Core Stability of Minimum Coloring Games

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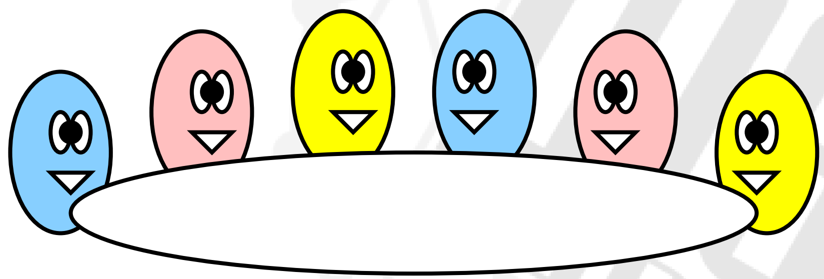
What is all about?

Framework: Several people are willing to work together...

- They want to have a largest possible benefit.
  . . . . optimization problem
- They want to allocate the benefit in a fair way.
  . . . . game-theoretic problem



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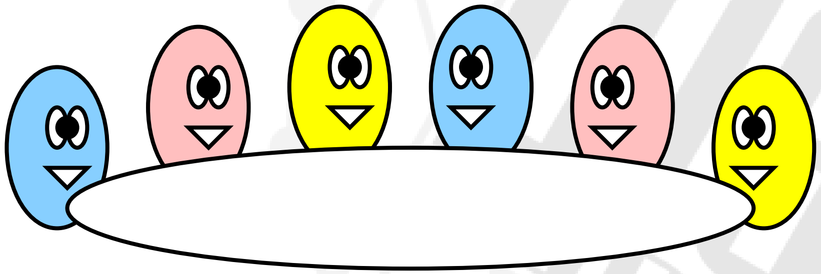
Game Theory?

- Noncooperative Game Theory
- Cooperative Game Theory



Cooperative games

Def.: A **cooperative game** (or a **game**) is a pair (N, γ) of

- a finite set N (set of **players**)
- a function $\gamma : 2^{\mathrm{N}} \to \mathbb{R}$ with $\gamma(\emptyset) = 0$ (**characteristic function**).

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Interpretation: For $S \subseteq N$,

$\gamma(S)$ represents $\left\{ \begin{array}{l} \text{the max. benefit gained by } S \\ \text{the min. cost owed by } S \end{array} \right\}$

when the players in S work in cooperation.

Goal: To allocate $\gamma(N)$ to each player in a "fair" way.

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This work: study on **"minimum coloring games."**

Definition: minimum coloring game

$G = (V, E)$ an undirected graph

- A **proper k-coloring** of G
  is a map $c : V \to \{1, \ldots, k\}$ s.t.
  if $\{u, v\} \in E$, then $c(u) \neq c(v)$.
- The **chromatic number** $\chi(G)$ of G
  $= \min\{\ k\ :\ $ a proper k-coloring of G exists $\}$.

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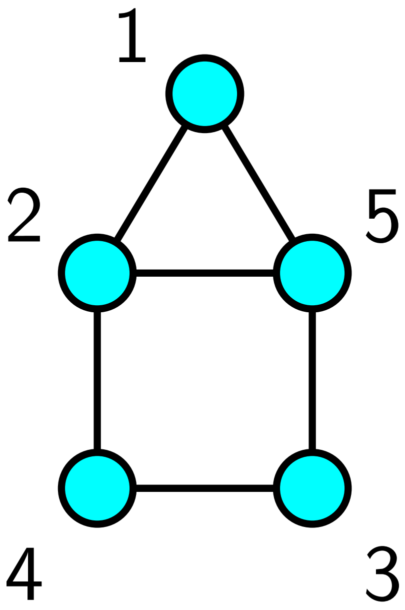
  $= \min\{\ k\ :\ $ a proper k-coloring of G exists $\}$.
- The **minimum coloring game** on G
  is a cooperative game (V, χ_G).

  $\chi_G : 2^V \to \mathbb{N}$ is defined as $\chi_G(S) = \chi(G[S])$,
  where $G[S]$ is the subgraph induced by $S \subseteq V$.

(Deng, Nagamochi & Ibaraki '99)

Example: minimum coloring game

$\chi_G(S) = \chi(G[S])$ for $S \subseteq V$.

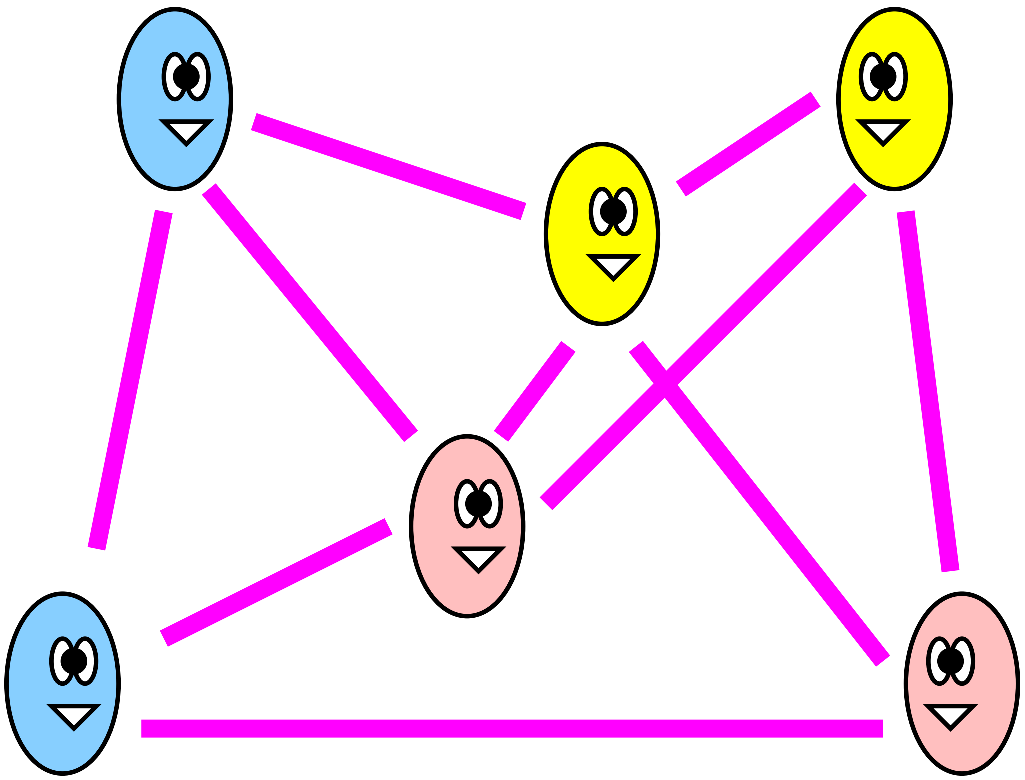


S	χ_G	S	χ_G	S	χ_G	S	χ_G
$\emptyset$	0	14	1	123	2	245	2
1	1	15	2	124	2	345	2
2	1	23	2	125	3	1234	2
3	1	24	1	134	2	1235	3
4	1	25	2	135	2	1245	3
5	1	34	2	145	2	1345	2
12	2	35	1	234	2	2345	2
13	1	45	2	235	2	12345	3

Goal: To allocate $\chi(G)$ to each vertex in a fair way.

Conflict graph: a model of conflict

- the vertices = the agents, the principals...
- the edges = between two in conflict.



min. coloring game:

a simplest model of the
fair cost allocation problem
in conflict situations

A bit of history

Interested in certain sets of fair allocations.

- **Stable set** (von Neumann & Morgenstern '44)
  - Quite useful
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Characterize games with stable cores.

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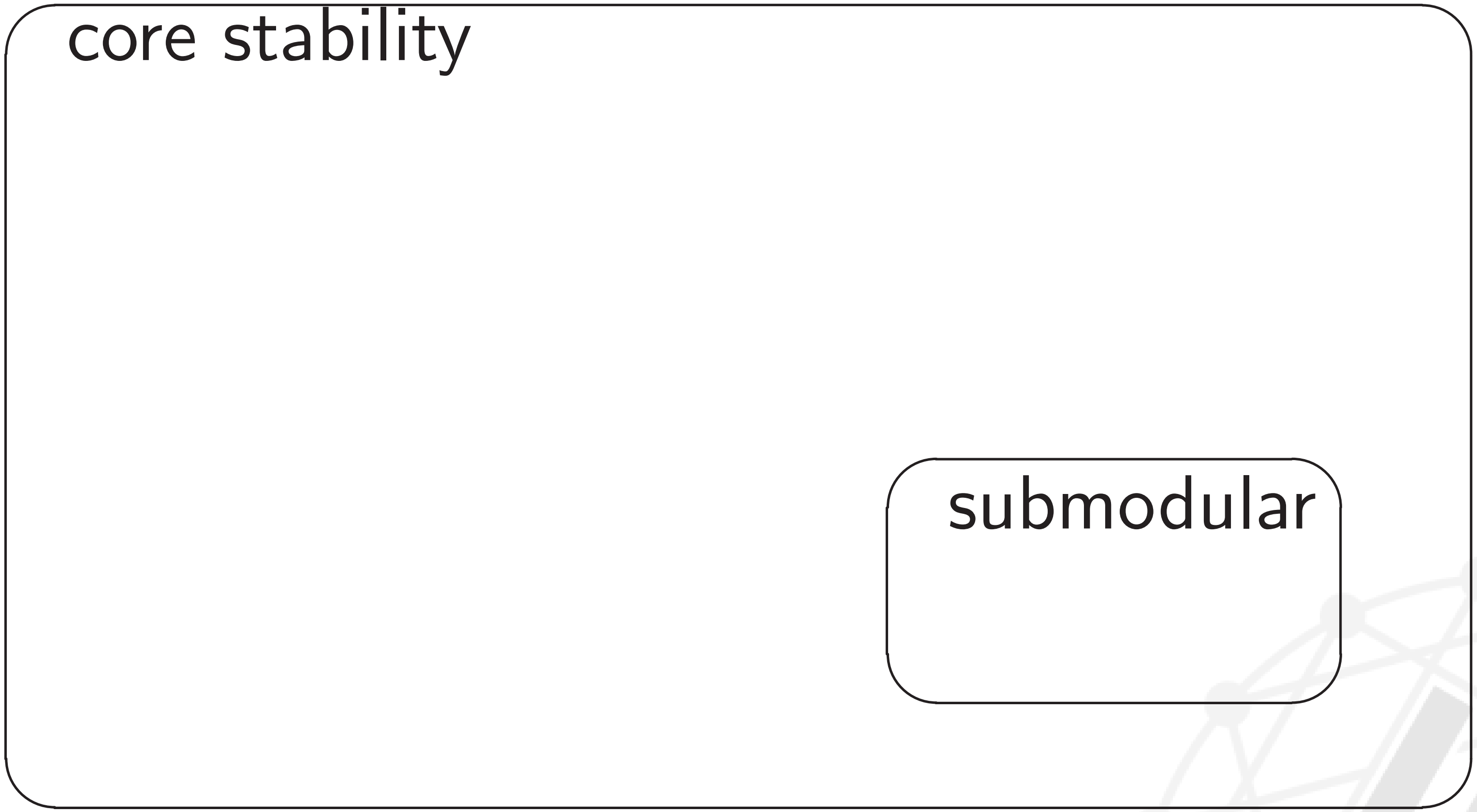
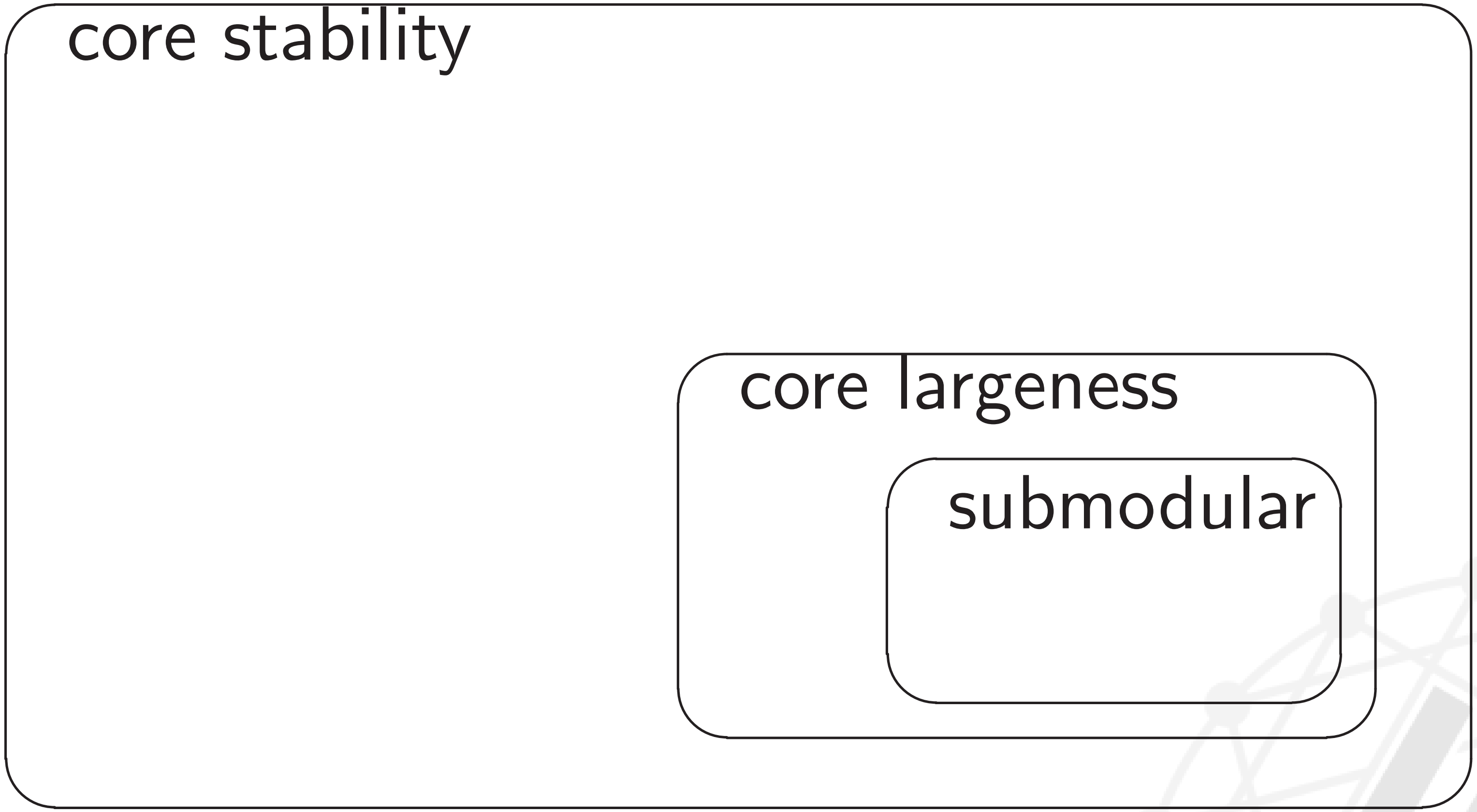
"Core Stability Problem" ...... Far from being solved

Core stability problem



Shapley ’71

Core stability problem



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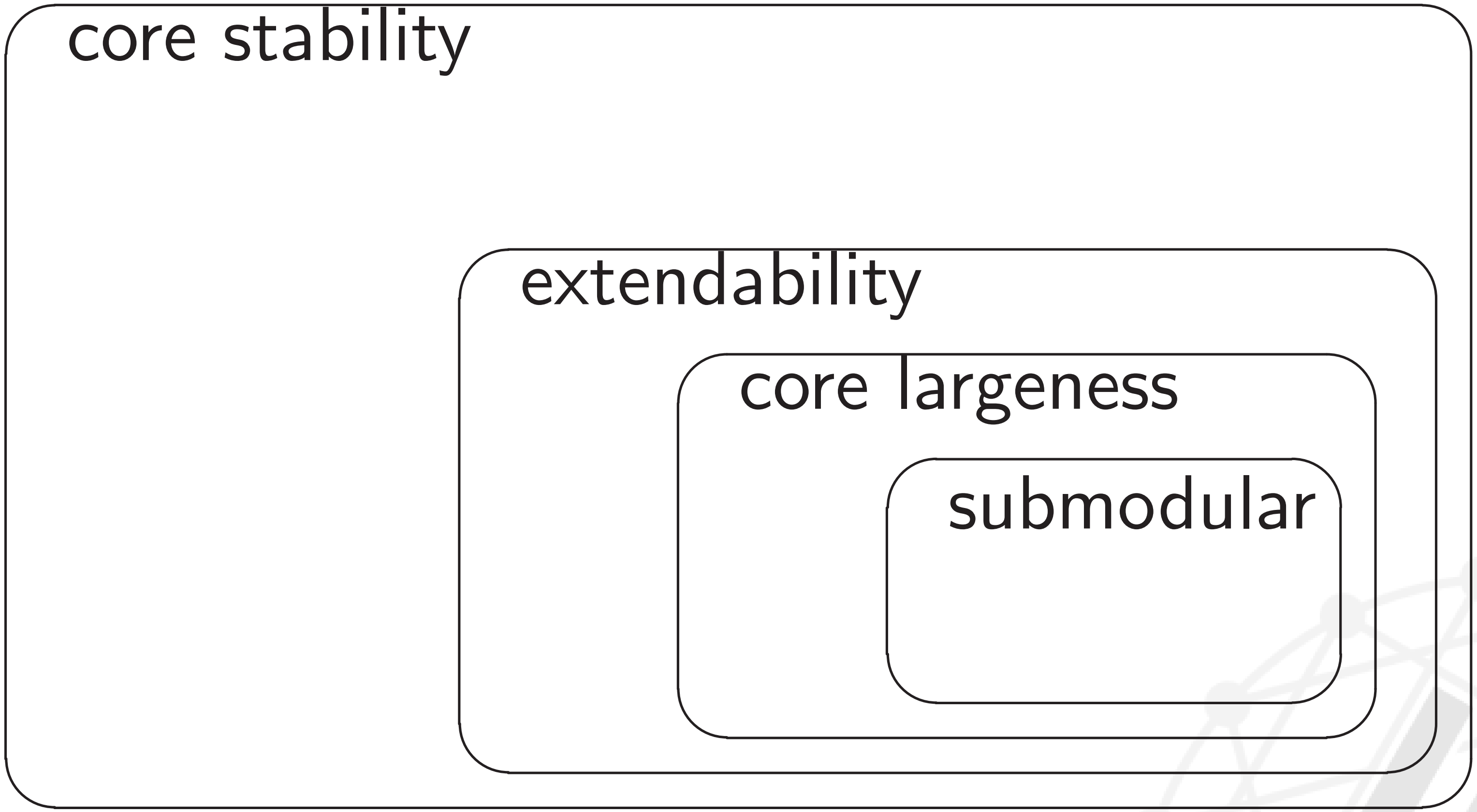

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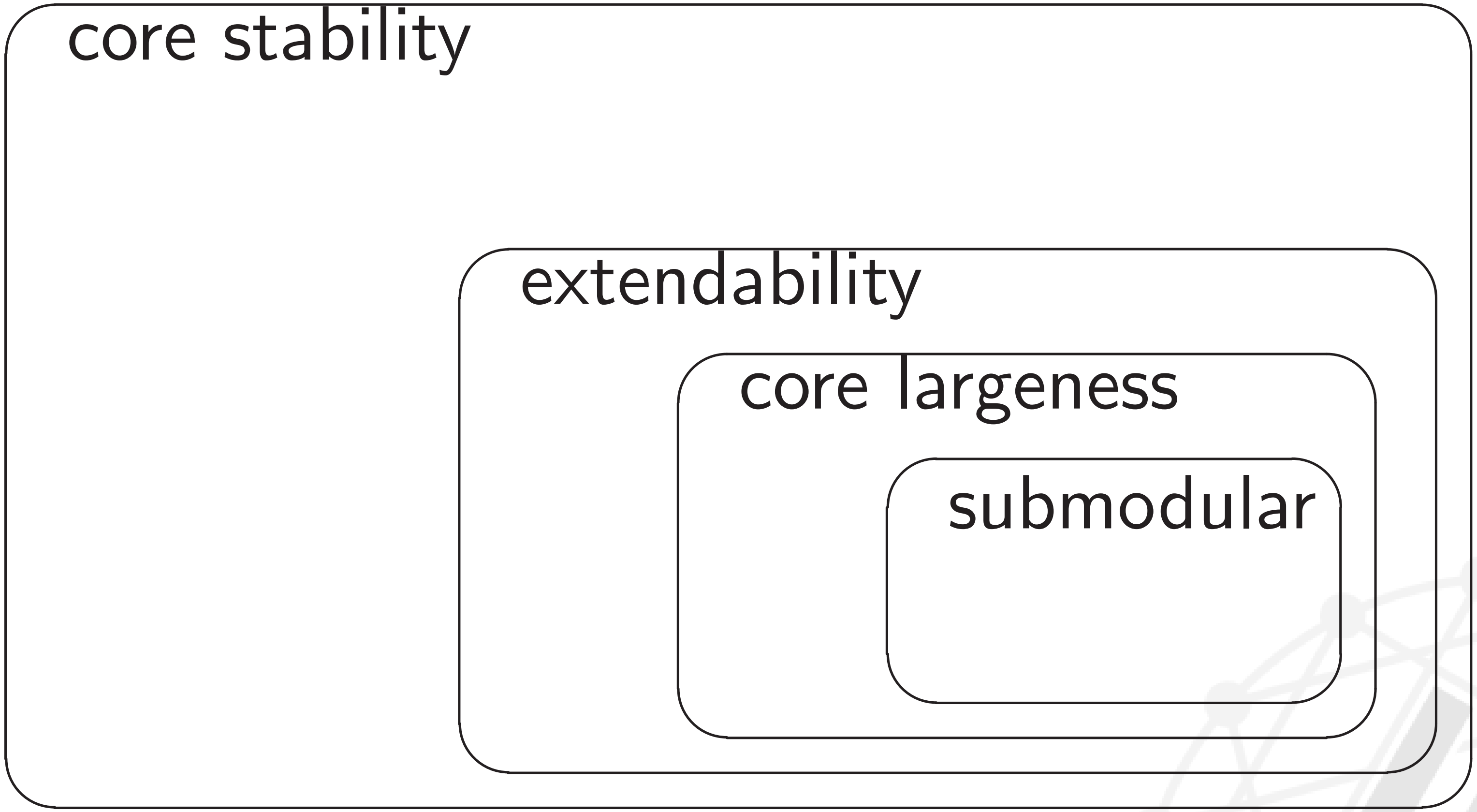

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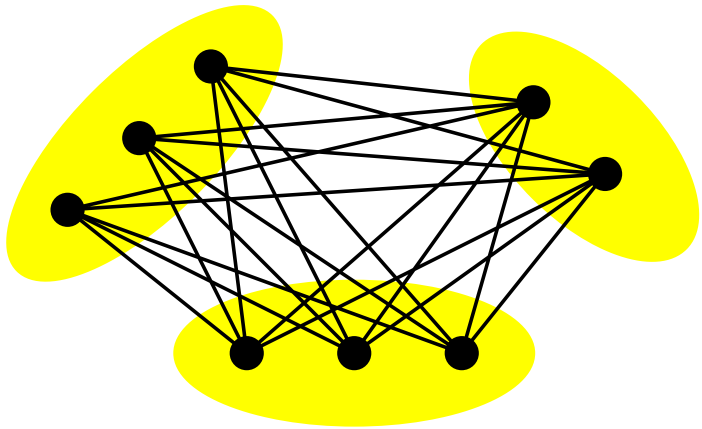
Sharkey '82
Kikuta & Shapley '86
van Gellekom, Potters & Reijnierse '99

Previous result

Thm (Okamoto '03)

The following are equivalent.

- The minimum coloring game on G is submodular.
- G is complete multipartite.

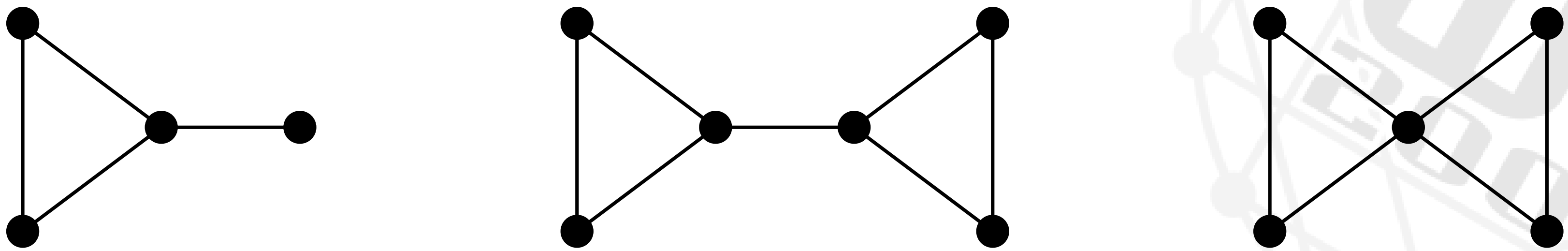


Result (1)

Thm **For a perfect graph** G**,**

(1) The following are equivalent.

- The minimum coloring game on G has a stable core.
- Every vertex of G belongs to a maximum clique.



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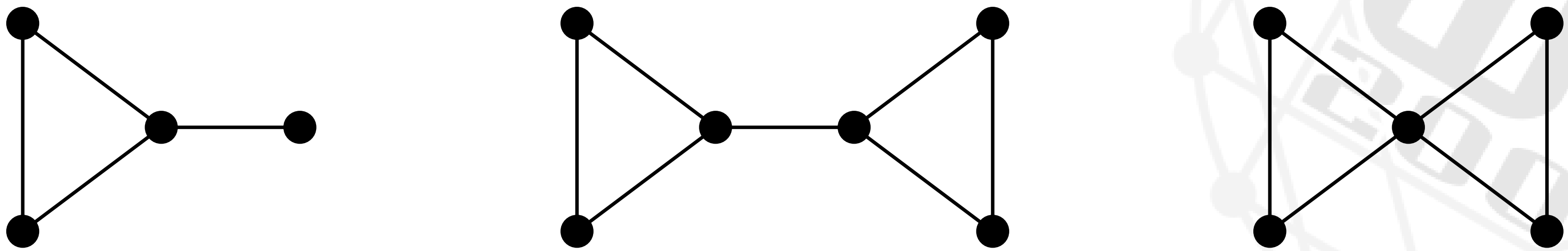
This condition can be checked in polynomial time.

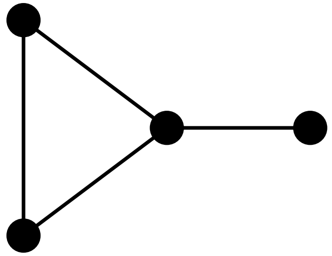


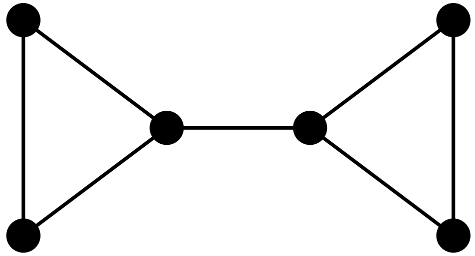
Result (2)

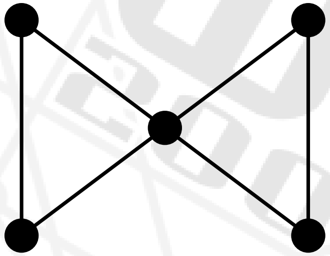
Thm **For a perfect graph** G**,**

(2) The following are equivalent.

- The minimum coloring game on G has a large core.
- The minimum coloring game on G is exact.
- The minimum coloring game on G is extendable.
- Every clique of G is contained in a maximum clique.







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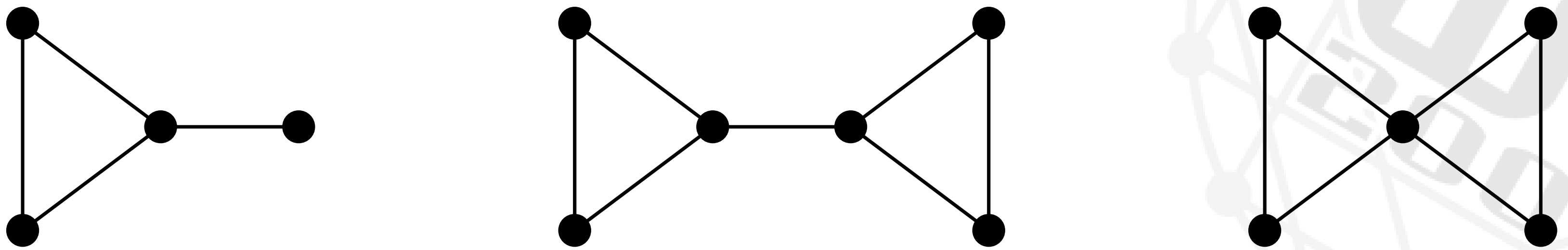
Checking this condition is coNP-complete.



We concentrate on Result (1).

- Cost allocation, Core
- Perfect graph
- Stable Core

Def.: A **cost allocation** for a game (N, γ) is a vector $z \in \mathbb{R}^N$ such that

$$\sum\{z[i] : i \in N\} = \gamma(N).$$

(Often called a **pre-imputation** in cooperative game theory)

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Interpretation:

$z[i] =$ the amount of cost the player i must pay when all players in N work together

Imputation

$\boxed{\text{Def.:}}$ A cost allocation $z \in \mathbb{R}^N$ for (N, γ) is an **imputation** if

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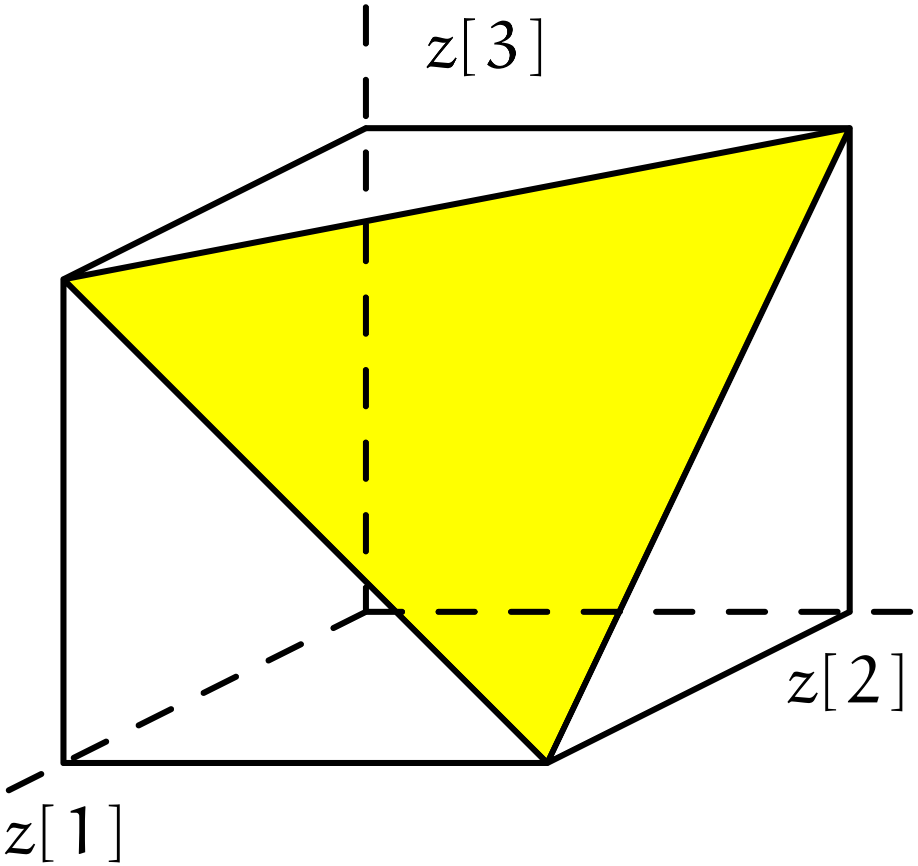
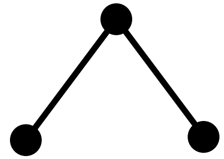
Interpretation: Each player $i \in N$ is satisfied with z

$z[i]$: cost owed by i when people in N work together

$\gamma(\{i\})$: cost owed by i when i works alone

Example: Imputation





$V = \{1, 2, 3\}$

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$$\mathrm{Imp} = \left\{ z \in \mathbb{R}^3 \;:\; \begin{array}{l} z[1] \le 1,\ z[2] \le 1,\ z[3] \le 1, \\ z[1] + z[2] + z[3] = 2 \end{array} \right\}$$

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Def.: A cost allocation $z \in \mathbb{R}^N$ for (N, γ) is a **core allocation** if

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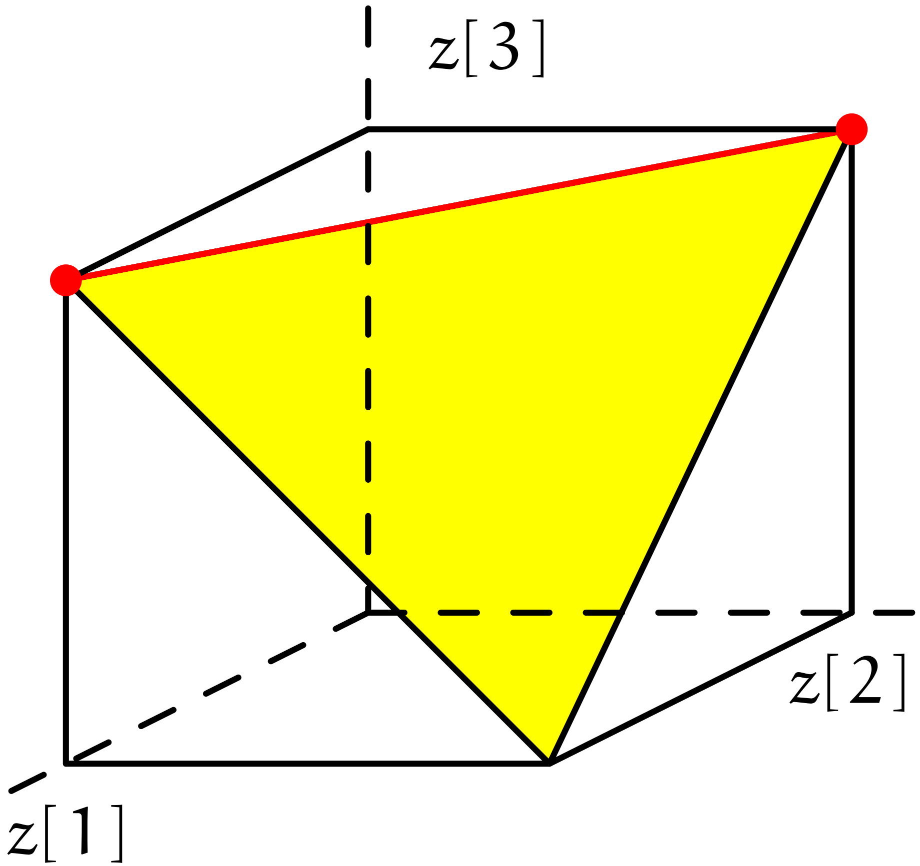
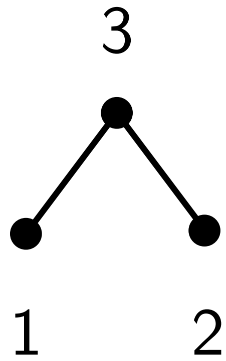
Interpretation: Each subset $S \subseteq N$ is satisfied with z

$\sum_{i \in S} z[i]$: cost owed by S when people in N work together

$\gamma(S)$: cost owed by S when people in S work together.

Example: Core





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$$\text{Core} = \left\{ z \in \mathbb{R}^3 \; : \; \begin{array}{l} z[1] \le 1,\; z[2] \le 1,\; z[3] \le 1, \\ z[1] + z[2] \le 1,\; z[1] + z[3] \le 2, \\ z[2] + z[3] \le 2,\; z[1] + z[2] + z[3] = 2 \end{array} \right\}$$

Perfect graph (Berge '60)

$\boxed{\text{Def.:}}$ A graph G is **perfect** if $\forall$ induced subgraph H of G

the size of maximum clique $=$ the chromatic number.

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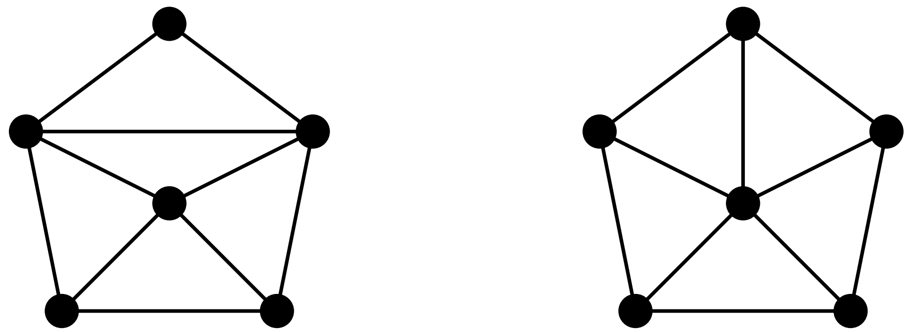

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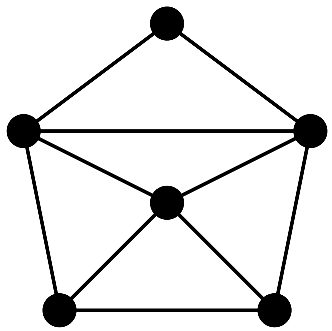

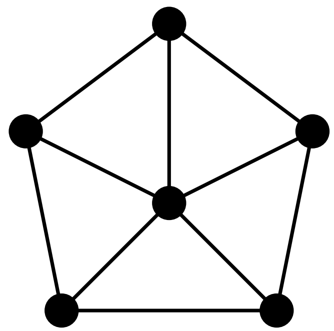

Examples of perfect graphs

- Bipartite graphs
- Complete multipartite graphs
- Interval graphs
- The complements of perfect graphs (Lovász '72)

Why perfect graphs??

When G is perfect,

- The chromatic number can be computed in poly time.
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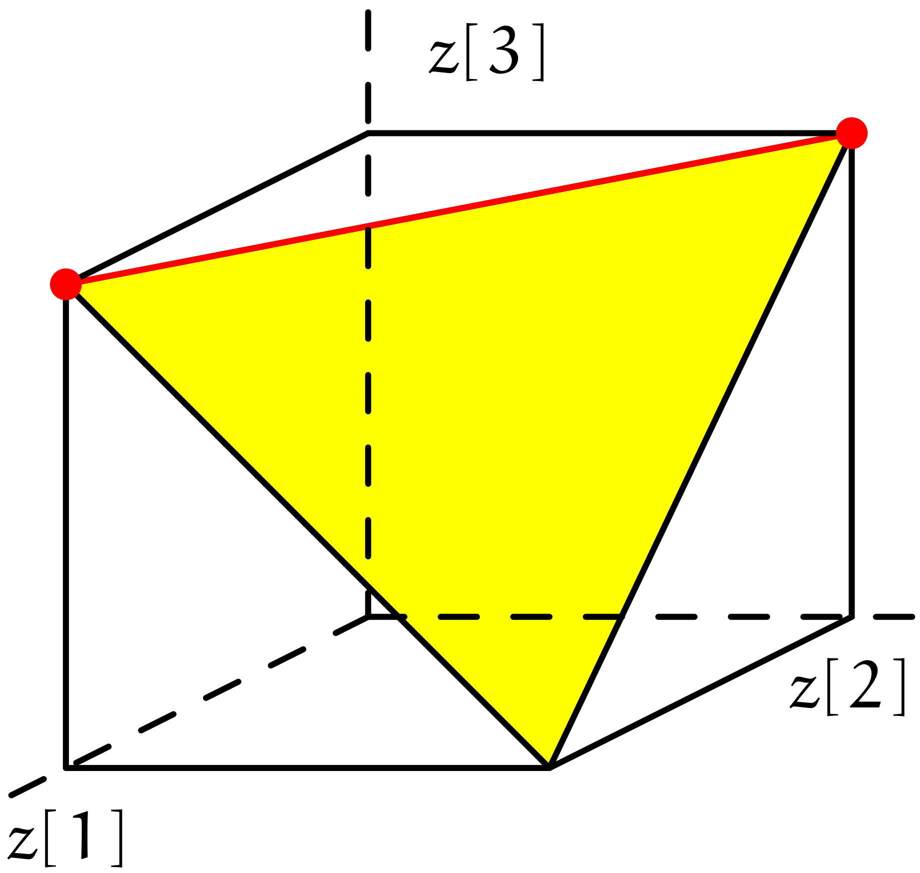
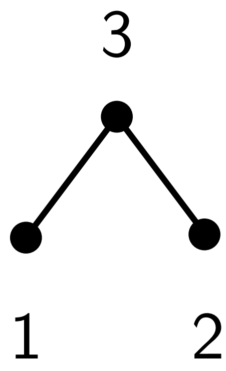
- Core = conv(the char vectors of maximum cliques of G).
  (Okamoto '03)

Example: Core





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$$\text{Core} = \text{conv}\{(1,0,1),\ (0,1,1)\}$$

Stable core (von Neumann & Morgenstern '44)

Def.: The core of (N, γ) is **stable** if

$\forall\, \mathbf{y} \in \text{Imp} \setminus \text{Core}$
$\exists\, \mathbf{x} \in \text{Core}$ and $S \subset N$ such that

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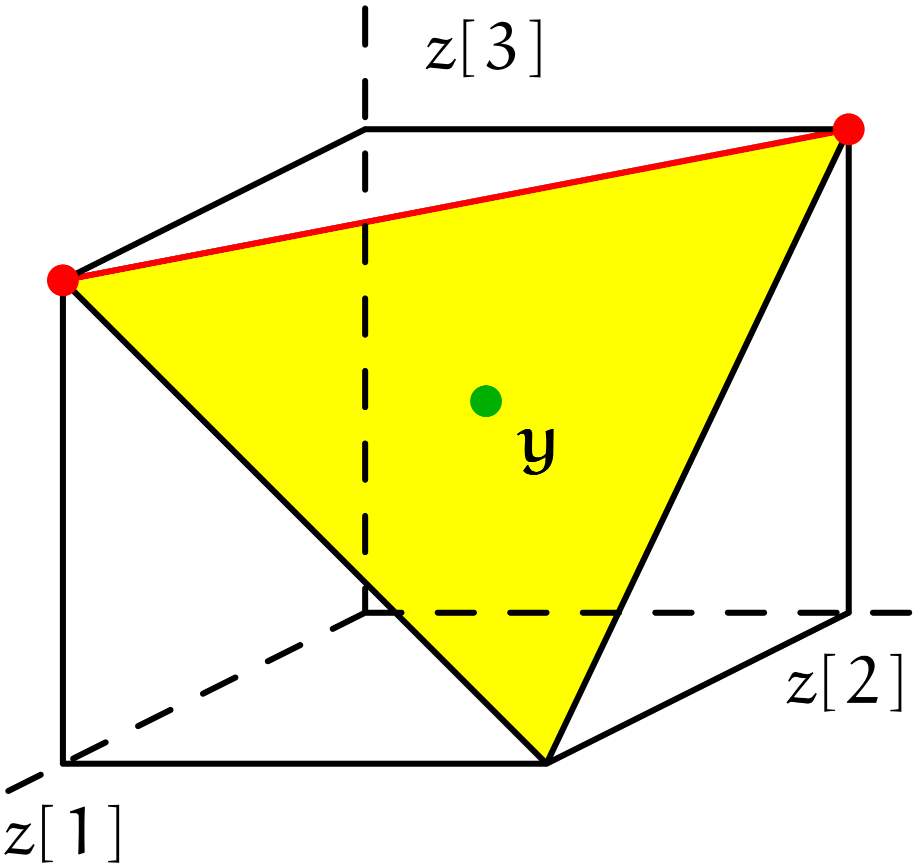
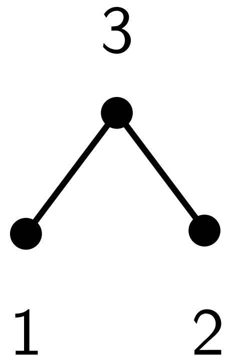
Remark:

More generally, a **stable set** can be defined.

Example: stable core



$\mathfrak{y} = (2/3, 2/3, 2/3)$



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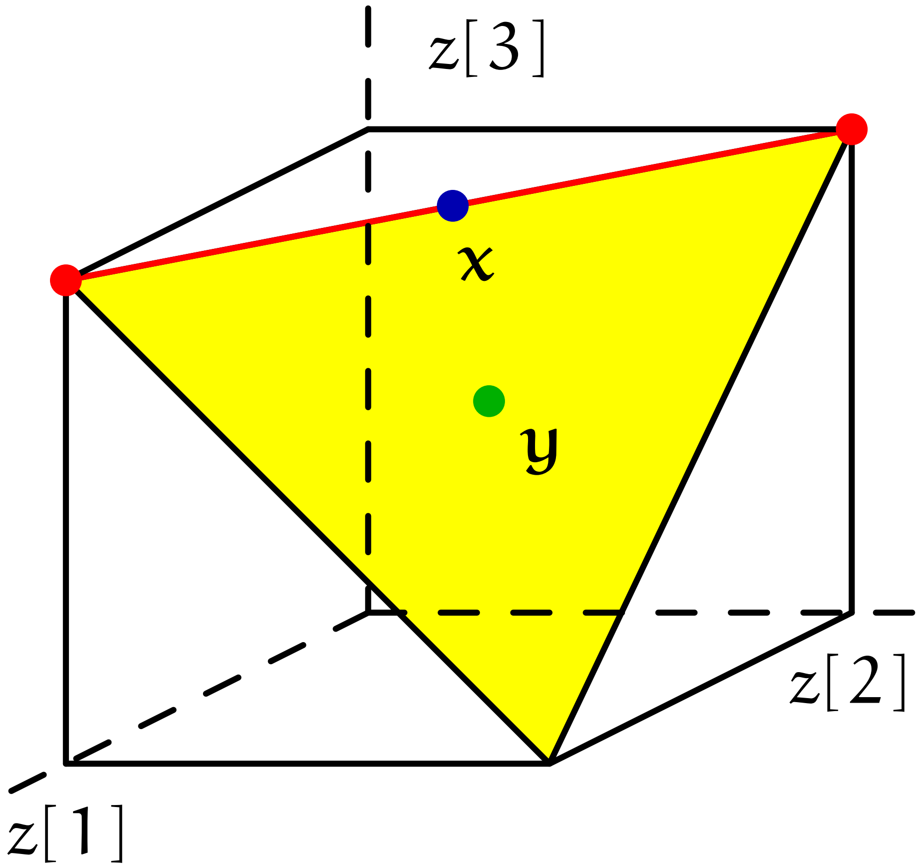

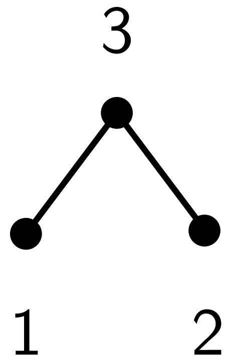

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$y = (2/3, 2/3, 2/3) \rightsquigarrow x = (1/2, 1/2, 0)$ and $S = \{1, 2\}$

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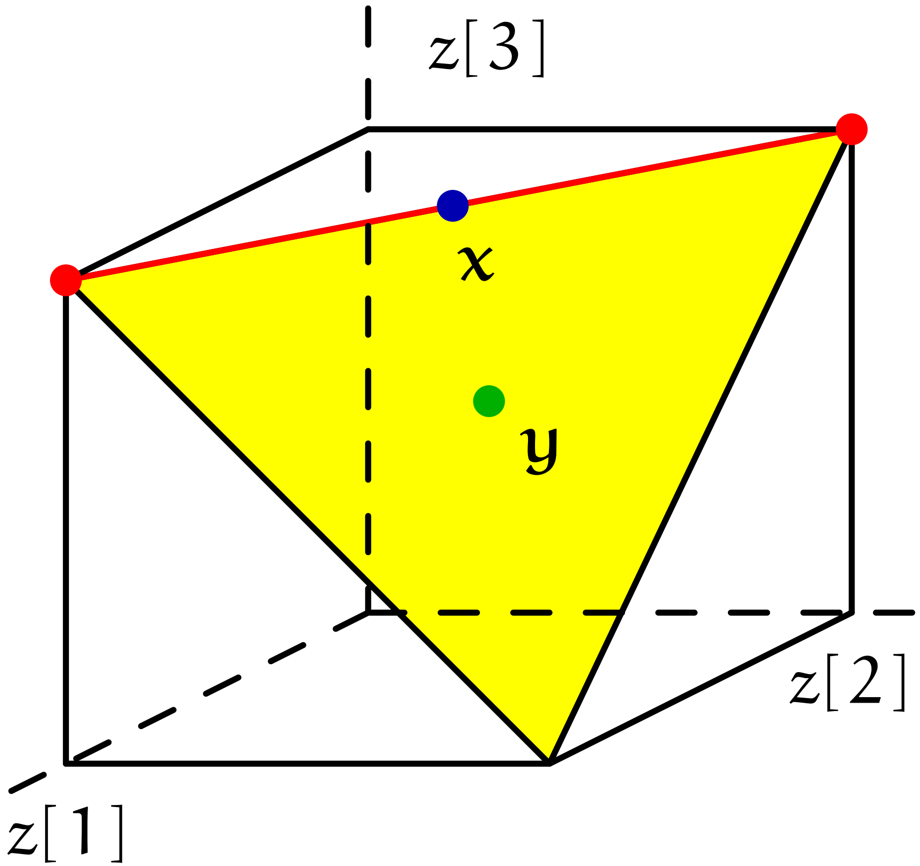

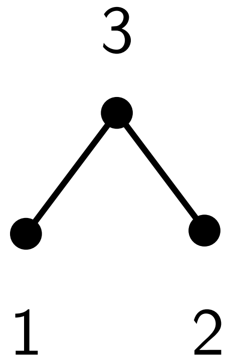

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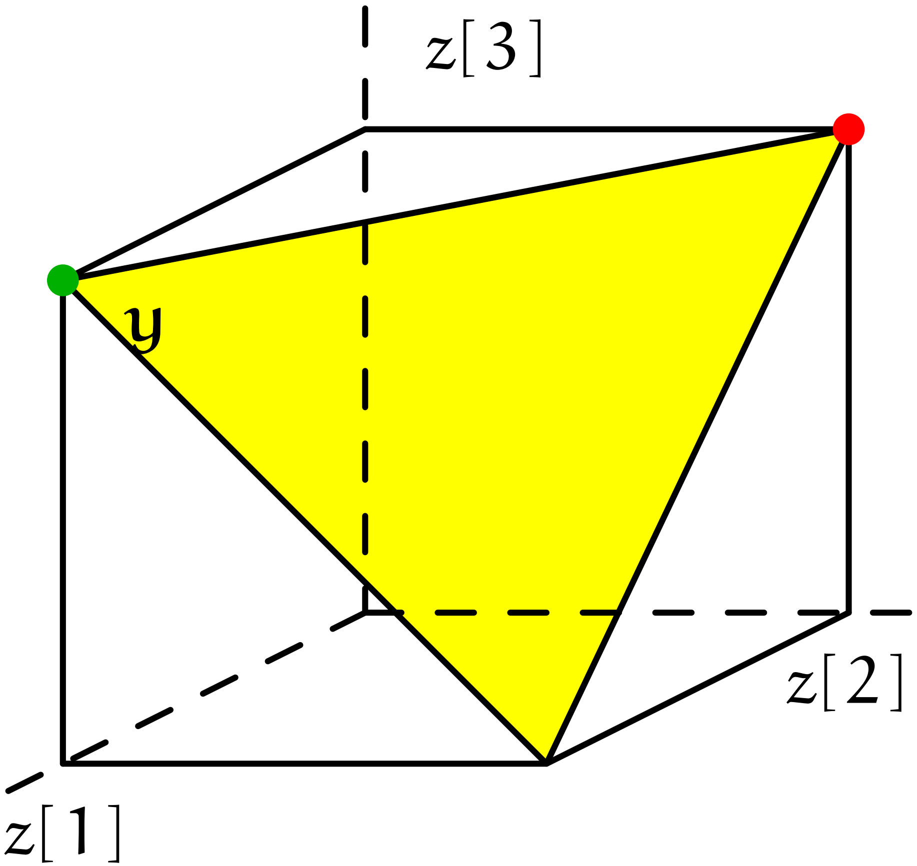
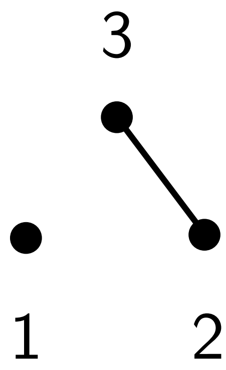
$y = (2/3, 2/3, 2/3) \rightsquigarrow x = (1/2, 1/2, 0)$ and $S = \{1, 2\}$

- $x[1] + [2] = 1 = \chi_G(\{1,2\})$,
- $x[1] < y[1]$ and $x[2] < y[2]$.

Example: unstable core



$y = (1, 0, 1)$



$V = \{1, 2, 3\}$

$\chi_G(\emptyset)$	0
$\chi_G(\{1\})$	1
$\chi_G(\{2\})$	1
$\chi_G(\{3\})$	1
$\chi_G(\{1,2\})$	1
$\chi_G(\{1,3\})$	1
$\chi_G(\{2,3\})$	2
$\chi_G(\{1,2,3\})$	2

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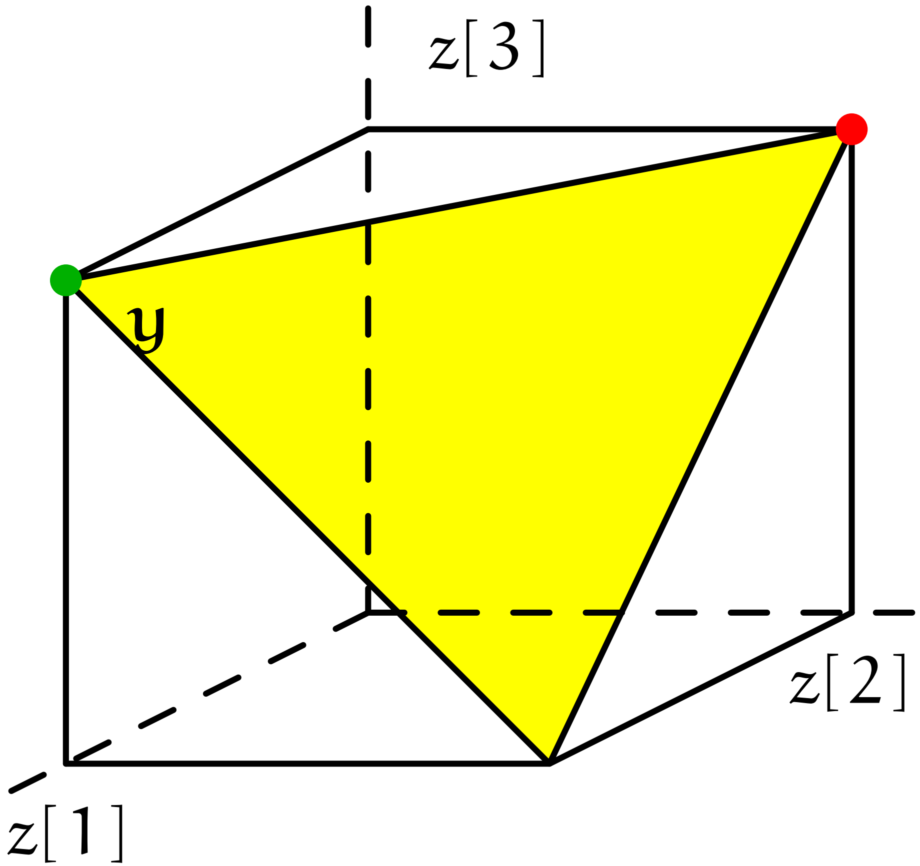

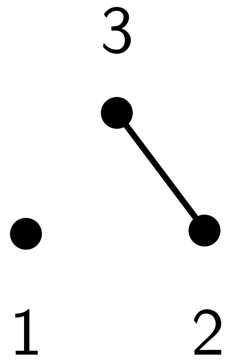

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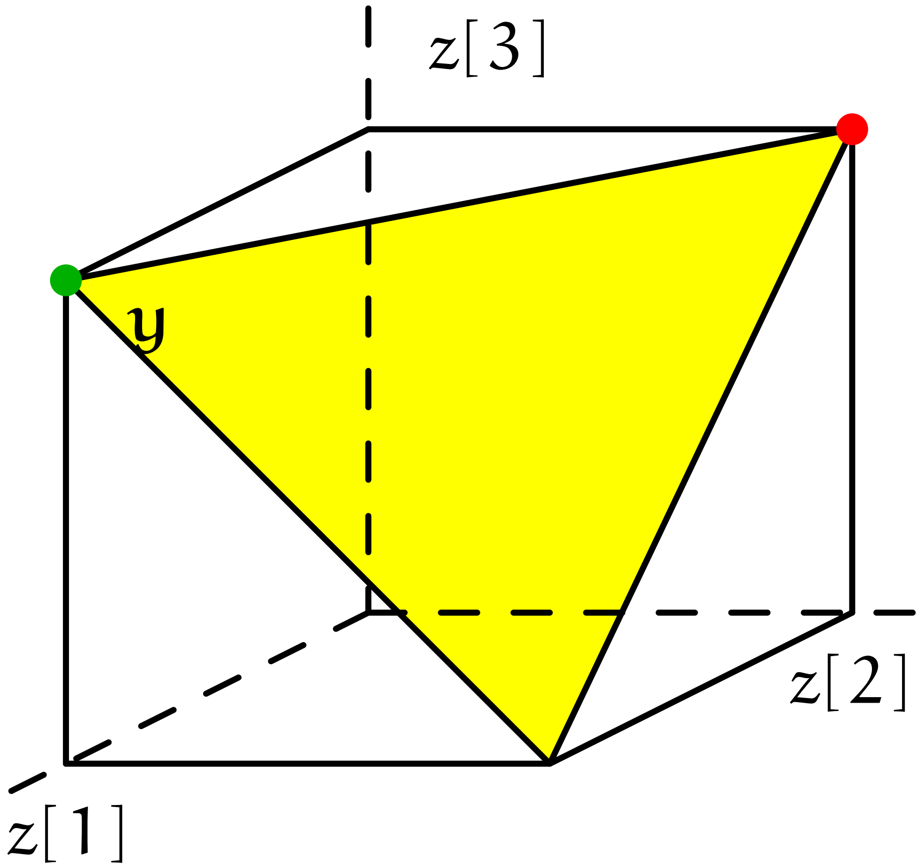
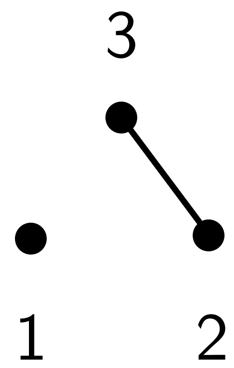
$\chi_G(\{2,3\})$	2
$\chi_G(\{1,2,3\})$	2

$$y = (1, 0, 1) \rightsquigarrow x = (0, 1, 1).$$

Example: unstable core





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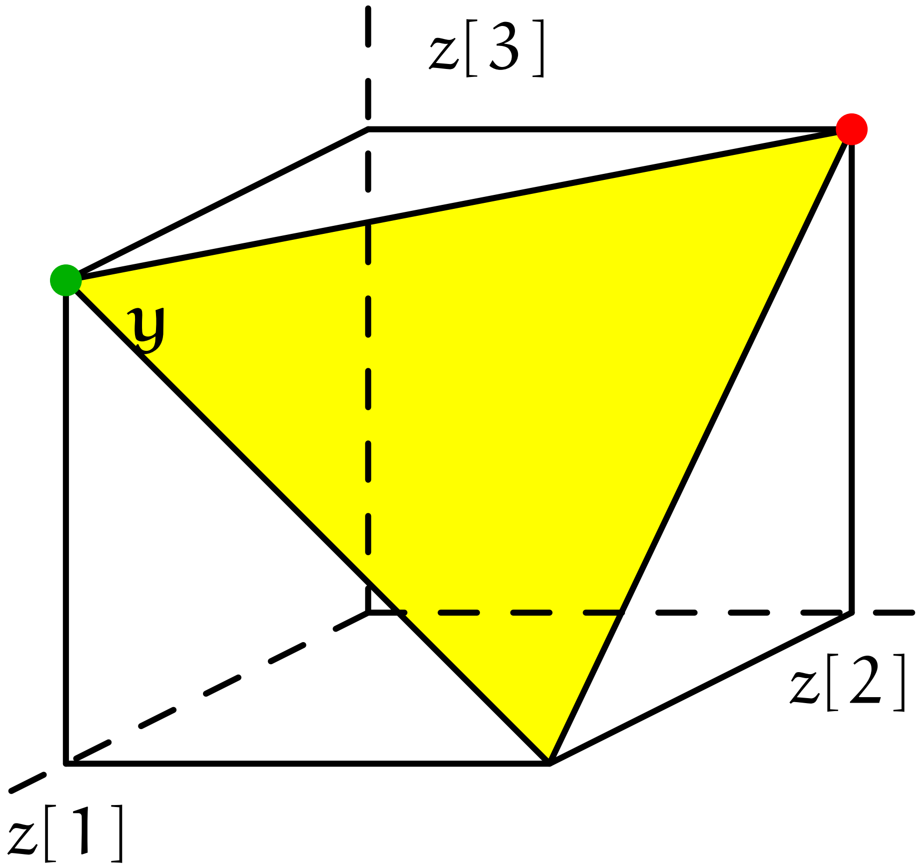
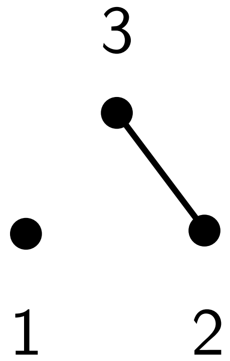
$\chi_G(\{2,3\})$	2
$\chi_G(\{1,2,3\})$	2

$\mathbf{y} = (1, 0, 1) \rightsquigarrow \mathbf{x} = (0, 1, 1)$. Need to find S s.t.

- $\sum\{x[i] : i \in S\} = \chi_G(S)$,
- $x[i] < y[i] \ \forall\, i \in S$.

Example: unstable core





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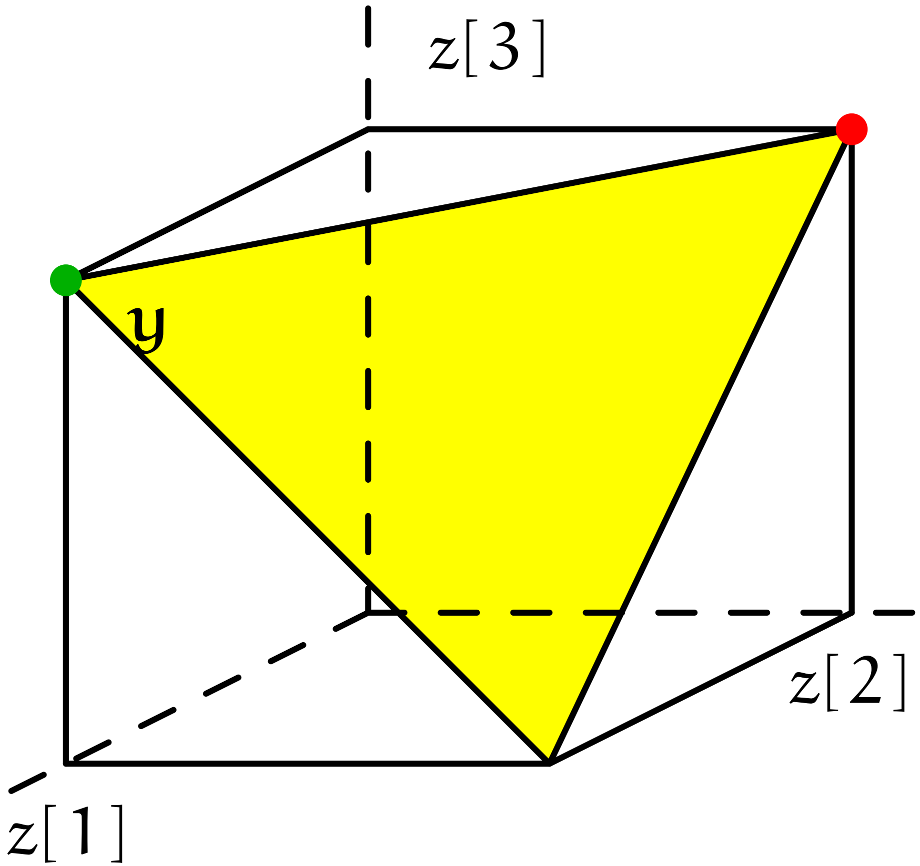
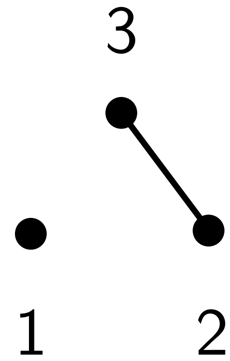
$y = (1, 0, 1) \rightsquigarrow x = (0, 1, 1)$. Need to find S s.t.

- $\sum\{x[i] : i \in S\} = \chi_G(S)$,
- $x[i] < y[i]\ \forall\, i \in S$. $\rightsquigarrow S = \{1\}$

Example: unstable core





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$\chi_G(\{2,3\})$	2
$\chi_G(\{1,2,3\})$	2

$\mathbf{y} = (1, 0, 1) \rightsquigarrow \mathbf{x} = (0, 1, 1)$. Need to find S s.t.

- $\sum\{x[i] : i \in S\} = \chi_G(S)$, $\rightsquigarrow x[1] = 0 \neq 1 = \chi_G(\{1\})$.
- $x[i] < y[i]\ \forall\, i \in S$. $\rightsquigarrow S = \{1\}$

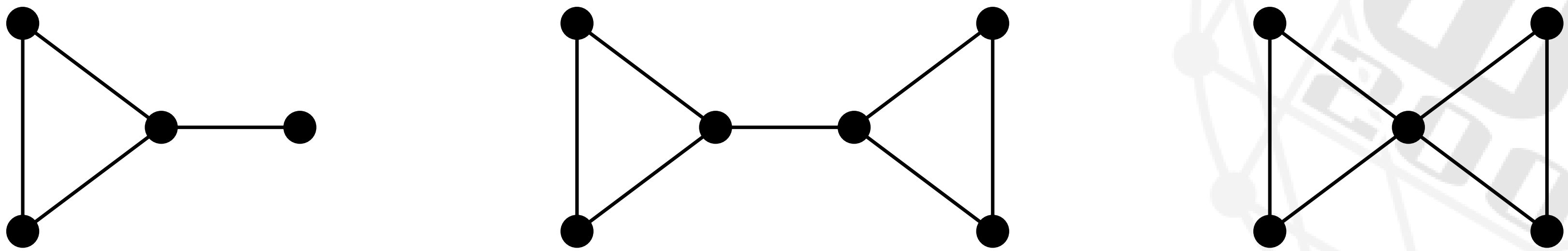
Result (1), again

Thm **For a perfect graph G,**

(1) The following are equivalent.

- The minimum coloring game on G has a stable core.
- Every vertex of G belongs to a maximum clique.

This condition can be checked in polynomial time.



Vielen Dank!

Thank you very much

Here are some extra slides which could be used for answering questions from the audience.

Polynomial-time algorithm for Result (1)

$\boxed{\text{Thm}}$ (Grötschel, Lovász & Schrijver '81)

A maximum weight clique of a perfect graph can be found in polynomial time.

Polynomial-time algorithm for Result (1)

Thm (Grötschel, Lovász & Schrijver '81)

A maximum weight clique of a perfect graph can be found in polynomial time.

Our Algorithm using the thm above

(1) For each vertex $v \in V$

- define a weight vector $\boldsymbol{w}^{(v)}$ as

$$w^{(v)}[u] = \begin{cases} \text{"large"} & u = v \\ \text{"small"} & u \neq v; \end{cases}$$

- Compute a maximum weight clique w.r.t. $\boldsymbol{w}^{(v)}$;

(2) If all of them are maximum-size cliques, return "YES;" otherwise return "NO."

Subgame

$\boxed{\text{Def.:}}$ For a game (N, γ) and $T \subseteq N$, define another game $(T, \gamma^{(T)})$ as

$$\gamma^{(T)}(S) = \gamma(S) \qquad \text{for all } S \subseteq T.$$

$(T, \gamma^{(T)})$ is called a **subgame**.

Extendability (Kikuta & Shapley '86)

Def.: A game (N, γ) is **extendable** if

$$\forall\, T \subseteq N \quad (T \neq \emptyset)$$
$$\forall\, \mathbf{y} \in \mathrm{Core}(T, \gamma^{(T)})$$
$$\exists\, \mathbf{x} \in \mathrm{Core}(N, \gamma) \text{ such that}$$
$$x_i = y_i \text{ for all } i \in T.$$

Interpretation:

Every core allocation of any subgame can be "extended" to a core allocation of the original game.

Large core (Sharkey '82)

$\boxed{\text{Def.:}}$ The core of (N, γ) is **large** if

$\forall\ \boldsymbol{y} \in \mathbb{R}^N$ such that

$$\sum\{y_i : i \in S\} \leq \gamma(S) \text{ for all } S \subseteq N$$

$\exists\ \boldsymbol{x} \in$ Core such that

$$\boldsymbol{y} \leq \boldsymbol{x}.$$

Exactness (Schmeidler '72)

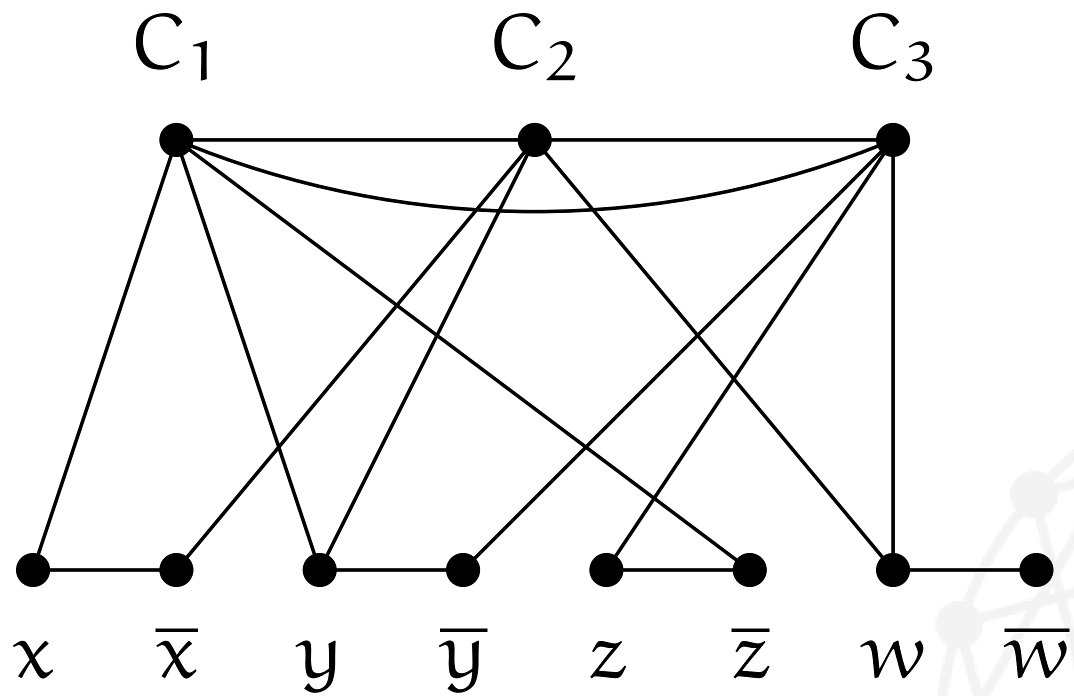
Def.: A game (N, γ) is **exact** if

$\forall\, S \subseteq N$
$\exists\, \boldsymbol{x} \in$ Core such that

$$\sum\{x_i : i \in S\} = \gamma(S).$$

Use the satisfiability problem (Zverovich '03)

Example: $\phi = (x \vee y \vee \bar{z}) \wedge (\bar{x} \vee y \vee w) \wedge (\bar{y} \vee z \vee w)$.



- $\omega(\overline{G}) = n + 1$. ($n := \#$ of var's in ϕ.)
- $\exists$ a maximal clique of size n in $\overline{G} \Leftrightarrow \phi$ satisfiable.