterized by the forbidden minor [35]:

Proposition 3.2. *$\mathcal{F}$ is the point-search antimatroid of a rooted digraph if and only if $\mathcal{F}$ does not contain a minor isomorphic to $D_5 = \{\emptyset, \{x\}, \{y\}, \{x, y\}, \{x, y, z\}\}$, as shown in Figure 3.3.*

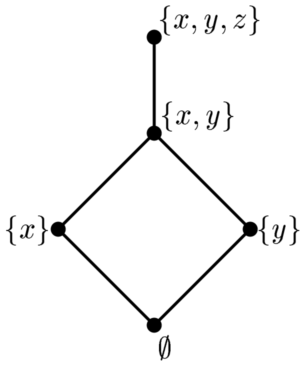

Figure 3.3: The forbidden minor D_5 of point-search antimatroids of rooted digraphs.

Hence, in order to characterize line-search antimatroids of rooted digraphs, we only need to characterize point-search antimatroids of irredundant rooted digraphs which violate the H-condition.

For example, an irredundant r-digraph $A = (V(A) \cup \{r\}, E(A))$ defined by

$$V(A) = \{a, b, c, d\}, \tag{3.4}$$

$$E(A) = \{(r, a), (r, b), (a, c), (b, c), (b, d)\}, \tag{3.5}$$

as shown in Figure 3.4, violates the H-condition.

Additionally, the following three kinds of irredundant r-digraphs $B, C_{m,n}, D_{l,m,n}$ also violate the H-condition; $B = (V(B) \cup \{r\}, E(B))$ is defined by

$$V(B) = \{a, b, c, d\}, \tag{3.6}$$

$$E(B) = \{(r, a), (r, b), (a, c), (b, c), (b, d), (c, d)\}, \tag{3.7}$$

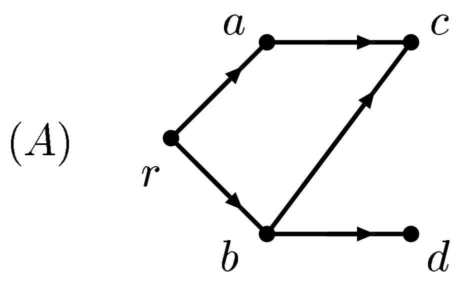

Figure 3.4: The r-digraph A violates the H-condition.

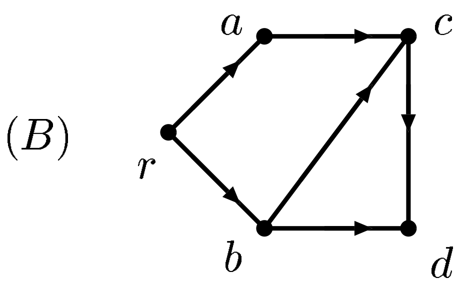

Figure 3.5: The r-digraph B violates the H-condition.

which is shown in Figure 3.5; $C_{m,n} = (V(C_{m,n}) \cup \{r\}, E(C_{m,n}))$ is defined by

$$\begin{aligned} V(C_{m,n}) &= \{a, b, c = x_0, d = y_0, e, x_1, \dots, x_m, y_1, \dots, y_n\}, \quad (3.8)\\ E(C_{m,n}) &= \{(r,a), (r,b), (a,c), (b,d), (c,x_1), (d,y_1), (e,c), (e,d), \quad (3.9)\\ &\quad (x_1,x_2), \dots, (x_{m-2}, x_{m-1}), (x_{m-1}, e),\\ &\quad (y_1,y_2), \dots, (y_{n-2}, y_{n-1}), (y_{n-1}, e)\}, \end{aligned}$$

where $m, n \geq 0$, which is shown in Figure 3.6; and $D_{l,m,n} = (V(D_{l,m,n}) \cup \{r\}, E(D_{l,m,n}))$ is defined by

$$\begin{aligned} V(D_{l,m,n}) &= \{a, b, c = x_0, d = y_0, e, f = z_0, x_1, \dots, x_l, y_1, \dots, y_m, \quad (3.10)\\ &\quad z_1, \dots, z_n\},\\ E(D_{l,m,n}) &= \{(r,a), (r,b), (a,c), (b,d), (c,x_1), (d,y_1), (e,c), (e,d), \quad (3.11)\\ &\quad (f,z_1), (x_1,x_2), \dots, (x_{l-2}, x_{l-1}), (x_{l-1}, f),\\ &\quad (y_1,y_2), \dots, (y_{m-2}, y_{m-1}), (y_{m-1}, f),\\ &\quad (z_1,z_2), \dots, (z_{n-2}, z_{n-1}), (z_{n-1}, e)\}, \end{aligned}$$

where $l, m, n \geq 0$, which is shown in Figure 3.7.

$(C_{m,n})$

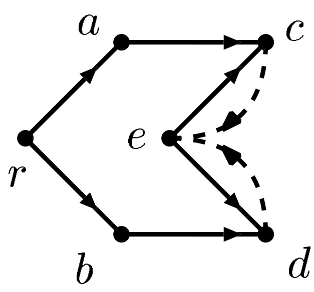

Figure 3.6: The r-digraph $C_{m,n}$ $(m, n \geq 0)$ violates the H-condition, where the broken arrows represent arbitrarily long paths.

$(D_{l,m,n})$

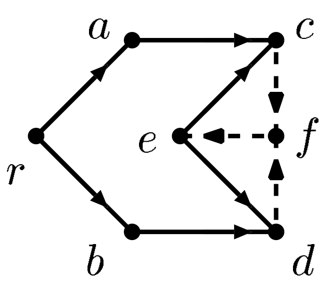

Figure 3.7: The r-digraph $D_{l,m,n}$ $(l, m, n \geq 0)$ violates the H-condition, where the broken arrows represent arbitrarily long paths.

Therefore, if an irredundant rooted digraph G is a rooted line graph, then G has no r-minor isomorphic to $A, B, C_{m,n}$ or $D_{l,m,n}$ $(l, m, n \geq 0)$. Indeed, these r-digraphs leads to the sufficiency.

Theorem 3.3. *Let G be an irredundant rooted digraph. Then, G is a rooted line graph if and only if G has no r-minor isomorphic to $A, B, C_{m,n}$ or $D_{l,m,n}$ $(l, m, n \geq 1)$.*

Proof. We only need to show the sufficiency. Let $G = (V \cup \{r\}, E)$ be an irredundant r-digraph containing four vertices x, y, z, w which violate the H-condition and is minor-minimal with respect to this property. Let $W = \{x, y, z, w\}$.

A vertex a is the *joint* of a straight path P from r to a vertex of W if a is the first vertex of W along the path P from r. Let T be the set of joints of straight paths in G. From the assumption, we have $T \neq \emptyset$ and there must exist a path supporting each of edges xy, zy, zw, which we denote by P, Q, R, respectively. We consider the following cases according to the size of T.

Case 1. $|T| = 1$. It is easily checked that this case leads to a contradiction.

Case 2. $|T| = 2$. This has the following six subcases.

Case 2-1. $T = \{x, y\}$. The path Q is not straight since Q must go through x or y. This is a contradiction.

Case 2-2. $T = \{x, z\}$. A path with the joint x supports the edge xy, and a path with the joint z supports the edge zw. From the minimality of G, the vertices of G must be $\{r, x, y, z, w\}$. If we consider all the possible edges among them, then we obtain A and B.

Case 2-3. $T = \{x, w\}$. Suppose that the path Q goes through x, then the edge xy is a shortcut. This is a contradiction. Therefore, Q must go through not x but w. Moreover, Q is $r \cdots w \cdots zy$ since Q does not go through y. If a path with the joint w has no vertex between r and w, then it is a shortcut of the path R. Therefore, it has an extra vertex p between r and w, namely the path is rpw, from the minimality of G. Moreover, the path with the joint x is rx from the minimality of G as r-minors. Since the path R does not go through w, it must go through x. We consider the subcases according to whether R goes through the edge xy or not.

Case 2-3-1. R **goes through** xy. R is $r \cdots xy \cdots z$. If there is a common vertex of the part $y \cdots z$ of R and the part $w \cdots z$ of Q except for z, then G must contain $D_{l,m,n}$ as a subgraph. Otherwise, G must contain $C_{m,n}$ as a subgraph.

Now we should check that if G has no r-minor isomorphic to $C_{m,n}$ or $D_{l,m,n}$, then G must have A or B as its r-minor, or it leads to a contradiction.

Case 2-3-1-1. $C_{m,n}$ **has extra edges.** Refer the definition (3.8, 3.9) of $C_{m,n}$.

Case 2-3-1-1-1. the edge cd **exists.** If we shrink a to r and we set $a = c$ and $c = x_1$, then we can reduce this case to A or B.

Case 2-3-1-1-2. the edge $x_i y_j$ **exists** $(0 < i < m, 0 < j < n)$. If we shrink a, b, c, x_1, $\ldots$, x_{i-1}, y_0, $\ldots$, y_{j-2} to r and we set $a = x_i$, $b = y_{j-1}$, $c = x_{i+1}$ and $d = y_j$, then we reduce this case to A or B.

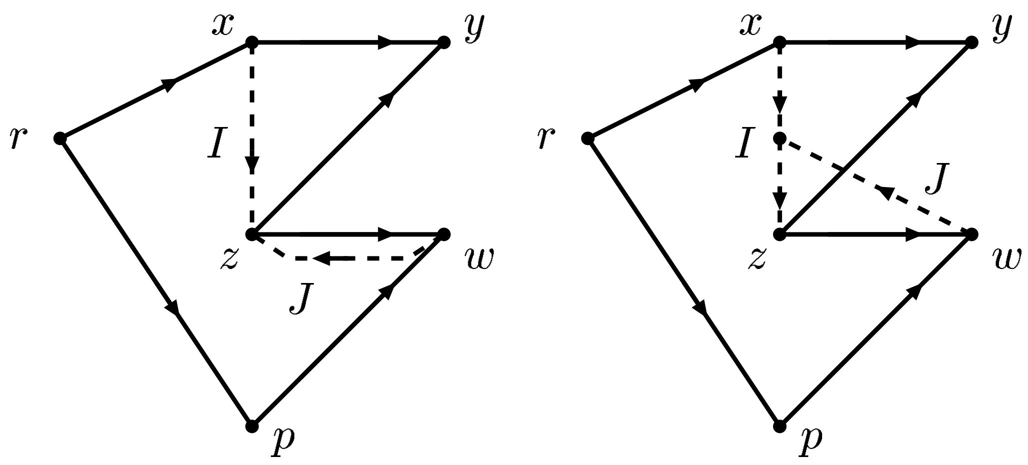

Figure 3.8: Case 2-3-2. Broken arrows represent arbitrarily long paths.

Case 2-3-1-1-3. the edge $x_i e$ exists. A contradiction since the edge $x_m e$ is redundant.

Case 2-3-1-2. $D_{l,m,n}$ has extra edges. We can check similarly to Case 2-3-1-1.

Case 2-3-2. R does not go through xy. Then, we obtain the graphs shown in Figure 3.8, where I is a path from x to z and J is a path from w to z. The left case shows that the common vertex of I and J is only z, and the right case shows that there are other common vertices of I and J.

Now we show that these graphs have A or B as an r-minor. We consider the left case. For the right case, it is similarly shown.

Case 2-3-2-1. the length of I is one, and the length of J is also one. If we shrink p to r, then it is reduced to B.

Case 2-3-2-2. the length of I is one, and the length of J is more than one. Let $J = wj_1j_2\dots j_hz$ for $h \geq 1$. If we shrink p, w, j_1, $\dots$, j_{h-1} to r, then it is reduced to B.

Case 2-3-2-3. the length of I is more than one, and the length of J is one. If we shrink p and w to r, then it is reduced to A.

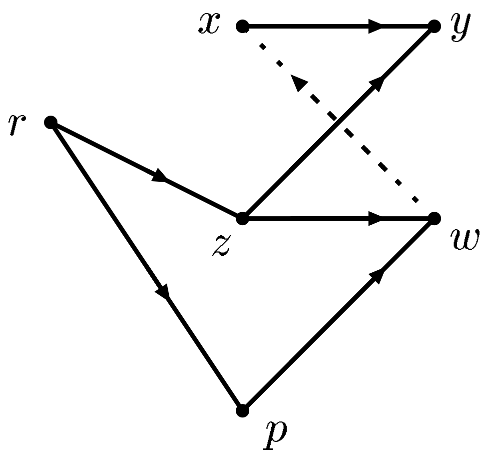

Figure 3.9: Case 2-6.

Case 2-3-2-4. the length of I is more than two, and the length of J is one. Let $I = xi_1i_2\dots i_kz$ for $k \geq 2$. If we delete $i_2, \dots, i_k$ and shrink p and w to r, then it is reduced to A.

Case 2-3-2-5. the lengths of both I and J are more than two. Let $I = xi_1i_2\dots i_kz$ for $k \geq 2$, and $J = wj_1\dots j_hz$ for $h \geq 1$. If we delete $i_2, \dots, i_k$ and shrink $p, w, j_1, \dots, j_h$ to r, then it is reduced to A.

Case 2-4. $T = \{y, w\}$. From the minimality and the irredundancy of G, the length of a path with the joint y is two, and let it be rpy. Similarly, the length of a path with the joint w is two, and let it be rqw. If $p = q$, then the three edges xy,zy and zw are always redundant. Therefore, we have $p \neq q$.

The path Q goes through neither x nor y. Therefore, Q is $rqw \cdots zy$.

The path R does not go through w. Hence, it must go through y. If we delete x, then it is reduced to $C_{m,n}$ or $D_{l,m,n}$.

Case 2-5. $T = \{y, z\}$. The path P does not go through y. Therefore, it must go through z. Then, it is a contradiction since the edge zy is a shortcut.

Case 2-6. $T = \{z, w\}$. Since the path Q does not go through z, it must go through w. From the minimality of G, the length of a path with the joint w is two, and the length of a path with the joint z is one. Now, we obtain the graph shown in Figure 3.9. Then, if we delete the vertices of the path $w \cdots x$ except for w, then it is reduced to A.

Case 3. $|T| = 3$. This has the following four subcases.

Case 3-1. $T = \{x, y, z\}$. The path P has the joint x. Moreover, the path Q has the joint z and supports the edge zw. Suppose that the length of a path Y with the joint y is one. Then the edges xy and zy are redundant. Therefore, the length of Y is more than one, that is, $Y = ry_1 \cdots y_k py$ for $k \geq 0$. Note that p is contained neither in P nor in Q.

Let $P = ru_1 \cdots u_l x$ and $Q = rv_1 \cdots v_m z$ for $l, m \geq 0$. If we delete p and shrink $u_1, \ldots, u_l, v_1, \ldots, v_m, y_1, \ldots, y_k$ to r, then it is reduced to A or B.

Case 3-2. $T = \{x, y, w\}$. Suppose that the length of a path Y with the joint y is one. Then the edges xy and zy are redundant. Therefore, the length of Y is more than one, that is, $Y = ry_1 \cdots y_k py$ for $k \geq 0$. If we delete x, then $\{p, y, z, w\}$ is the set of vertices which violates the H-condition. Therefore, it is reduced to Case 2-3.

Case 3-3. $T = \{x, z, w\}$. The path P has the joint x. Moreover, the path Q has the joint z and supports the edge zw. Suppose that the length of a path Y with the joint z is one. Then the edge zw is redundant. Therefore, the length of Y is more than one, that is, $Y = ry_1 \cdots y_k pw$ for $k \geq 0$. Note that p is contained neither in P nor in Q.

Let $P = ru_1 \cdots u_l x$ and $Q = rv_1 \cdots v_m z$ for $l, m \geq 0$. If we delete p, and shrink $u_1, \ldots, u_l, v_1, \ldots, v_m, y_1, \ldots, y_k$ to r, then it is reduced to A or B.

Case 3-4. $T = \{y, z, w\}$. The path Q has the joint z and supports the edge zw. Let Y be the path with the joint y. Note that the length of Y is more than one. Similarly, let W be the path with the joint w, then its length is more than one. The path P supporting the edge xy has the joint w. Let y be the vertex of Y which precedes p and q be the vertex of W which precedes q. Suppose that $p = q$, and consider the path P supporting the edge xy. The joint of P is not y. If the joint of P is z, then the edge zy is a shortcut of P. If the joint of P is w, then the edge py is a shortcut of P. Therefore, we have $p \neq q$.

Let $Y = ry_1 \cdots y_l py$, $W = rw_1 \cdots w_m qw$ and $Q = rq_1 \cdots q_n z$ for $l, m, n \geq 0$. If we delete p and x, and shrink $y_1, \ldots, y_l, w_1, \ldots, w_m, q_1, \ldots, q_n$ to r, then it is reduced to A or B.

Case 4. $|T| = 4$**.** It is easily checked that this case is reduced to Case 3-1 or Case 3-3. $\square$

Robertson–Seymour [39] have shown the Graph Minor Theorem, that is, in every infinite set of graphs there are two graphs such that one is a minor of the other. From this theorem, we conclude that every minor-closed property of graphs can be characterized by finitely many forbidden minors. But for antimatroids, Theorem 3.3 implies that there exists an infinite set of antimatroids such that any of those is not a proper minor of others.

Chapter 4

An Antimatroidal Analogue of Dilworth's Decomposition Theorem

4.1 Dilworth-type antimatroids

Let us recall Dilworth's decomposition theorem.

Let $P = (E, \leq)$ be a poset. A subset $X \subseteq E$ is a *chain* of P if any two distinct elements of X are comparable. Similarly, a subset $X \subseteq E$ is an *antichain* of P if any two distinct elements of X are incomparable. We denote the family of the chains of P by $\mathcal{C}(P)$, and the family of the antichains of P by $\mathcal{A}(P)$.

Proposition 4.1 (Dilworth's decomposition theorem [9]). *Let $P = (E, \leq)$ be a poset. Then,*

$$\max\{|A| : A \in \mathcal{A}(P)\} = \min\left\{t : C_1, C_2, \dots, C_t \in \mathcal{C}(P), \bigcup_{i=1}^{t} C_i = E\right\}, \quad (4.1)$$

that is, the maximum size of antichains of P is equal to the minimum number of chains of P which cover E.

Now, we consider an extension of Dilworth's decomposition theorem to antimatroids. First, we have to consider antimatroidal analogues of chains and antichains.

Let $\mathcal{F}$ be an antimatroid on E, and $A \subseteq E$. The *trace* on A is defined by

$$\mathcal{F} : A = \{A \cap X : X \in \mathcal{F}\}, \quad (4.2)$$

which means that, if we restrict E to A and consider shelling processes on A in the same way as we obtain $\mathcal{F}$ on E, then we obtain $\mathcal{F} : A$. Notice that the trace of an antimatroid is also an antimatroid on A. A subset $A \subseteq E$ is *free* if

$$\mathcal{F} : A = 2^A, \tag{4.3}$$

while A is *linear* if

$$\mathcal{F} : A = \{\emptyset\} \cup \{\{e_1, \ldots, e_i\} : i = 1, \ldots, k\} \tag{4.4}$$

for some numbering $A = \{e_1, \ldots, e_k\}$.

Remark 4.2. *Let $\mathcal{F}$ be the poset shelling of a poset $P = (E, \leq)$, and $A \subseteq E$. A is a free set or a linear set of $\mathcal{F}$ if and only if it is an antichain or a chain of P, respectively.*

Hence, Dilworth's decomposition theorem (Proposition 4.1) is restated as follows: for any poset shelling, the maximum size of free sets is equal to the minimum number of linear sets which cover the ground set.

For an antimatroid $\mathcal{F}$, we denote the family of the free sets by $\mathrm{Free}(\mathcal{F})$, and the family of the linear sets by $\mathrm{Lin}(\mathcal{F})$. Traces and free sets may be first introduced in [27], while linear sets did not appear in the past literature. We investigate some properties of free sets and linear sets.

Lemma 4.3. *Let $\mathcal{F}$ be an antimatroid on E. Then,*

$$X \in \mathrm{Free}(\mathcal{F}) \text{ and } X' \subseteq X \text{ imply } X' \in \mathrm{Free}(\mathcal{F}), \tag{4.5}$$

$$X \in \mathrm{Lin}(\mathcal{F}) \text{ and } X' \subseteq X \text{ imply } X' \in \mathrm{Lin}(\mathcal{F}). \tag{4.6}$$

Proof. Let $X \in \mathrm{Free}(\mathcal{F})$ and $X' \subseteq X$. Then,

$$\begin{aligned} \{Y \cap X' : Y \in \mathcal{F}\} &= \{(Y \cap X) \cap X' : Y \in \mathcal{F}\} \\ &= \{Z \cap X' : Z \in \mathcal{F} : X\}. \end{aligned}$$

Therefore, $\mathcal{F} : X' = (\mathcal{F} : X) : X'$. Since $2^E : X = \{Y \cap X : Y \in 2^E\} = 2^X$ for any $A \subseteq E$, we have $\mathcal{F} : X' = (\mathcal{F} : X) : X' = 2^X : X' = 2^{X'}$ for $X \in \mathrm{Free}(\mathcal{F})$. Hence $X' \in \mathrm{Free}(\mathcal{F})$. We have a similar discussion on $\mathrm{Lin}(\mathcal{F})$. ☐

Lemma 4.4. *Let $\mathcal{F}$ be an antimatroid on E. Then, for any $X \in \mathrm{Free}(\mathcal{F})$ and any $Y \in \mathrm{Lin}(\mathcal{F})$, we have $|X \cap Y| \leq 1$.*

Proof. By Lemma 4.3, we have $X \cap Y \in \mathrm{Free}(\mathcal{F})$. Similarly, $X \cap Y \in \mathrm{Lin}(\mathcal{F})$. Therefore, $\mathcal{F} : X \cap Y = 2^{X \cap Y} = \{\emptyset\} \cup \{\{e_1, \dots, e_i\} : i = 1, \dots, k\}$ for $X \cap Y = \{e_1, \dots, e_k\}$. Hence, $|X \cap Y|$ must be at most one. $\square$

Remark 4.5. *Let $\mathcal{F}$ be an antimatroid on E, and $A \subseteq E$. Then, it holds that*

$$\mathrm{Lin}(\mathcal{F}) = \{Y \subseteq E : |X \cap Y| \leq 1 \text{ for all } X \in \mathrm{Free}(\mathcal{F})\}.$$

In general, a pair of two families with this property is called anti-blocking pair [19, 22]. Although it is an important property, we do not use this property in this chapter.

Lemma 4.6. *Let $\mathcal{F}$ be an antimatroid on E, and $A \subseteq E$. Then,*

$$|A| \leq 1 \iff A \in \mathrm{Lin}(\mathcal{F}) \text{ and } A \in \mathrm{Free}(\mathcal{F}), \tag{4.7}$$

$$|A| = 2 \implies \text{either } A \in \mathrm{Lin}(\mathcal{F}) \text{ or } A \in \mathrm{Free}(\mathcal{F}). \tag{4.8}$$

Proof. It is easily checked from the definitions of free sets and linear sets. $\square$

Lemma 4.7. *Let $\mathcal{F}$ be an antimatroid on E. Then,*

$$\max\{|X| : X \in \mathrm{Free}(\mathcal{F})\} \leq \min\left\{t : Y_1, \dots, Y_t \in \mathrm{Lin}(\mathcal{F}), \bigcup_{i=1}^{t} Y_i = E\right\}. \tag{4.9}$$

Proof. Without loss of generality, we assume that $Y_1, \dots, Y_t \in \mathrm{Lin}(\mathcal{F})$ are all disjoint by Lemma 4.3. For any $X \in \mathrm{Free}(\mathcal{F})$ and for any disjoint $Y_1, \dots, Y_t \in \mathrm{Lin}(\mathcal{F})$ such that $\bigcup Y_i = E$,

$$|X| \leq \sum_{i=1}^{t} |X \cap Y_i| \leq \sum_{i=1}^{t} 1 = t,$$

using Lemma 4.4. $\square$

We now introduce Dilworth-type antimatroids. Let $\mathcal{F}$ be an antimatroid on E, $\mathrm{Free}(\mathcal{F})$ be the family of the free sets of $\mathcal{F}$, and $\mathrm{Lin}(\mathcal{F})$ be the family of the linear sets of $\mathcal{F}$. The antimatroid $\mathcal{F}$ is *Dilworth-type* if it satisfies the following equality:

$$\max\{|X| : X \in \mathrm{Free}(\mathcal{F})\} = \min\left\{t : Y_1, \dots, Y_t \in \mathrm{Lin}(\mathcal{F}), \bigcup_{i=1}^{t} Y_i = E\right\}. \tag{4.10}$$

Poset shellings (Example 2.5) are Dilworth-type by Remark 4.2. However, there are some antimatroids which are not Dilworth-type. Now, we observe the point-searches (Example 2.9) of two rooted directed graphs; one is Dilworth-type while the other is not.

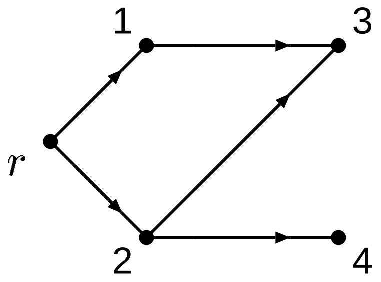

Figure 4.1: This graph yields a Dilworth-type antimatroid.

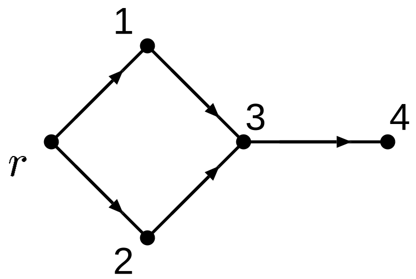

Figure 4.2: This graph does not yield Dilworth-type antimatroid.

Example 4.8. Let G be a directed graph with the vertices $\{r, 1, 2, 3, 4\}$ and the edges $\{(r, 1), (r, 2), (1, 3), (2, 3), (2, 4)\}$ (Figure 4.1), and $\mathcal{F}$ be the point-search of G. Then,

$$\begin{aligned} \mathrm{Lin}(\mathcal{F}) &= \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{2, 4\}\}, \\ \mathrm{Free}(\mathcal{F}) &= \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{3, 4\}, \{1, 3, 4\}\}. \end{aligned}$$

We can check that $\mathcal{F}$ is Dilworth-type.

Example 4.9. Let G be a directed graph with the vertices $\{r, 1, 2, 3, 4\}$ and the edges $\{(r, 1), (r, 2), (1, 3), (2, 3), (3, 4)\}$ (Figure 4.2), and $\mathcal{F}$ be the point-search of G. Then,

$$\begin{aligned} \mathrm{Lin}(\mathcal{F}) &= \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{3, 4\}\}, \\ \mathrm{Free}(\mathcal{F}) &= \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\}. \end{aligned}$$

We can check that $\mathcal{F}$ is not Dilworth-type.

In the next section, we introduce circuits of antimatroids and show that an antimatroid is not Dilworth-type if all of its circuits have cardinality more than two.

4.2 Circuits of antimatroids and Dilworth-type antimatroids

We now introduce the concept of circuits of antimatroids. Let $\mathcal{F}$ be an antimatroid on E. A subset $C \subseteq E$ is a *circuit* of $\mathcal{F}$ if C is a minimal non-free set, that is, C is not free and every proper subset of C is free. Let us see some examples. If $\mathcal{F}$ is a convex shelling on the 2-dimensional Euclidean plane (Example 2.4), then the circuits are triples of colinear points and quadruples in which one point is in the interior of the triangle spanned by the other three. If $\mathcal{F}$ is a shelling of a poset $P = (E, \leq)$ (Example 2.5), then the circuits are pairs $\{a, b\}$ such that $a < b$. If $\mathcal{F}$ is a vertex shelling of a tree $G = (V, E)$ (Example 2.6), then the circuits are triples of vertices lying on a path. If $\mathcal{F}$ is a double shelling of a poset $P = (E, \leq)$ (Example 2.7), then the circuits are triples $\{a, b, c\}$ such that $a < b < c$. Note that we have the characterization of antimatroids via circuits [6] analogous to the case of matroids, although it is not a topic for this thesis.

Let C be a circuit of an antimatroid $\mathcal{F}$ on E. Then, it is known that there exists a unique element $r \in C$ such that $\mathcal{F} : C = 2^C \setminus \{\{r\}\}$ (see [27, 28], etc.). We call the element r the *root* of the circuit C. Sometimes we denote a circuit C with its root r by (C, r) in order to specify the root of C.

Lemma 4.10. *Let $\mathcal{F}$ be an antimatroid on E and $C \subseteq E$ be a circuit of $\mathcal{F}$. Then $|C| = 2$ if and only if $C \in \mathrm{Lin}(\mathcal{F})$.*

Proof. Set $C = \{r, e\}$ with the root r. From the definition of the root, $\mathcal{F} : \{r, e\} = \{\emptyset, \{e\}, \{r, e\}\}$. Therefore, $\{r, e\}$ is linear. The converse can be similarly checked. □

We see an interesting fact in [27]. An antimatroid $\mathcal{F}$ on E is *Boolean* if $\mathcal{F} = 2^E$. Otherwise, it is called non-Boolean.

Proposition 4.11 (Characterization of poset shellings via circuits [27]). *Let $\mathcal{F}$ be a non-Boolean antimatroid on E. Then, $\mathcal{F}$ is a poset shelling if and only if every circuit of $\mathcal{F}$ has cardinality two.*

From Remark 4.2 and Proposition 4.11, we can see that, if every circuit of $\mathcal{F}$ has cardinality two, then $\mathcal{F}$ is Dilworth-type.

Now we characterize the antimatroids whose circuits have cardinality more than two, and show that they are not Dilworth-type. An antimatroid $\mathcal{F}$ on E is *coatomic* if $E \setminus \{e\} \in \mathcal{F}$ for all $e \in E$, which means that we can leave any element as the last in the shelling process for a coatomic antimatroid. There are many coatomic antimatroids. For example, convex shellings of points on the Euclidean space (Example 2.4), vertex/edge shellings of trees (Example 2.6), double shellings of posets (Example 2.7), simplicial shellings of triangulated graphs (Example 2.8) are coatomic.

Lemma 4.12 (Characterization of coatomic antimatroids via circuits). *Let $\mathcal{F}$ be a non-Boolean antimatroid on E. Then, $\mathcal{F}$ is a coatomic antimatroid if and only if every circuit of $\mathcal{F}$ has cardinality more than two.*

Proof. Let $\mathcal{F}$ be coatomic, that is, $E \setminus \{e\} \in \mathcal{F}$ for any $e \in E$. Suppose that $\{a, b\}$ is a circuit of $\mathcal{F}$. From Lemma 4.10, $\{a, b\}$ is linear. Therefore, $\mathcal{F} : \{a, b\} = \{\emptyset, \{a\}, \{a, b\}\}$. This implies $E \setminus \{a\} \notin \mathcal{F}$, which is a contradiction.

Conversely, let $\mathcal{F}$ be not coatomic, that is, $E \setminus \{e\} \notin \mathcal{F}$ for some $e \in E$. Here, we only have to show that $\{a, e\}$ is a circuit for some $a \in E$, that is, $\{a, e\}$ is linear (Lemma 4.10). Suppose that $\{a, e\}$ is free for any $a \in E \setminus \{e\}$. Then, for each $a \in E \setminus \{e\}$, there exists $A \in \mathcal{F}$ such that $e \notin A$ and $a \in A$. This implies that $E \setminus \{e\} = \bigcup\{A \in \mathcal{F} : e \notin A\} \in \mathcal{F}$, which is a contradiction. □

Lemma 4.13. *Let $\mathcal{F}$ be a coatomic antimatroid on E and $A \subseteq E$. Then $A \in \mathrm{Lin}(\mathcal{F})$ if and only if $|A| \leq 1$.*

Proof. It is checked directly from Lemmas 4.6, 4.10 and 4.12. □

Theorem 4.14. *Any non-Boolean coatomic antimatroid is not Dilworth-type.*

Proof. Let $\mathcal{F}$ be a non-Boolean coatomic antimatroid on E. From Lemma 4.13,

$$\min\left\{t : Y_1, \ldots, Y_t \in \mathrm{Lin}(\mathcal{F}), \bigcup_{i=1}^{t} Y_i = E\right\} = |E|.$$

On the other hand, suppose that $\max\{|X| : X \in \mathrm{Free}(\mathcal{F})\} = |E|$. Then $\mathcal{F}$ must be Boolean, which is a contradiction. □

4.3 Summary of this chapter

We summarize the results of this chapter in Table 4.1.

We consider point-searches of rooted digraphs (Example 2.9) such that some circuits of them have size two and some of them have size more than two. In this

Table 4.1: Summary of this chapter

Size of circuits	Dilworth-type	Examples
all two	yes	poset shellings
all more than two	no	convex shellings of points on $\mathbb{R}^n$, shellings of trees, double shellings of posets, simplicial shellings of triangulated graphs, etc.
otherwise	yes/no	point-searches of rooted digraphs, etc.

class we have both Dilworth-type antimatroids and not Dilworth-type ones. Indeed, Example 4.8 is Dilworth-type, and the family of its circuits is $\{\{1,2,3\},\{2,4\}\}$. On the other hand, Example 4.9 is not Dilworth-type, and the family of its circuits is $\{\{1,2,3\},\{1,2,4\},\{3,4\}\}$.

We finally remark that we have not yet known a characterization of Dilworth-type antimatroids and any algorithmic properties of them.

Chapter 5

Submodular-Type Optimization for Convex Geometries

5.1 Introduction to this chapter

The theory of submodular optimization has been developed since the greedy-algorithmic characterization of submodular functions by Edmonds [13]. See [18] for a comprehensive survey of this field. In particular, Faigle–Kern [15, 16] considered the dual greedy algorithm and submodular functions over the family of the antichains of a partially ordered set, or a poset. Krüger [29] showed that submodular-type optimization problems on the family of the antichains of a poset are characterized by b-submodularity.

In this chapter, we extend the result of Krüger [29] to convex geometries. In researches of optimization problems for antimatroids, bottleneck-type optimization problems over them are characterized by a greedy algorithm [5, 34]. These results generalize a result of Lawler [31]. We consider another optimization problem for them, which is linear optimization over their extreme sets.

See Chapter 2 for the definitions of a closure space and a convex geometry. We call an element a of a subset $A \subseteq E$ an *extreme point* of A if $a \notin \tau(A \setminus \{a\})$. The set of the extreme points of A is denoted by $\mathrm{ex}(A)$, which is called the *extreme set* of A. Note that $\mathrm{ex}(A) \subseteq A$. For a convex geometry $\mathcal{L}$ we denote $\mathrm{ex}(\mathcal{L}) = \{\mathrm{ex}(A) : A \in \mathcal{L}\}$. We can easily check that $\emptyset \in \mathrm{ex}(\mathcal{L})$.

Now we see some examples of extreme sets. If $\mathcal{L}$ is a convex shelling on E (Example 2.4), then $\mathrm{ex}(A) = \{x \in A : x \text{ is a vertex of conv.hull}(A)\}$ for $A \subseteq E$.

If $\mathcal{L}$ is a poset shelling on E (Example 2.5), $\mathrm{ex}(A)$ is the family of the maximal elements of A for $A \subseteq E$.

Convex geometries have an important property known as the *finite Minkowski-Krein-Milman property* shown in the following proposition.

Proposition 5.1 ([12, 28]). *A closure space $\mathcal{L}$ is a convex geometry if and only if $A = \tau(\mathrm{ex}(A))$ for all closed set $A \subseteq E$.*

A convex geometry $\mathcal{L}$ forms a lattice with respect to set-inclusion [12, 28]. Then the join $\vee_{\mathcal{L}}$ and the meet $\wedge_{\mathcal{L}}$ on $\mathcal{L}$ are defined as follows: for all $A, B \in \mathcal{L}$

$$A \vee_{\mathcal{L}} B = \tau(A \cup B), \tag{5.1}$$

$$A \wedge_{\mathcal{L}} B = A \cap B. \tag{5.2}$$

By the finite Minkowski-Krein-Milman property (Proposition 5.1), we can show that the map $\mathrm{ex} : \mathcal{L} \to \mathrm{ex}(\mathcal{L})$ is bijective. We call the map $\mathrm{ex} : \mathcal{L} \to \mathrm{ex}(\mathcal{L})$ the *extreme operator*. Therefore, by setting a partial order $\preceq$ on $\mathrm{ex}(\mathcal{L})$ so that

$$X \preceq Y \text{ if and only if } \tau(X) \subseteq \tau(Y) \tag{5.3}$$

for all $X, Y \in \mathrm{ex}(\mathcal{L})$, $\mathrm{ex}(\mathcal{L})$ is a lattice isomorphic to $\mathcal{L}$. Then, we have the join $\vee_{\mathrm{ex}(\mathcal{L})}$ and the meet $\wedge_{\mathrm{ex}(\mathcal{L})}$:

$$A \vee_{\mathrm{ex}(\mathcal{L})} B = \mathrm{ex}(\tau(A) \vee_{\mathcal{L}} \tau(B)), \tag{5.4}$$

$$A \wedge_{\mathrm{ex}(\mathcal{L})} B = \mathrm{ex}(\tau(A) \wedge_{\mathcal{L}} \tau(B)). \tag{5.5}$$

Note that $X \prec Y$ means $X \preceq Y$ and $X \neq Y$.

Remark that for a convex geometry $\mathcal{L}$ with the closure operator τ, the restriction of τ to $\mathrm{ex}(\mathcal{L})$ is a lattice-isomorphism from $\mathrm{ex}(\mathcal{L})$ to $\mathcal{L}$, whose inverse operation is ex.

When there is no risk of confusion, we use $\vee$ instead of $\vee_{\mathrm{ex}(\mathcal{L})}$, and $\wedge$ instead of $\wedge_{\mathrm{ex}(\mathcal{L})}$.

We introduce two new operators on the family $\mathrm{ex}(\mathcal{L})$ of the extreme sets of a convex geometry $\mathcal{L}$. For $X, Y \in \mathrm{ex}(\mathcal{L})$, the *reduced meet* $X \sqcap Y$ is defined as $X \sqcap Y = (X \wedge Y) \cap (X \cup Y)$, and the *residue* $X \diamond Y$ is defined as $X \diamond Y = (X \cap Y) \setminus (X \vee Y)$. Remark that $X \sqcap Y$ and $X \diamond Y$ are also in $\mathrm{ex}(\mathcal{L})$. This remark is led from Lemma 5.6 proved later.

Let $\mathcal{L}$ be a convex geometry on E, $c \in \mathbb{R}_+^E$ be a non-negative vector and $f : \mathrm{ex}(\mathcal{L}) \to \mathbb{R}$ be a set function on $\mathrm{ex}(\mathcal{L})$ with $f(\emptyset) = 0$. Then we consider the following linear programming (P):

$$\text{maximize} \quad \sum_{e \in E} c(e)x(e) \tag{5.6}$$

$$\text{subject to} \quad \sum_{e \in X} x(e) \leq f(X) \quad (X \in \mathrm{ex}(\mathcal{L})). \tag{5.7}$$

The problem (P) leads to the dual problem (D):

$$\text{minimize} \quad \sum_{X \in \mathrm{ex}(\mathcal{L})} f(X)y(X) \tag{5.8}$$

$$\text{subject to} \quad \sum_{\substack{X \in \mathrm{ex}(\mathcal{L}) \\ \text{s.t. } e \in X}} y(X) = c(e) \quad (e \in E), \tag{5.9}$$

$$y(X) \geq 0 \quad (X \in \mathrm{ex}(\mathcal{L})). \tag{5.10}$$

It is easily checked that the problem (P) always has an optimum. In order to solve the dual problem (D) Faigle–Kern [15] introduced a greedy algorithm (G) shown below, which is known as *Faigle–Kern's dual greedy algorithm.*

`Algorithm` Faigle–Kern's dual greedy algorithm (G)

`Input` $c \in \mathbb{R}_+^E$

`Output` $y \in \mathbb{R}^{\mathrm{ex}(\mathcal{L})}$ (an optimum of (D)) and π (a permutation of E)

`Initialization:`

    $y(X) \leftarrow 0 \quad (\forall X \in \mathrm{ex}(\mathcal{L}))$;

    $T \leftarrow E$;

    $w(e) \leftarrow c(e) \quad (\forall e \in E)$;

    $\pi \leftarrow \emptyset$;

`Iteration:`

    `while` $T \neq \emptyset$ `do`

        determine $e \in \mathrm{ex}(T)$ s.t. $w(e) = \min\{w(e') : e' \in \mathrm{ex}(T)\}$;

        $y(\mathrm{ex}(T)) \leftarrow w(e)$;

$\qquad \pi \leftarrow e\pi;$

$\qquad w(x) \leftarrow w(x) - w(e) \quad (\forall x \in \mathrm{ex}(T));$

$\qquad T \leftarrow T \setminus \{e\}.$

`end of while`

For the algorithm (G) and submodular functions, the next theorem is important.

Theorem 5.2 (Krüger [29], also see Ando [1]). *Let $\mathcal{L}$ be a poset shelling on E. Then for $f : \mathrm{ex}(\mathcal{L}) \to \mathbb{R}$ with $f(\emptyset) = 0$, the greedy algorithm* (G) *gives an optimum of the problem* (D) *for all $c \in \mathbb{R}_+^E$ if and only if f is b-submodular, that is, for all $X, Y \in \mathrm{ex}(\mathcal{L})$*

$$f(X) + f(Y) \geq f(X \vee Y) + f(X \sqcap Y). \tag{5.11}$$

In order to consider optimization problems for all convex geometries, we extend Krüger's b-submodular function to a slightly different submodular function in our sense, which we call a c-submodular function. Let $\mathcal{L}$ be a convex geometry on E. Then a function $f : \mathrm{ex}(\mathcal{L}) \to \mathbb{R}$ is *c-submodular* if, for all $X, Y \in \mathrm{ex}(\mathcal{L})$ satisfying the condition that $\chi^X + \chi^Y = \chi^{X \vee Y} + \chi^{X \sqcap Y} + \chi^{X \diamond Y}$,

$$f(X) + f(Y) \geq f(X \vee Y) + f(X \sqcap Y) + f(X \diamond Y), \tag{5.12}$$

where $\chi^X \in \{0, 1\}^E$ is the characteristic vector of X, that is,

$$\chi^X(e) = \begin{cases} 1 & (e \in X), \\ 0 & (e \notin X). \end{cases} \tag{5.13}$$

The next theorem shows the relationship between b-submodularity and c-submodularity.

Theorem 5.3. *Let $\mathcal{L}$ be a convex geometry. Then the class of c-submodular functions on $\mathrm{ex}(\mathcal{L})$ coincides with the class of b-submodular functions if and only if $\mathcal{L}$ is a poset shelling.*

Theorem 5.3 is proved directly from Theorem 5.34 proved later.

We extend Theorem 5.2 to the next theorem, which is the main theorem in this chapter.

Theorem 5.4. *Let $\mathcal{L}$ be a convex geometry on E. Then for $f : \mathrm{ex}(\mathcal{L}) \to \mathbb{R}$ with $f(\emptyset) = 0$, the greedy algorithm* (G) *gives an optimum of the problem* (D) *for all $c \in \mathbb{R}_+^E$ if and only if f is c-submodular.*

In Section 5.2, we review some properties of convex geometries. The main theorem is proved in Section 5.3, and in Section 5.4 we discuss some consequences from the main theorem.

5.2 Some properties of the extreme sets

In this section, we review some properties of convex geometries which we need for the proof of the main theorem (Theorem 5.4). Particularly, we investigate the lattice of the extreme sets of a convex geometry.

The next lemma is a characterization of extreme points of a closure space, which is frequently used in this chapter.

Lemma 5.5. *Let $\mathcal{L}$ be a closure space on E, and $A \subseteq E$ be a closed set. An element $a \in A$ is an extreme point of A if and only if $A \setminus \{a\}$ is a closed set.*

Proof. The definition of extreme points directly leads to the if-part. We now show the only-if-part. That is, we must show that: if a is an extreme point of A, then $\tau(A \setminus \{a\}) = A \setminus \{a\}$. By the definition of the closure operator, $\tau(A \setminus \{a\}) \supseteq A \setminus \{a\}$. Suppose that $e \in \tau(A \setminus \{a\})$. Then, $e \in \tau(A)$ by the monotonicity of τ. Since a is an extreme point of A, we have $e \neq a$. Since A is a closed set, we have $e \in A \setminus \{a\}$. Therefore, $\tau(A \setminus \{a\}) \subseteq A \setminus \{a\}$. □

Lemma 5.6. *For a convex geometry $\mathcal{L}$, $A \in \mathrm{ex}(\mathcal{L})$ and $B \subseteq A$ imply $B \in \mathrm{ex}(\mathcal{L})$.*

Proof. If we suppose that $A \in \mathrm{ex}(\mathcal{L})$, then $a \notin \tau(\tau(A) \setminus \{a\})$ for any $a \in A$ by the definition of extreme points. Take $b \in B$ for $B \subseteq A$. Now we show that $b \notin \tau(\tau(B) \setminus \{b\})$. Since $b \in B$, we have $b \in A$ and $b \notin \tau(\tau(A) \setminus \{b\})$. Moreover, by the fact that $B \subseteq A$ and the monotonicity of the closure operator, $\tau(\tau(B) \setminus \{b\}) \subseteq \tau(\tau(A) \setminus \{b\})$. Therefore, $b \notin \tau(\tau(B) \setminus \{b\})$. □

From Lemma 5.6 we can see that $X \sqcap Y, X \diamond Y \in \mathrm{ex}(\mathcal{L})$ for any $X, Y \in \mathrm{ex}(\mathcal{L})$ where $\mathcal{L}$ is a convex geometry.

Now, we show some lemmas about $\mathrm{ex}(\mathcal{L})$ as a lattice. Let $\preceq$ be the partial order on $\mathrm{ex}(\mathcal{L})$ defined by (5.3).

Lemma 5.7. *Let $\mathcal{L}$ be a convex geometry. Then, $X \subseteq Y$ implies $X \preceq Y$ for all $X, Y \in \mathrm{ex}(\mathcal{L})$.*

Proof. Suppose that $X \subseteq Y$. Then, $\tau(X) \subseteq \tau(Y)$ by the monotonicity of the closure operator. By (5.3), we conclude that $X \preceq Y$ $\square$

Lemma 5.8. *Let $\mathcal{L}$ be a convex geometry. For $X, Y \in \mathrm{ex}(\mathcal{L})$ with $X \preceq Y$ and for $e \in Y$ with $e \in \tau(X)$, we have $e \in X$.*

Proof. By Lemma 5.5, $\tau(Y) \setminus \{e\}$ is a convex set. Therefore, $(\tau(Y) \setminus \{e\}) \cap \tau(Z)$ is also a convex set by the definition of the closed set. Since $X \preceq Y$, we have $\tau(X) \subseteq \tau(Y)$. Hence, $(\tau(Y) \setminus \{e\}) \cap \tau(X) = \tau(X) \setminus \{e\}$. Using Lemma 5.5, we have $e \in X$. $\square$

Lemma 5.9. *If $X, Y \in \mathrm{ex}(\mathcal{L})$, then $X \diamond Y \subseteq X \cap Y \subseteq X \sqcap Y$.*

Proof. By the definition of the residue, we can easily show that $X \diamond Y \subseteq X \cap Y$. Suppose that $e \in X \cap Y$. Then, since $e \in X$ and $e \in Y$, we have $e \in \tau(X)$ and $e \in \tau(Y)$. Therefore, $e \in \tau(X) \cap \tau(Y)$, and by Lemma 5.8 $e \in \mathrm{ex}(\tau(X) \cap \tau(Y))$, that is, $e \in X \wedge Y$. Since $e \in X \cup Y$, we conclude that $X \cap Y \subseteq X \sqcap Y$. $\square$

Lemmas 5.7 and 5.9 lead to the next important lemma.

Lemma 5.10. *Let $\mathcal{L}$ be a convex geometry. Then, $X \diamond Y \preceq X \sqcap Y \preceq X \vee Y$ for all $X, Y \in \mathrm{ex}(\mathcal{L})$.*

When we see convex geometries as lattices, they belong to the class of lower semimodular lattices [12]. This means that all the maximal chains of $\mathrm{ex}(\mathcal{L})$ have the same length $|E|$. See [43] for more discussion on semimodular lattices.

Let $\mathcal{L}$ be a convex geometry on E. Since we have $X \subseteq \tau(X)$ for all $X \subseteq E$, every extreme set $Z \in \mathrm{ex}(\mathcal{L})$ satisfies either of the followings for each $a \in E$:

$$a \notin Z, a \in \tau(Z) \quad \text{(This case is indicated by } \textsf{(in)}\text{);} \tag{5.14}$$

$$a \in Z \quad \text{(This case is indicated by } \textsf{(ex)}\text{);} \tag{5.15}$$

$$a \notin \tau(Z) \quad \text{(This case is indicated by } \textsf{(out)}\text{).} \tag{5.16}$$

A more precise notion is given as the following lemma.

Lemma 5.11. *Let $\mathcal{L}$ be a convex geometry on E. If we take any maximal chain $\mathcal{C} : \emptyset = X_0 \prec X_1 \prec X_2 \prec X_3 \prec \cdots \prec X_{n-1} \prec X_n = \mathrm{ex}(E)$ of $\mathrm{ex}(\mathcal{L})$ where $n = |E|$, then for any $a \in E$ there exist natural numbers l, m such that $l \le m$ and*

$$0 \le i < l \Leftrightarrow a \notin \tau(X_i), \tag{5.17}$$

$$l \le i < m \Leftrightarrow a \in X_i, \tag{5.18}$$

$$m \le i \le n \Leftrightarrow a \notin X_i,\ a \in \tau(X_i). \tag{5.19}$$

Proof. It is straightforward from the isomorphism between $\mathcal{L}$ and $\mathrm{ex}(\mathcal{L})$, Proposition 2.1 and Lemmas 5.7 and 5.8. □

This leads to the following remark.

Remark 5.12. *Let $X, Y \in \mathcal{L}$. If either of X and Y is in the case* (in) *for $a \in E$, then $X \vee Y$ is also in the case* (in) *for the same $a \in E$. This fact is indicated in Table 5.1 as $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ and $\mathfrak{D}$.*

Proof. Without loss of generality, assume that X is in the case (in) for $a \in E$. We take a maximal chain $\mathcal{C} : \emptyset = X_0 \prec X_1 \prec X_2 \prec X_3 \prec \cdots \prec X_{n-1} \prec X_n = \mathrm{ex}(E)$ of $\mathrm{ex}(\mathcal{L})$ containing X and $X \vee Y$. Suppose that $X_i = X$ and $X_j = X \vee Y$, with $i \le j$. From the assumption, X satisfies that $a \notin X, a \in \tau(X)$. Then, by Lemma 5.11, $X \vee Y$ also satisfies that $a \notin X \vee Y$ and $a \in \tau(X \vee Y)$ for the same $a \in E$. This means that $X \vee Y$ is in the case (in). □

Similarly, we can obtain some consequences. We arrange them in Table 5.1.

Table 5.1 shows all the possible cases. We have some additional remarks, which we can easily check.

Table 5.1: All the possible cases

	$\mathfrak{A}$	$\mathfrak{B}$	$\mathfrak{C}$	$\mathfrak{D}$	$\mathfrak{E}$	$\mathfrak{F}$	$\mathfrak{G}$	$\mathfrak{H}$	$\mathfrak{I}$	$\mathfrak{J}$
X	(in)	(in)	(in)	(in)	(ex)	(ex)	(ex)	(ex)	(out)	(out)
Y	(in)	(in)	(ex)	(out)	(ex)	(ex)	(out)	(out)	(out)	(out)
$X \vee Y$	(in)	(in)	(in)	(in)	(in)	(ex)	(in)	(ex)	(in)	(out)
$X \wedge Y$	(in)	(ex)	(ex)	(out)	(ex)	(ex)	(out)	(out)	(out)	(out)

Remark 5.13. *$X, Y \in \mathrm{ex}(\mathcal{L})$ are in the case $\mathfrak{B}$ for $a \in E$ if and only if $a \in (X \wedge Y) \setminus (X \sqcap Y)$.*

Remark 5.14. *$X, Y \in \mathrm{ex}(\mathcal{L})$ are in the case $\mathfrak{E}$ for $a \in E$ if and only if $a \in X \diamond Y$.*

Remark 5.15. *$X, Y \in \mathrm{ex}(\mathcal{L})$ are in the case $\mathfrak{G}$ for $a \in E$ if and only if $\chi^X(a) + \chi^Y(a) > \chi^{X\vee Y}(a) + \chi^{X\sqcap Y}(a) + \chi^{X\diamond Y}(a)$. Otherwise, $\chi^X(a) + \chi^Y(a) = \chi^{X\vee Y}(a) + \chi^{X\sqcap Y}(a) + \chi^{X\diamond Y}(a)$.*

5.3 Proof of the main theorem

In this section, we give the proof of Theorem 5.4. First, we make some remarks on the greedy algorithm (G).

Remark 5.16. *During the execution of (G), $w(e) \geq 0$ holds for all $e \in E$. Therefore, $y(\mathrm{ex}(T))$ in line 9 is always non-negative.*

Proof. Since $c(e) \geq 0$ for all $e \in E$, we have $w(e) \geq 0$ for all $e \in E$ at the completion of the step `Initialize`. During the execution of `Iteration`, if $w(x) < w(e)$ in line 11 of (G), then the substituted $w(x)$ becomes negative. However, by the choice of e in line 8, it is ensured that $w(x) \geq w(e)$ for all $x \in \mathrm{ex}(T)$. Therefore, $w(x)$ must be non-negative. $\square$

Remark 5.17. *During the execution of `Iteration` in (G), $\mathrm{ex}(T) \neq \emptyset$ holds.*

Proof. Since the convex geometry $\mathcal{L}$ and the family $\mathrm{ex}(\mathcal{L})$ of extreme sets of $\mathcal{L}$ are lattice-isomorphic and $\mathrm{ex}(\emptyset) = \emptyset$, we have $\mathrm{ex}(X) \neq \emptyset$ for any non-empty $X \in \mathcal{L}$. Moreover, since `Iteration` terminates if $T = \emptyset$, we have $\mathrm{ex}(T) \neq \emptyset$ during the execution of `Iteration`. $\square$

We mention the following remark without proof, which can be easily checked.

Remark 5.18. *Suppose that there exist more than one elements $e \in \mathrm{ex}(T)$ such that $w(e)$ is minimum in line 8 of (G) at an `Iteration` step. Then, whichever one of the candidates we choose at the step, the output y is the same at the termination of (G).*

Now, we prove some lemmas for the main theorem.

Lemma 5.19. *Let $n = |E|$. The sequence $\pi = e_1 e_2 \cdots e_n$ of the elements in E produced as the output of* (G) *satisfies the next condition: $e_i \in \tau(\{e_1, \ldots, e_j\})$ implies $i \leq j$. In particular, $\{e_1, \ldots, e_j\}$ is a convex set.*

Proof. Suppose that $i > j$. Since $\{e_1, \ldots, e_j\} \subseteq \{e_1, \ldots, e_{i-1}\}$ and τ is monotone, $\tau(\{e_1, \ldots, e_j\}) \subseteq \tau(\{e_1, \ldots, e_{i-1}\})$. Since we choose e_i as e in line 8 of (G) when $T = \{e_1, \ldots, e_i\}$, we have $e_i \in \mathrm{ex}(\{e_1, \ldots, e_i\})$. Hence $e_i \notin \tau(\{e_1, \ldots, e_{i-1}\})$, and we have $e_i \notin \tau(\{e_1, \ldots, e_j\})$. $\square$

By Lemma 5.19, we set $A_i^\pi = \mathrm{ex}(\{e_1, e_2, \ldots, e_i\})$ for π. Let x^π be the unique solution of the system of the following linear equations:

$$\sum_{e \in A_i^\pi} x^\pi(e) = f(A_i^\pi) \quad (i = 1, \ldots, n). \tag{5.20}$$

We call x^π the *greedy vector* to π. Notice that $A_i^\pi \prec A_j^\pi$ for all $i < j$. If $\mathcal{C}^\pi : \emptyset \prec A_1^\pi \prec A_2^\pi \prec \cdots \prec A_n^\pi$ is a maximal chain of $\mathrm{ex}(\mathcal{L})$, then we call $\mathcal{C}^\pi$ the *greedy chain* to π.

Lemma 5.20. *There exists $l \in \{i, \ldots, n\}$ for all $i \in \{1, \ldots, n\}$ such that $e_i \in A_j^\pi$ if and only if $j \in \{i, \ldots, l\}$.*

Proof. It is straightforward from Lemma 5.11. $\square$

Lemma 5.21. *The output $y \in \mathbb{R}_+^{\mathrm{ex}(\mathcal{L})}$ of* (G) *coincides with the unique solution of the following system of equations:*

$$\sum_{\substack{A \in \{A_1^\pi, \ldots, A_n^\pi\} \\ s.t.\ e_i \in A}} y^\pi(A) = c(e_i) \quad (i \in \{1, \ldots, n\}) \tag{5.21}$$

$$y^\pi(A) = 0 \quad (A \in \mathrm{ex}(\mathcal{L}) \setminus \{A_1^\pi, \ldots, A_n^\pi\}). \tag{5.22}$$

We call y^π the dual greedy vector to π.

Proof. From the algorithm (G), we have $y^\pi(A) = 0$ for all $A \in \mathrm{ex}(\mathcal{L}) \setminus \{A_1^\pi, \ldots, A_n^\pi\}$. For $i \in \{1, \ldots, n\}$, by Lemma 5.20

$$\sum_{\substack{A \in \{A_1^\pi, \ldots, A_n^\pi\} \\ \text{s.t. } e_i \in A}} y^\pi(A) = \sum_{i \leq j \leq l} y^\pi(A_j^\pi) = \sum_{i \leq j \leq l} w(e_j) = c(e_i). \tag{5.23}$$

The uniqueness is obtained from the form of the equations, determined by Lemma 5.20. $\square$

Lemma 5.22. *The greedy vector x^π and the dual greedy vector y^π satisfy complementary slackness, that is, $y^\pi(A) > 0$ implies $\sum\{x^\pi(e) : e \in A\} = f(A)$ for all $A \in \mathrm{ex}(\mathcal{L})$.*

Proof. Since $A \in \{A_1^\pi, \dots, A_n^\pi\}$, it holds from the definitions of the greedy vector and the dual greedy vector. □

Next, we show the feasibility of the greedy vector x^π and the dual greedy vector y^π.

Lemma 5.23. *The dual greedy vector y^π is a feasible solution of the problem* (D).

Proof. From the definition of the dual greedy vector, $y^\pi \geq 0$. By Lemma 5.21, we can easily checked that $\sum\{y^\pi(X) : X \in \mathcal{L} \text{ s.t. } e \in X\} = c(e)$. □

Define $\mathrm{atom}(\mathcal{L}) = \{e \in E : \{e\} \in \mathcal{L}\}$ for a convex geometry $\mathcal{L}$ on E.

For a convex geometry $\mathcal{L}$ and $A \in \mathcal{L}$, define the *contraction* of $\mathcal{L}$ by A as $\mathcal{L}/A = \{X \setminus A : X \in \mathcal{L}, A \subseteq X\}$. And for a convex geometry $\mathcal{L}$ and $E \setminus A \in \mathcal{L}$, define the *deletion* of $\mathcal{L}$ by A as $\mathcal{L} \setminus A = \{X \in \mathcal{L} : X \subseteq E \setminus A\}$. Notice that every contraction of a convex geometry is also a convex geometry and so is every deletion.

Let $\mathcal{L}$ be a convex geometry on E satisfying $|E| > 1$. For $e \in \mathrm{atom}(\mathcal{L})$, we define $f' : \mathrm{ex}(\mathcal{L}/\{e\}) \to \mathbb{R}$ as

$$f'(X) = \begin{cases} f(X \cup \{e\}) - f(\{e\}) & (X \cup \{e\} \in \mathrm{ex}(\mathcal{L})), \\ f(X) & (X \cup \{e\} \notin \mathrm{ex}(\mathcal{L})). \end{cases} \tag{5.24}$$

Lemma 5.24. *Let $e \in \mathrm{atom}(\mathcal{L})$ and $f : \mathrm{ex}(\mathcal{L}) \to \mathbb{R}$ be c-submodular. Then, $f'(X) \leq f(X)$ for all $X \in \mathrm{ex}(\mathcal{L}/\{e\})$.*

Proof. Suppose that $X \cup \{e\} \notin \mathrm{ex}(\mathcal{L})$. Then $f'(X) = f(X)$ by the definition of f'.

Suppose that $X \cup \{e\} \in \mathrm{ex}(\mathcal{L})$. Since f is c-submodular and $e \notin X$, we have $f(X) + f(\{e\}) \geq f(X \cup \{e\})$. Therefore, $f'(X) = f(X \cup \{e\}) - f(\{e\}) \leq f(X)$. □

Lemma 5.25. *Let $\mathcal{L}$ be a convex geometry, and suppose that $e \in \mathrm{atom}(\mathcal{L})$ and $X \in \mathrm{ex}(\mathcal{L}/\{e\})$. $X \cup \{e\} \in \mathrm{ex}(\mathcal{L})$ if and only if $e \notin \tau(X)$, where τ is the closure operator on $\mathcal{L}$.*

Proof. It is easily checked from the definition of extreme points. □

Now we consider the relationship between $\mathrm{ex}(\mathcal{L}/\{e\})$ and $\mathrm{ex}(\mathcal{L})$. We use τ' and ex' for the closure operator and the extreme operator on $\mathrm{ex}(\mathcal{L}/\{e\})$, respectively, because of the distinction from the operators on $\mathrm{ex}(\mathcal{L})$. We define the map $\iota : \mathrm{ex}(\mathcal{L}/\{e\}) \to \mathrm{ex}(\mathcal{L})$ as $\iota(X) = \mathrm{ex}(\tau'(X) \cup \{e\})$ for all $X \in \mathrm{ex}(\mathcal{L}/\{e\})$. Then, by Lemma 5.25, we have

$$\iota(X) = \begin{cases} X \cup \{e\} & (X \cup \{e\} \in \mathrm{ex}(\mathcal{L})), \\ X & (X \cup \{e\} \notin \mathrm{ex}(\mathcal{L})). \end{cases} \tag{5.25}$$

The join, the meet, the reduced meet, and the residue on $\mathrm{ex}(\mathcal{L}/\{e\})$ are denoted by $\vee'$, $\wedge'$, $\sqcap'$ and $\diamond'$, respectively. Then, we have

$$\iota(X \vee' Y) = \iota(X) \vee \iota(Y), \tag{5.26}$$

$$\iota(X \wedge' Y) = \iota(X) \wedge \iota(Y). \tag{5.27}$$

Therefore, $\iota : \mathrm{ex}(\mathcal{L}/\{e\}) \to \mathrm{ex}(\mathcal{L})$ is a lattice-embedding.

The next is a key lemma for the main theorem.

Lemma 5.26. *Let $|E| > 1$ and $e \in \mathrm{atom}(\mathcal{L})$. If $f : \mathrm{ex}(\mathcal{L}) \to \mathbb{R}$ is c-submodular, then $f' : \mathrm{ex}(\mathcal{L}/\{e\}) \to \mathbb{R}$ defined above is also c-submodular.*

Proof. Since $e \in \tau(\iota(X)), \tau(\iota(Y))$ for all $X, Y \in \mathrm{ex}(\mathcal{L}/\{e\})$, we only have to consider the cases $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{E}$, $\mathfrak{F}$ in Table 5.1 with respect to $X \cup \{e\}$, $Y \cup \{e\}$, $(X \vee' Y) \cup \{e\}$ and $(X \wedge' Y) \cup \{e\}$.

Case : $\mathfrak{A}$. That is, $X \cup \{e\} \notin \mathrm{ex}(\mathcal{L})$, $Y \cup \{e\} \notin \mathrm{ex}(\mathcal{L})$, $(X \vee' Y) \cup \{e\} \notin \mathrm{ex}(\mathcal{L})$ and $(X \wedge' Y) \cup \{e\} \notin \mathrm{ex}(\mathcal{L})$. By (5.25), $\iota(X) = X$, $\iota(Y) = Y$, $\iota(X \vee' Y) = X \vee' Y$ and $\iota(X \wedge' Y) = X \wedge' Y$. Moreover, by (5.26), $\iota(X \vee' Y) = \iota(X) \vee \iota(Y) = X \vee Y$. Similarly, by (5.27), $\iota(X \wedge' Y) = \iota(X) \wedge \iota(Y) = X \wedge Y$. Therefore we have $X \sqcap' Y = X \sqcap Y$ and $X \diamond' Y = X \diamond Y$. Hence, if $\chi^X + \chi^Y = \chi^{X \vee' Y} + \chi^{X \sqcap' Y} + \chi^{X \diamond' Y}$ for $\mathrm{ex}(\mathcal{L}/\{e\})$, then we also have $\chi^X + \chi^Y = \chi^{X \vee Y} + \chi^{X \sqcap Y} + \chi^{X \diamond Y}$ for $\mathrm{ex}(\mathcal{L})$. By Lemma 5.24,

$$\begin{aligned} f'(X) + f'(Y) &= f(X) + f(Y) \\ &\geq f(X \vee Y) + f(X \sqcap Y) + f(X \diamond Y) \\ &\geq f'(X \vee Y) + f'(X \sqcap Y) + f'(X \diamond Y) \\ &= f'(X \vee' Y) + f'(X \sqcap' Y) + f'(X \diamond' Y). \end{aligned}$$

Case : $\mathfrak{B}$. That is, $X \cup \{e\} \notin \mathrm{ex}(\mathcal{L})$, $Y \cup \{e\} \notin \mathrm{ex}(\mathcal{L})$, $(X \vee' Y) \cup \{e\} \notin \mathrm{ex}(\mathcal{L})$ and $(X \wedge' Y) \cup \{e\} \in \mathrm{ex}(\mathcal{L})$. By (5.25), $\iota(X) = X$, $\iota(Y) = Y$, $\iota(X \vee' Y) = X \vee' Y$

and $\iota(X \wedge' Y) = (X \wedge' Y) \cup \{e\}$. Also, by (5.26), $\iota(X \vee' Y) = \iota(X) \vee \iota(Y) = X \vee Y$. Similarly, by (5.27), $\iota(X \wedge' Y) = \iota(X) \wedge \iota(Y) = X \wedge Y$. Therefore, we have $X \diamond' Y = X \diamond Y$. Moreover, since $e \notin X \cup Y$,

$$\begin{aligned} X \sqcap Y &= (X \wedge Y) \cap (X \cup Y) = \iota(X \wedge' Y) \cap (X \cup Y) \\ &= ((X \wedge' Y) \cup \{e\}) \cap (X \cup Y) = (X \wedge' Y) \cap (X \cup Y) \\ &= X \sqcap' Y. \end{aligned}$$

Hence, if $\chi^X + \chi^Y = \chi^{X \vee' Y} + \chi^{X \sqcap' Y} + \chi^{X \diamond' Y}$ for $\mathrm{ex}(\mathcal{L}/\{e\})$, then we have $\chi^X + \chi^Y = \chi^{X \vee Y} + \chi^{X \sqcap Y} + \chi^{X \diamond Y}$ for $\mathrm{ex}(\mathcal{L})$. By Lemma 5.24,

$$\begin{aligned} f'(X) + f'(Y) &= f(X) + f(Y) \\ &\geq f(X \vee Y) + f(X \sqcap Y) + f(X \diamond Y) \\ &\geq f'(X \vee Y) + f'(X \sqcap Y) + f'(X \diamond Y) \\ &= f'(X \vee' Y) + f'(X \sqcap' Y) + f'(X \diamond' Y). \end{aligned}$$

Case : $\mathfrak{C}$. *That is,* $X \cup \{e\} \notin \mathrm{ex}(\mathcal{L})$, $Y \cup \{e\} \in \mathrm{ex}(\mathcal{L})$, $(X \vee' Y) \cup \{e\} \notin \mathrm{ex}(\mathcal{L})$ *and* $(X \wedge' Y) \cup \{e\} \in \mathrm{ex}(\mathcal{L})$. By (5.25), $\iota(X) = X$, $\iota(Y) = Y \cup \{e\}$, $\iota(X \vee' Y) = X \vee' Y$ and $\iota(X \wedge' Y) = (X \wedge' Y) \cup \{e\}$. Also, by (5.26), $\iota(X \vee' Y) = \iota(X) \vee \iota(Y) = X \vee (Y \cup \{e\})$. Similarly, by (5.27), $\iota(X \wedge' Y) = \iota(X) \wedge \iota(Y) = X \wedge (Y \cup \{e\})$. Therefore, $X \vee' Y = X \vee (Y \cup \{e\})$ and $(X \wedge' Y) \cup \{e\} = X \wedge (Y \cup \{e\})$. Since $Y \subseteq Y \cup \{e\}$ and Lemma 5.7, we have $Y \preceq Y \cup \{e\}$. Since $Y \cup \{e\} \subseteq \tau(Y) \cup \{e\} \subseteq \tau(X) \cup \tau(Y) \subseteq \tau(\tau(X) \cup \tau(Y)) = \tau(X \vee Y)$, we have $\tau(Y \cup \{e\}) \subseteq \tau(X \vee Y)$. Therefore, $Y \cup \{e\} \preceq X \vee Y$. Hence, we also have $X \vee Y = X \vee (Y \cup \{e\}) = X \vee' Y$, and $X \diamond (Y \cup \{e\}) = X \diamond Y = X \diamond' Y$. Moreover, we have

$$\begin{aligned} (X \sqcap' Y) \cup \{e\} &= ((X \wedge' Y) \cap (X \cup Y)) \cup \{e\} \\ &= ((X \wedge' Y) \cup \{e\}) \cap ((X \cup Y) \cup \{e\}) \\ &= (X \wedge (Y \cup \{e\})) \cap (X \cup (Y \cup \{e\})) = X \sqcap (Y \cup \{e\}). \end{aligned}$$

Now, if $\chi^X + \chi^Y = \chi^{X \vee' Y} + \chi^{X \sqcap' Y} + \chi^{X \diamond' Y}$ on $\mathrm{ex}(\mathcal{L}/\{e\})$, then we also have $\chi^X + \chi^{Y \cup \{e\}} = \chi^{X \vee (Y \cup \{e\})} + \chi^{X \sqcap (Y \cup \{e\})} + \chi^{X \diamond (Y \cup \{e\})}$ on $\mathrm{ex}(\mathcal{L})$. By Lemma 5.24,

$$\begin{aligned} f'(X) + f'(Y) &= f(X) + f(Y \cup \{e\}) - f(\{e\}) \\ &\geq f(X \vee (Y \cup \{e\}) + f(X \sqcap (Y \cup \{e\})) + f(X \diamond (Y \cup \{e\})) \\ &\qquad - f(\{e\}) \\ &= f(X \vee' Y) + f((X \sqcap' Y) \cup \{e\}) + f(X \diamond' Y) - f(\{e\}) \\ &\geq f'(X \vee' Y) + f'(X \sqcap' Y) + f'(X \diamond' Y). \end{aligned}$$

Case : $\mathfrak{E}$. *That is,* $X \cup \{e\} \in \mathrm{ex}(\mathcal{L})$, $Y \cup \{e\} \in \mathrm{ex}(\mathcal{L})$, $(X \vee' Y) \cup \{e\} \notin \mathrm{ex}(\mathcal{L})$ *and* $(X \wedge' Y) \cup \{e\} \in \mathrm{ex}(\mathcal{L})$. By (5.25), $\iota(X) = X \cup \{e\}$, $\iota(Y) = Y \cup \{e\}$, $\iota(X \vee' Y) = X \vee' Y$ and $\iota(X \wedge' Y) = (X \wedge' Y) \cup \{e\}$. Also, by (5.26), $\iota(X \vee' Y) = \iota(X) \vee \iota(Y) = (X \cup \{e\}) \vee (Y \cup \{e\})$. Similarly, by (5.27), $\iota(X \wedge' Y) = \iota(X) \wedge \iota(Y) = (X \cup \{e\}) \wedge (Y \cup \{e\})$. Therefore $X \vee' Y = (X \cup \{e\}) \vee (Y \cup \{e\})$ and $(X \wedge' Y) \cup \{e\} = (X \cup \{e\}) \wedge (Y \cup \{e\})$. Similarly as the case $\mathfrak{C}$, since $X \preceq X \cup \{e\} \preceq X \vee Y$ and $Y \preceq Y \cup \{e\} \preceq X \vee Y$, we have $X \vee Y = (X \cup \{e\}) \vee (Y \cup \{e\}) = X \vee' Y$. Moreover, we have

$$\begin{aligned}
(X \sqcap' Y) \cup \{e\} &= ((X \wedge' Y) \cap (X \cup Y)) \cup \{e\} \\
&= ((X \wedge' Y) \cup \{e\}) \cap ((X \cup Y) \cup \{e\}) \\
&= ((X \cup \{e\}) \wedge (Y \cup \{e\})) \cap ((X \cup \{e\}) \cup (Y \cup \{e\})) \\
&= (X \cup \{e\}) \sqcap (Y \cup \{e\}), \\
(X \diamond' Y) \cup \{e\} &= ((X \cap Y) \setminus (X \vee' Y)) \cup \{e\} \\
&= ((X \cap Y) \setminus (X \vee Y)) \cup \{e\} \\
&= ((X \cap Y) \cup \{e\}) \setminus (X \vee Y) \\
&= ((X \cup \{e\}) \cap (Y \cup \{e\})) \setminus ((X \cup \{e\}) \vee (Y \cup \{e\})) \\
&= (X \cup \{e\}) \diamond (Y \cup \{e\}).
\end{aligned}$$

Hence, if $\chi^X + \chi^Y = \chi^{X \vee' Y} + \chi^{X \sqcap' Y} + \chi^{X \diamond' Y}$ on $\mathrm{ex}(\mathcal{L}/\{e\})$, then we also have $\chi^{X \cup \{e\}} + \chi^{Y \cup \{e\}} = \chi^{(X \cup \{e\}) \vee (Y \cup \{e\})} + \chi^{(X \cup \{e\}) \sqcap (Y \cup \{e\})} + \chi^{(X \cup \{e\}) \diamond (Y \cup \{e\})}$ on $\mathrm{ex}(\mathcal{L})$. By Lemma 5.24,

$$\begin{aligned}
f'(X) + f'(Y) &= f(X \cup \{e\}) + f(Y \cup \{e\}) - 2f(\{e\}) \\
&\geq f((X \cup \{e\}) \vee (Y \cup \{e\})) + f((X \cup \{e\}) \sqcap (Y \cup \{e\})) \\
&\qquad + f((X \cup \{e\}) \diamond (Y \cup \{e\})) - 2f(\{e\}) \\
&= f(X \vee' Y) + f((X \sqcap' Y) \cup \{e\}) + f((X \diamond' Y) \cup \{e\}) \\
&\qquad - 2f(\{e\}) \\
&\geq f'(X \vee' Y) + f'(X \sqcap' Y) + f'(X \diamond' Y).
\end{aligned}$$

Case : $\mathfrak{F}$. *That is,* $X \cup \{e\} \in \mathrm{ex}(\mathcal{L})$, $Y \cup \{e\} \in \mathrm{ex}(\mathcal{L})$, $(X \vee' Y) \cup \{e\} \in \mathrm{ex}(\mathcal{L})$ *and* $(X \wedge' Y) \cup \{e\} \in \mathrm{ex}(\mathcal{L})$. By (5.25), $\iota(X) = X \cup \{e\}$, $\iota(Y) = Y \cup \{e\}$, $\iota(X \vee' Y) = (X \vee' Y) \cup \{e\}$ and $\iota(X \wedge' Y) = (X \wedge' Y) \cup \{e\}$. Also, by (5.26), $\iota(X \vee' Y) = \iota(X) \vee \iota(Y) = (X \cup \{e\}) \vee (Y \cup \{e\})$. Similarly, by (5.27), $\iota(X \wedge' Y) = \iota(X) \wedge \iota(Y) = (X \cup \{e\}) \wedge (Y \cup \{e\})$. Therefore, $(X \vee' Y) \cup \{e\} = (X \cup \{e\}) \vee (Y \cup \{e\})$ and $(X \wedge' Y) \cup \{e\} = (X \cup \{e\}) \wedge (Y \cup \{e\})$. Similarly as the case $\mathfrak{E}$, we have $(X \vee Y) \cup \{e\} = (X \cup \{e\}) \vee (Y \cup \{e\}) = (X \vee' Y) \cup \{e\}$, $(X \cup \{e\}) \sqcap (Y \cup \{e\}) = (X \sqcap' Y) \cup \{e\}$ and $(X \cup \{e\}) \diamond (Y \cup \{e\}) = X \diamond' Y$. Hence, if $\chi^X + \chi^Y = \chi^{X \vee' Y} + \chi^{X \sqcap' Y} + \chi^{X \diamond' Y}$ on $\mathrm{ex}(\mathcal{L}/\{e\})$, then we also have $\chi^{X \cup \{e\}} + \chi^{Y \cup \{e\}} = \chi^{(X \cup \{e\}) \vee (Y \cup \{e\})} + \chi^{(X \cup \{e\}) \sqcap (Y \cup \{e\})} + \chi^{(X \cup \{e\}) \diamond (Y \cup \{e\})}$ on $\mathrm{ex}(\mathcal{L})$. By Lemma 5.24, we have

$$
\begin{aligned}
f'(X) + f'(Y) &= f(X \cup \{e\}) + f(Y \cup \{e\}) - 2f(\{e\}) \\
&\geq f((X \cup \{e\}) \vee (Y \cup \{e\})) + f((X \cup \{e\}) \sqcap (Y \cup \{e\})) \\
&\qquad + f((X \cup \{e\}) \diamond (Y \cup \{e\})) - 2f(\{e\}) \\
&= f((X \vee' Y) \cup \{e\}) + f((X \sqcap' Y) \cup \{e\}) + f(X \diamond' Y) \\
&\qquad - 2f(\{e\}) \\
&\geq f'(X \vee' Y) + f'(X \sqcap' Y) + f'(X \diamond' Y).
\end{aligned}
$$

It concludes the proof. $\square$

Now, we prove the feasibility of the greedy vector x^π.

Lemma 5.27. *If f is c-submodular, then the greedy vector x^π is a feasible solution of the problem* (P).

Proof. We will prove this by induction on $|E|$.

It trivially holds for the case of $|E| = 1$. For the output $\pi = e_1 e_2 \cdots e_n$ of (G), set $\pi' = e_2 \cdots e_n$. Define $f' : \mathrm{ex}(\mathcal{L}/\{e_1\}) \to \mathbb{R}$ as

$$
f'(A) = \begin{cases} f(A \cup \{e_1\}) - f(\{e_1\}) & (A \cup \{e_1\} \in \mathrm{ex}(\mathcal{L})), \\ f(A) & (\text{otherwise}). \end{cases}
$$

Note that $e_1 \in \mathrm{atom}(\mathcal{L})$ and that f' is c-submodular by Lemma 5.26. Let $x^{\pi'}$ be the unique solution of the system of the following equations:

$$
\sum_{e \in A_i^\pi \setminus \{e_1\}} x^{\pi'}(e) = f'(A_i^\pi \setminus \{e_1\}) \quad (i = 2, \ldots, n).
$$

Notice that $A_i^\pi = \mathrm{ex}(\{e_1, \ldots, e_i\})$ and that $x^{\pi'}(e) = x^\pi(e)$ for all $e \in \{e_2, \ldots, e_n\}$. Assume that $\sum\{x^{\pi'}(e) : e \in X\} \leq f'(X)$ for all $X \in \mathrm{ex}(\mathcal{L}/\{e_1\})$ by the induction hypothesis.

If $|E| = n > 1$, we consider the following three cases for each $X \in \mathrm{ex}(\mathcal{L})$.

Case 1 : $e_1 \in X$.

$$\begin{aligned}\sum_{e \in X} x^\pi(e) &= x^\pi(e_1) + \sum_{e \in X \setminus \{e_1\}} x^\pi(e) = x^\pi(e_1) + \sum_{e \in X \setminus \{e_1\}} x^{\pi'}(e) \\ &\leq x^\pi(e_1) + f'(X \setminus \{e_1\}) \leq x^\pi(e_1) + f(X) - f(\{e_1\}) \\ &= x^\pi(e_1) + f(X) - x^\pi(e_1) = f(X).\end{aligned}$$

Case 2 : $e_1 \notin X$ *and* $X \cup \{e_1\} \in \mathrm{ex}(\mathcal{L})$. Since $X \sqcap \{e_1\} = \emptyset$ and $X \diamond \{e_1\} = \emptyset$,

$$\begin{aligned}\sum_{e \in X} x^\pi(e) &= \sum_{e \in X} x^{\pi'}(e) \leq f'(X) = f(X \cup \{e_1\}) - f(\{e_1\}) \\ &\leq f(X) - f(X \sqcap \{e_1\}) - f(X \diamond \{e_1\}) = f(X).\end{aligned}$$

Case 3 : $e_1 \notin X$ *and* $X \cup \{e_1\} \notin \mathrm{ex}(\mathcal{L})$. Since $X \in \mathrm{ex}(\mathcal{L}/\{e_1\})$,

$$\sum_{e \in X} x^\pi(e) = \sum_{e \in X} x^{\pi'}(e) \leq f'(X) = f(X).$$

It concludes the proof. □

From the discussion above, we show the if-part of Theorem 5.4. Now we show the only-if-part, and it completes the proof of Theorem 5.4.

Proof of Theorem 5.4. We obtain the if-part from Lemmas 5.21, 5.22, 5.23 and 5.27. Now we prove the only-if-part. By Lemma 5.10, we have $X \diamond Y \preceq X \sqcap Y \preceq X \vee Y$ for all $X, Y \in \mathrm{ex}(\mathcal{L})$ satisfying $\chi^X + \chi^Y = \chi^{X \vee Y} + \chi^{X \sqcap Y} + \chi^{X \diamond Y}$. Hence, there exists a maximal chain $\mathcal{C}$ of $\mathrm{ex}(\mathcal{L})$ containing these three sets. Let π be the permutation whose greedy chain is $\mathcal{C}$. Consider the greedy vector x^π. From the assumption that the algorithm (G) is valid, for all $A \in \mathcal{C}$,

$$\sum_{e \in A} x^\pi(e) = f(A).$$

Hence,

$$\begin{aligned}f(X \vee Y) + f(X \sqcap Y) + f(X \diamond Y) &= \sum_{e \in X \vee Y} x^\pi(e) + \sum_{e \in X \sqcap Y} x^\pi(e) + \sum_{e \in X \diamond Y} x^\pi(e) \\ = \sum_{e \in X} x^\pi(e) + \sum_{e \in Y} x^\pi(e) \leq f(X) + f(Y).\end{aligned}$$

We have the c-submodularity of f and this completes the proof. □

5.4 Consequences from the main theorem

In this section, we discuss some consequences obtained from the main theorem.

5.4.1 The Lovász extensions of c-submodular functions

Ando [1] gives the characterization of b-submodular functions by the Lovász extension. Now, we give the characterization of c-submodular functions by the Lovász extension.

To do that, we need the next lemma.

Lemma 5.28. *For any $c \in \mathbb{R}_+^E$ there uniquely exist a chain $\tilde{\mathcal{C}} : \emptyset \prec A_1 \prec A_2 \prec \cdots \prec A_m$ of $\mathrm{ex}(\mathcal{L})$ and positive numbers $\lambda_i > 0$ $(i = 1, \ldots, m)$ such that*

$$c = \sum_{i=1}^{m} \lambda_i \chi^{A_i}. \tag{5.28}$$

Proof. If we consider the greedy chain $\mathcal{C}^\pi$ and the dual greedy vector y^π, then $\tilde{\mathcal{C}} = \{A \in \mathcal{C}^\pi : y^\pi(A) > 0\}$ is the chain above.

The uniqueness is shown by induction on the size of the ground set E. Suppose that the statement holds for any proper subset of E. Let $Z = \{e \in E : c(e) > 0\}$. Assume that c is represented as (5.28). We consider the following two cases.

Case 1: $\tau(Z) = E$. We now show that $A_m = \mathrm{ex}(Z)$. For any $A \in \mathrm{ex}(\mathcal{L})$, we have $A \preceq \mathrm{ex}(E) = \mathrm{ex}(Z)$. Set A to A_m, then $A \preceq \mathrm{ex}(Z)$. Suppose that $A \prec \mathrm{ex}(Z)$. That is, there exists $e \in \mathrm{ex}(Z) \setminus A_m$. Therefore, we have $e \notin A_i$ for all $i \in \{1, \ldots, m\}$ from Lemma 5.11. This contradicts the fact $c(e) > 0$.

Secondly, we show that $\lambda_m = \min\{c(e) : e \in \mathrm{ex}(Z)\}$. Since $e \in A_m \setminus A_{m-1}$ for some $e \in \mathrm{ex}(Z)$ and $e \notin A_i$ for all $i \in \{1, \ldots, m-1\}$, we have $c(e) = \lambda_m$. Suppose that e is not a minimizer of $\{c(e) : e \in \mathrm{ex}(Z)\}$. That is, there exists $e' \in \mathrm{ex}(Z)$ such that $c(e') < c(e)$. Then, since $A_m = \mathrm{ex}(Z)$,

$$\begin{aligned} c(e') &= \sum \{\lambda_i : i \in \{1, \ldots, m\} \text{ s.t. } e' \in A_i\} \\ &= \lambda_m + \sum \{\lambda_i : i \in \{1, \ldots, m-1\} \text{ s.t. } e' \in A_i\} \\ &> \lambda_m = c(e). \end{aligned}$$

This leads to a contradiction.

Now, we have $A_m = \mathrm{ex}(Z)$ and $\lambda_m = \min\{c(e) : e \in \mathrm{ex}(Z)\}$. Let $\bar{e}$ be a minimizer of $\{c(e) : e \in \mathrm{ex}(Z)\}$, and consider the deletion $\mathcal{L} \setminus \{\bar{e}\}$. From the induction hypothesis, we have the uniqueness.

Case 2: $\tau(Z) \subsetneq E$. Delete $E \setminus \tau(Z)$ from E and consider $\mathcal{L} \setminus (E \setminus \tau(Z))$. Then it concludes the proof by the induction hypothesis. $\square$

When $c \in \mathbb{R}_+^E$ is decomposed as (5.28) for a function $f : \mathrm{ex}(\mathcal{L}) \to \mathbb{R}$, then we define $\hat{f} : \mathbb{R}_+^E \to \mathbb{R}$ as

$$\hat{f}(c) = \sum_{i=1}^{m} \lambda_i f(A_i), \tag{5.29}$$

and $\hat{f}$ is called the *Lovász extension* of f. The name of the Lovász extension is derived from Lovász [32]. Notice that $\hat{f}$ is positively homogeneous, that is, $\hat{f}(\mu c) = \mu \hat{f}(c)$ for all positive number $\mu > 0$.

Theorem 5.29. *Let $\mathcal{L}$ be a convex geometry on E. $f : \mathrm{ex}(\mathcal{L}) \to \mathbb{R}$ is c-submodular if and only if the Lovász extension $\hat{f} : \mathbb{R}_+^E \to \mathbb{R}$ defined by (5.29) is convex.*

Proof. Suppose that f is c-submodular. By Theorem 5.4

$$\hat{f}(c) = \max\left\{\sum_{e \in E} c(e)x(e) : \sum_{e \in X} x(e) \leq f(X) \quad \text{for all } X \in \mathrm{ex}(\mathcal{L})\right\}. \tag{5.30}$$

Therefore $\hat{f}$ is convex.

Conversely, suppose that $\hat{f}$ is convex. For all $X, Y \in \mathrm{ex}(\mathcal{L})$

$$\hat{f}(\chi^X) + \hat{f}(\chi^Y) \geq 2\hat{f}\left(\frac{\chi^X + \chi^Y}{2}\right) = \hat{f}(\chi^X + \chi^Y).$$

Therefore, if X and Y satisfy $\chi^X + \chi^Y = \chi^{X \vee Y} + \chi^{X \sqcap Y} + \chi^{X \diamond Y}$,

$$f(X) + f(Y) = \hat{f}(\chi^X) + \hat{f}(\chi^Y) \geq \hat{f}(\chi^X + \chi^Y) = \hat{f}(\chi^{X \vee Y} + \chi^{X \sqcap Y} + \chi^{X \diamond Y}).$$

Since $X \diamond Y \preceq X \sqcap Y \preceq X \vee Y$ by Lemma 5.10, $\hat{f}(\chi^{X \vee Y} + \chi^{X \sqcap Y} + \chi^{X \diamond Y}) = f(X \vee Y) + f(X \sqcap Y) + f(X \diamond Y)$. Hence f is c-submodular. $\square$

5.4.2 Totally dual integrality of the problem (P)

We consider the totally dual integrality of the system of the constraint inequalities (5.7) of the problem (P). For the problem (P), let c and f be rational. Then, the system of the constraint inequalities of the problem (P) is *totally dual integral* if the integrality of c implies the integrality of all λ_i $(i = 1, \ldots, m)$ in the unique representation (5.28). For the totally dual integrality, see [42] etc.

If c is integral for the problem (P), then the greedy vector y^π must be integral from the description of the greedy algorithm (G). Moreover, by Lemma 5.28, we have $\lambda_i = y^\pi(A_i)$ for any $i \in \{1, \ldots, m\}$. This fact and Theorem 5.4 lead to the following theorem.

Theorem 5.30. *If $f : \mathrm{ex}(\mathcal{L}) \to \mathbb{R}$ is c-submodular, the system of the constraint inequality of the problem* (P) *is totally dual integral.*

5.4.3 Characterization of poset shellings by b-submodular functions

Krüger [29] asserted that for a poset shelling $\mathcal{L}$ the greedy algorithm (G) works if and only if a function $f : \mathrm{ex}(\mathcal{L}) \to \mathbb{R}$ is b-submodular. Now, we prove that the greedy algorithm (G) works for all b-submodular functions if and only if $\mathcal{L}$ is a poset shelling as Theorem 5.34. Theorem 5.3 noted in Section 5.1 is a direct consequence of Theorems 5.4 and 5.34.

The next proposition is an important characterization of poset shellings.

Proposition 5.31 ([28]). *A convex geometry is a poset shelling if and only if it is closed under union.*

Using this proposition, we have another characterization of poset shellings, which is related to the residue.

Lemma 5.32. *A convex geometry $\mathcal{L}$ on E is a poset shelling if and only if $X \diamond Y = \emptyset$ for all $X, Y \in \mathrm{ex}(\mathcal{L})$.*

Proof. Suppose that $\mathcal{L}$ is not a poset shelling. Then by Proposition 5.31, there exist $X, Y \in \mathrm{ex}(\mathcal{L})$ such that $\tau(X) \cup \tau(Y) \subsetneq \tau(\tau(X) \cup \tau(Y)) = \tau(X \vee Y)$. That is, there exists some $a \in E$ such that $a \in \tau(X \vee Y) \setminus (\tau(X) \cup \tau(Y))$. Set $A = \mathrm{ex}(\tau(X) \cup \{a\})$ and $B = \mathrm{ex}(\tau(Y) \cup \{a\})$. Now we show that $A \diamond B \neq \emptyset$.

Since $a \in \tau(X \vee Y) \setminus (\tau(X) \cup \tau(Y))$, we have $a \notin \tau(X) = \tau(\tau(X)) = \tau((\tau(X) \cup \{a\}) \setminus \{a\})$. Therefore, a is an extreme point of $\tau(X) \cup \{a\}$, and we obtain that $a \in \mathrm{ex}(\tau(X) \cup \{a\}) = A$. Similarly, we obtain that $a \in B$. Hence, $a \in A \cap B$.

Now suppose that $a \in X \vee Y$. Then, $\tau(X \vee Y) \setminus \{a\} \in \mathcal{L}$ by the definition of extreme points. Since $a \notin \tau(X)$, we have $\tau(X) = \tau(X) \setminus \{a\} \subseteq \tau(X \vee Y) \setminus \{a\}$. Similarly, $\tau(Y) \subseteq \tau(X \vee Y) \setminus \{a\}$. This contradicts the minimality of $X \vee Y$. Therefore $a \notin X \vee Y$.

We have $A \preceq X \vee Y$ since $\tau(A) = \tau(\mathrm{ex}(\tau(X) \cup \{a\})) \subseteq \tau(\tau(X) \cup \{a\}) \subseteq \tau(\tau(X) \cup \tau(Y) \cup \{a\}) = \tau(\tau(X) \cup \tau(Y)) = \tau(X \vee Y)$. Since $\tau(X) \subseteq \tau(\tau(X) \cup \{a\}) = \tau(A)$, $X \preceq A$. Similarly, $Y \preceq B \preceq X \vee Y$. Therefore, $X \vee Y = A \vee B$, and we obtain that $a \notin A \vee B$. Hence we have $a \in (A \cap B) \setminus (A \vee B) = A \diamond B$.

Conversely, suppose that $X \diamond Y \neq \emptyset$ for some $X, Y \in \mathrm{ex}(\mathcal{L})$. That is, there exists $a \in E$ such that $a \in X$, $a \in Y$ and $a \notin X \vee Y$. By Lemma 5.5, $\tau(X) \setminus \{a\} \in \mathcal{L}$ and $\tau(Y) \setminus \{a\} \in \mathcal{L}$. Now suppose that $\mathcal{L}$ is a poset shelling. Then, by Proposition 5.31 we obtain that $(\tau(X) \setminus \{a\}) \cup (\tau(Y) \setminus \{a\}) \in \mathcal{L}$. Therefore, $(\tau(X) \cup \tau(Y)) \setminus \{a\} \in \mathcal{L}$. By Proposition 5.31 and $\tau(X \vee Y) = \tau(X) \cup \tau(Y) \in \mathcal{L}$, we have $a \in X \vee Y$. This is a contradiction. ☐

Next, we prove the following lemma for the theorem of this subsection.

Lemma 5.33. *Let $\mathcal{L}$ be a convex geometry on E. Suppose that $\mathcal{L}$ is not a poset shelling. Then, there exist $X, Y \in \mathrm{ex}(\mathcal{L})$ such that $X \diamond Y \neq \emptyset$ and $\chi^X + \chi^Y = \chi^{X \vee Y} + \chi^{X \sqcap Y} + \chi^{X \diamond Y}$.*

Proof. Since $\mathcal{L}$ is not a poset shelling, there exist $X, Y \in \mathrm{ex}(\mathcal{L})$ such that $X \diamond Y \neq \emptyset$ by Lemma 5.32. Set X^*, Y^* as

$$(X^*, Y^*) \in \arg\min\{|\tau(X \vee Y) \setminus \tau(X)| + |\tau(X \vee Y) \setminus \tau(Y)| : X \diamond Y \neq \emptyset\}.$$

By Remark 5.15, $\chi^{X^*} + \chi^{Y^*} \geq \chi^{X^* \vee Y^*} + \chi^{X^* \sqcap Y^*} + \chi^{X^* \diamond Y^*}$. Now suppose that $\chi^{X^*} + \chi^{Y^*} > \chi^{X^* \vee Y^*} + \chi^{X^* \sqcap Y^*} + \chi^{X^* \diamond Y^*}$. Then, by Remark 5.15, without loss of generality, we have $b \in X^*$, $b \notin \tau(Y^*)$, $b \notin X^* \vee Y^*$ and $b \in \tau(X^* \vee Y^*)$ for some $b \in E$. By Lemma 5.11, there exists $Y' \in \mathrm{ex}(\mathcal{L})$ such that $b \in Y'$ and $Y^* \prec Y' \prec X^* \vee Y^*$. Therefore $X^* \vee Y^* = X^* \vee Y'$. Hence, we have $|\tau(X^* \vee Y^*) \setminus \tau(X^*)| + |\tau(X^* \vee Y^*) \setminus \tau(Y^*)| > |\tau(X^* \vee Y') \setminus \tau(X^*)| + |\tau(X^* \vee Y') \setminus \tau(Y')|$, since $b \notin \tau(X^* \vee Y^*) \setminus \tau(Y')$. This contradicts the minimality of X^* and Y^*. ☐

Theorem 5.34. *Let $\mathcal{L}$ be a convex geometry. Algorithm* (G) *gives an optimum of the problem* (D) *for all b-submodular functions if and only if $\mathcal{L}$ is a poset shelling.*

Proof. The if-part is Theorem 5.2. We show the only-if-part. Suppose that $\mathcal{L}$ is not a poset shelling. Then, we set $\tilde{f}$ as

$$\tilde{f}(X) = \begin{cases} 0 & (X = \emptyset), \\ 1 & (X \neq \emptyset). \end{cases}$$

We can easily check that $\tilde{f}$ is b-submodular but not c-submodular using Lemma 5.33. From Theorem 5.4, Algorithm (G) does not always give an optimum for $\tilde{f}$. $\square$

Chapter 6

Core Stability of Games on Convex Geometries

6.1 Introduction to this chapter

The concepts of core and stability consist of a part of the main topics of cooperative game theory [36]. We consider a game on a convex geometry, introduced by Bilbao [2] as a model of cooperation structures in which only certain coalitions are allowed to form. We show that if a game on a convex geometry is quasi-convex, then the core is a unique stable set of the game.

Recall the definition of a convex geometry. Let $N = \{1, 2, \ldots, n\}$ be a finite set. A *convex geometry* $\mathcal{L}$ on N is a family of subsets of N with the properties

$$\emptyset \in \mathcal{L}, N \in \mathcal{L}, \tag{6.1}$$

$$S, T \in \mathcal{L} \text{ imply } S \cap T \in \mathcal{L}, \tag{6.2}$$

$$S \in \mathcal{L} \setminus \{N\} \text{ implies } S \cup \{i\} \in \mathcal{L} \text{ for some } i \in N \setminus S. \tag{6.3}$$

For a convex geometry $\mathcal{L}$ on N, we call an element i of $S \in \mathcal{L}$ an *extreme point* of S if $S \setminus \{i\} \in \mathcal{L}$. The set of the extreme points of S is denoted by $\mathrm{ex}(S)$, called the *extreme set* of S.

A *cooperative game* on a convex geometry $\mathcal{L}$ on N is a function $v : \mathcal{L} \to \mathbb{R}$ with $v(\emptyset) = 0$. The *players* are the elements of N and the *coalitions* are the members of $\mathcal{L}$.

This includes some situations introduced former. Indeed, if $\mathcal{L} = 2^N$, then every subset of the players forms a coalition. If $\mathcal{L}$ is defined via a partially ordered set

$P = (N, \leq)$ as

$$\mathcal{L} = \{S \subseteq N : S \text{ is an order ideal of } P\}, \tag{6.4}$$

then $\mathcal{L}$ forms a convex geometry, which is a model introduced by Faigle–Kern [14]. Besides, let G be a connected graph with the vertex set N. Myerson [33] introduced a communication game. Define $\mathcal{L}$ as

$$\mathcal{L} = \{S \subseteq N : G[S] \text{ is connected }\}, \tag{6.5}$$

where $G[S]$ means the subgraph induced by S. If G is a block-graph, that is, every 2-connected component forms a complete graph, then $\mathcal{L}$ is a convex geometry.

6.2 The cores and the stable sets

In this section, we introduce some solution concepts. Let $\mathcal{L}$ be a convex geometry on the players N, and $v : \mathcal{L} \to \mathbb{R}$ be a game on $\mathcal{L}$. The set of *pre-imputations* of the game v on $\mathcal{L}$ is

$$I^*(\mathcal{L}, v) = \left\{ x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N) \right\}, \tag{6.6}$$

and the set of *imputations* is

$$I(\mathcal{L}, v) = \{x \in I^*(\mathcal{L}, v) : x_i \geq v(\{i\}) \text{ for all } \{i\} \in \mathcal{L}\}. \tag{6.7}$$

We may say that a pre-imputation is a *Pareto-optimal* or *efficient* pay-off vector, and that an imputation is an *individually rational* pre-imputation. Note that the set of imputations is possibly empty and not necessarily bounded.

The *core* of the game v on $\mathcal{L}$ is defined as

$$\mathrm{Core}(\mathcal{L}, v) = \left\{ x \in I^*(\mathcal{L}, v) : \sum_{i \in S} x_i \geq v(S) \text{ for all } S \in \mathcal{L} \right\}. \tag{6.8}$$

We may also say that an imputation in the core is *coalitionally rational*. For the core, the next proposition is known.

Proposition 6.1 ([3]). *Let $\mathcal{L}$ be a convex geometry on N, and v be a game on $\mathcal{L}$.*

1. *The core* $\mathrm{Core}(\mathcal{L}, v)$ *is either empty or a pointed polyhedron.*

2. *Moreover, if* $\{i\} \in \mathcal{L}$ *for all* $i \in N$*, then* $\mathrm{Core}(\mathcal{L}, v)$ *is bounded.*

Next, we introduce the stable set. Let $x, y \in I(\mathcal{L}, v)$ for a game v on a convex geometry $\mathcal{L}$, and $S \in \mathcal{L}$. We say that y *dominates* x *via* S if

$$\sum_{i \in S} y_i \leq v(S) \quad \text{and} \tag{6.9}$$

$$y_i > x_i \quad \text{for all } i \in S, \tag{6.10}$$

and denote it by $x \prec_S y$. A *stable set* [47] of a game v on a convex geometry $\mathcal{L}$ is a set $\mathrm{Stable}(\mathcal{L}, v) \subseteq I(\mathcal{L}, v)$ of imputations satisfying the following two conditions:

1. for all $y \in I(\mathcal{L}, v) \setminus \mathrm{Stable}(\mathcal{L}, v)$, there exist $x \in \mathrm{Stable}(\mathcal{L}, v)$ and $S \in \mathcal{L}$ such that $y \prec_S x$,

2. for all $x, y \in \mathrm{Stable}(\mathcal{L}, v)$ and $S \in \mathcal{L}$, $x \not\prec_S y$ and $y \not\prec_S x$.

The first condition is called the *external stability*, while the second one is called the *internal stability*.

Proposition 6.2. *1. A stable set is not necessarily unique and possibly empty.*

2. If two collections of imputations are distinct stable sets, then they do not include each other.

3. If the core is nonempty and so is any stable set of the game, then the core is always included in every stable set.

A well-known fact on stable sets is that, if $\mathcal{L} = 2^N$ and a game v is convex, then the core is a unique stable set [41], where a game $v : 2^N \to \mathbb{R}$ is *convex* if we have

$$v(S) + v(T) \leq v(S \cap T) + v(S \cup T) \tag{6.11}$$

for all $S, T \in 2^N$. In the next section, we show that if a game on a general convex geometry is quasi-convex, then the stable set is unique and coincides with the core.

6.3 Quasi-convex games and core stability

A game $v : \mathcal{L} \to \mathbb{R}$ is *quasi-convex* if for all $S, T \in \mathcal{L}$ with $S \cup T \in \mathcal{L}$, we have

$$v(S) + v(T) \leq v(S \cap T) + v(S \cup T). \tag{6.12}$$

The quasi-convexity may be first defined in [3]. Obviously, if $\mathcal{L} = 2^N$, then a quasi-convex game is convex.

Let $\mathcal{L}$ be a convex geometry, v be a game on $\mathcal{L}$ and

$$\mathcal{C} : \emptyset = S_0 \subsetneq S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_{n-1} \subsetneq S_n = N \tag{6.13}$$

be a maximal chain of $\mathcal{L}$. Then, the *marginal worth vector* $a^{\mathcal{C}} \in \mathbb{R}^N$ with respect to the maximal chain $\mathcal{C}$ in the game v is defined by

$$a_i^{\mathcal{C}} = v(S_{\mathcal{C}}(i)) - v(S_{\mathcal{C}}(i) \setminus \{i\}) \quad \text{for all } i \in N, \tag{6.14}$$

where $S_{\mathcal{C}}(i) = \bigcap\{S \in \mathcal{C} : i \in S\}$, that is, $S_{\mathcal{C}}(i)$ is the minimal set of $\mathcal{C}$ containing i. Note that $i \in \mathrm{ex}(S_{\mathcal{C}}(i))$ from the minimality of $S_{\mathcal{C}}(i)$.

Lemma 6.3 ([3, Theorem 5]). *A game v on a convex geometry $\mathcal{L}$ is quasi-convex if and only if $a^{\mathcal{C}}$ is in* $\mathrm{Core}(\mathcal{L}, v)$ *for every maximal chain $\mathcal{C}$ of $\mathcal{L}$.*

Now, we show the main result.

Theorem 6.4. *Let $\mathcal{L}$ be a convex geometry on the players N, and $v : \mathcal{L} \to \mathbb{R}$ be a quasi-convex game on $\mathcal{L}$. Then* $\mathrm{Core}(\mathcal{L}, v) = \mathrm{Stable}(\mathcal{L}, v)$.

Proof. First, we show that the core satisfies the internal stability. Suppose that $x, y \in \mathrm{Core}(\mathcal{L}, v)$ satisfy $x \prec_S y$ for some $S \in \mathcal{L}$. Namely, $\sum\{y_i : i \in S\} \leq v(S)$ and $y_i > x_i$ for all $i \in S$. Then we have $\sum\{x_i : i \in S\} < v(S)$, which contradicts the fact that x is in the core.

Next, we consider the external stability. Take an arbitrary $y \in I(\mathcal{L}, v) \setminus \mathrm{Core}(\mathcal{L}, v)$. Define $g : \mathcal{L} \setminus \{\emptyset\} \to \mathbb{R}$ as

$$g(S) = \frac{v(S) - \sum_{i \in S} y_i}{|S|}. \tag{6.15}$$

Let S^* be a maximizer of g. Since $y \notin \mathrm{Core}(\mathcal{L}, v)$ and S^* is a maximizer of g, we have $g(S^*) > 0$. We pick up a maximal chain $\mathcal{C}$ of $\mathcal{L}$ such that $S^* \in \mathcal{C}$. Now, define x as

$$x_i = \begin{cases} y_i + g(S^*) & (i \in S^*), \\ a_i^{\mathcal{C}} & (i \in N \setminus S^*). \end{cases} \tag{6.16}$$

We have $x_i > y_i$ for all $i \in S^*$. Moreover,

$$\sum_{i \in S^*} x_i = \sum_{i \in S^*} y_i + |S^*| g(S^*) = v(S^*), \tag{6.17}$$

$$\sum_{i \in N} x_i = v(S^*) + \sum_{i \in N \setminus S^*} a_i^{\mathcal{C}} = \sum_{i \in S^*} a_i^{\mathcal{C}} + \sum_{i \in N \setminus S^*} a_i^{\mathcal{C}} = \sum_{i \in N} a_i^{\mathcal{C}} = v(N). \tag{6.18}$$

Therefore, we have $y \prec_{S^*} x$. It remains to show that $x \in \mathrm{Core}(\mathcal{L}, v)$.

Take an arbitrary $T \in \mathcal{L}$. Then,

$$\sum_{i \in T} x_i = \sum_{i \in S^* \cap T} x_i + \sum_{i \in T \setminus S^*} x_i \tag{6.19}$$

$$= \sum_{i \in S^* \cap T} y_i + |S^* \cap T| g(S^*) + \sum_{i \in T \setminus S^*} a_i^{\mathcal{C}} \tag{6.20}$$

$$\geq \sum_{i \in S^* \cap T} y_i + |S^* \cap T| g(S^* \cap T) + \sum_{i \in T \setminus S^*} a_i^{\mathcal{C}} \tag{6.21}$$

$$= v(S^* \cap T) + \sum_{i \in T \setminus S^*} a_i^{\mathcal{C}}. \tag{6.22}$$

Let us recall that $S_{\mathcal{C}}(i)$ is the minimal set of $\mathcal{C}$ containing i. From (6.3), there must be a numbering $T \setminus S^* = \{i_1, i_2, \ldots, i_m\}$ such that $S_{\mathcal{C}}(i_1) \subsetneq S_{\mathcal{C}}(i_2) \subsetneq \cdots \subsetneq S_{\mathcal{C}}(i_m)$. Notice that $S^* \subsetneq S_{\mathcal{C}}(i_1)$. Set $S_j = S_{\mathcal{C}}(i_j) \cap T \in \mathcal{L}$ for all $j \in \{1, \ldots, m\}$, and $S_0 = S^* \cap T$. Then, we have $S_j = \{i_1, \ldots, i_j\}$ and $i_j \in \mathrm{ex}(S_j)$. Since $S_{\mathcal{C}}(i_j) = (S_{\mathcal{C}}(i_j) \setminus \{i_j\}) \cup S_j$, $S_{j-1} = (S_{\mathcal{C}}(i_j) \setminus \{i_j\}) \cap S_j$, and v is quasi-convex,

$$a_{i_j}^{\mathcal{C}} = v(S_{\mathcal{C}}(i_j)) - v(S_{\mathcal{C}}(i_j) \setminus \{i_j\}) \geq v(S_j) - v(S_{j-1}), \tag{6.23}$$

and we have

$$\sum_{i \in T \setminus S^*} a_i^{\mathcal{C}} \geq \sum_{j=1}^{m} (v(S_j) - v(S_{j-1})) = v(T) - v(S^* \cap T). \tag{6.24}$$

Hence,

$$v(S^* \cap T) + \sum_{i \in T \setminus S^*} a_i^{\mathcal{C}} \geq v(S^* \cap T) + v(T) - v(S^* \cap T) = v(T). \tag{6.25}$$

This concludes that $x \in \mathrm{Core}(\mathcal{L}, v)$. Moreover, from Proposition 6.2 we have $\mathrm{Core}(\mathcal{L}, v) = \mathrm{Stable}(\mathcal{L}, v)$. $\square$

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