The traveling salesman problem with few inner points

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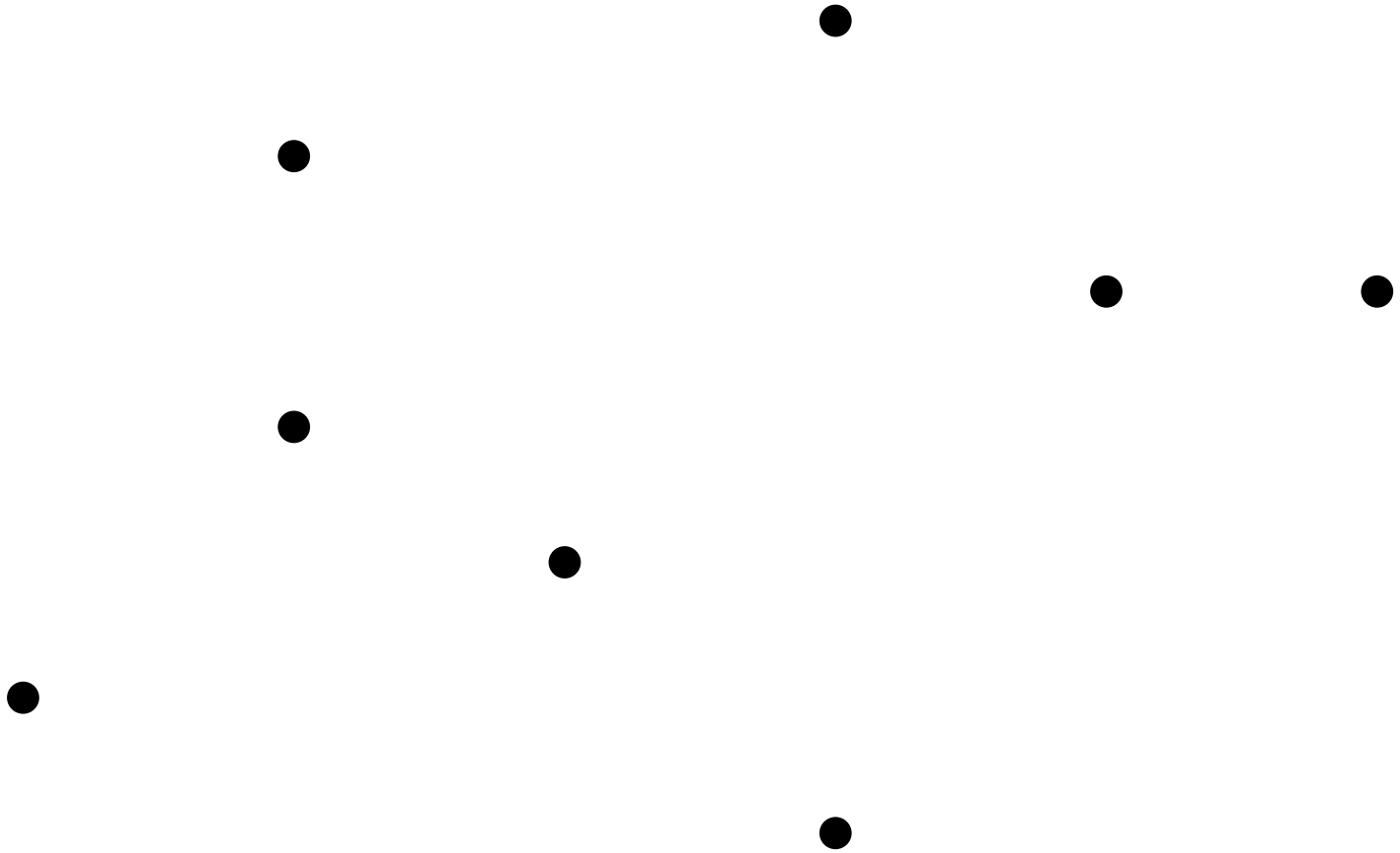
* Supported by the Berlin-Zürich Joint Graduate Program

CGC
Combinatorics
Geometry
Computation

Given: finite set of points on $\mathbb{R}^2$
Find: a minimum-length tour



The 2DTSP

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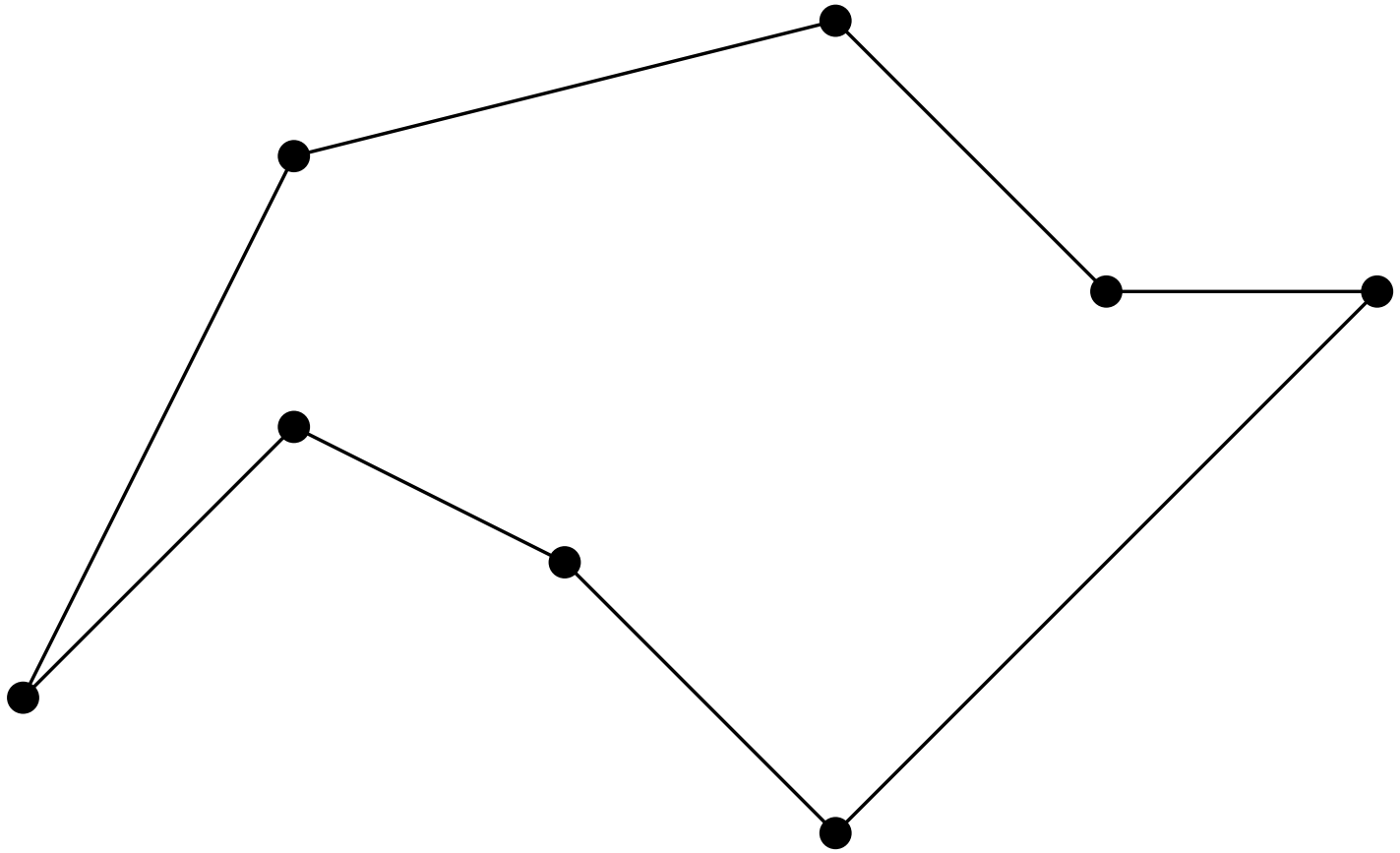
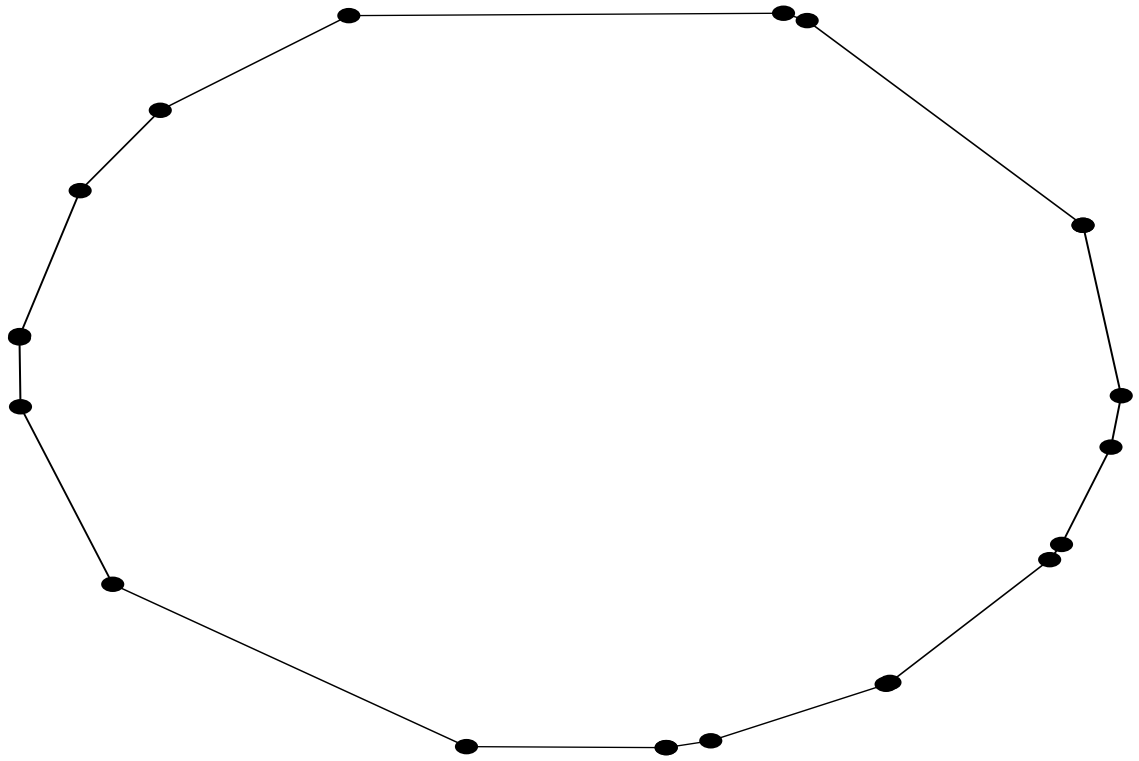


Known facts about 2DTSP

- In general, it is NP-hard.

  (Garey, Graham & Johnson '76
  Papadimitriou '77)

- When the points are in convex position, the problem is easy.



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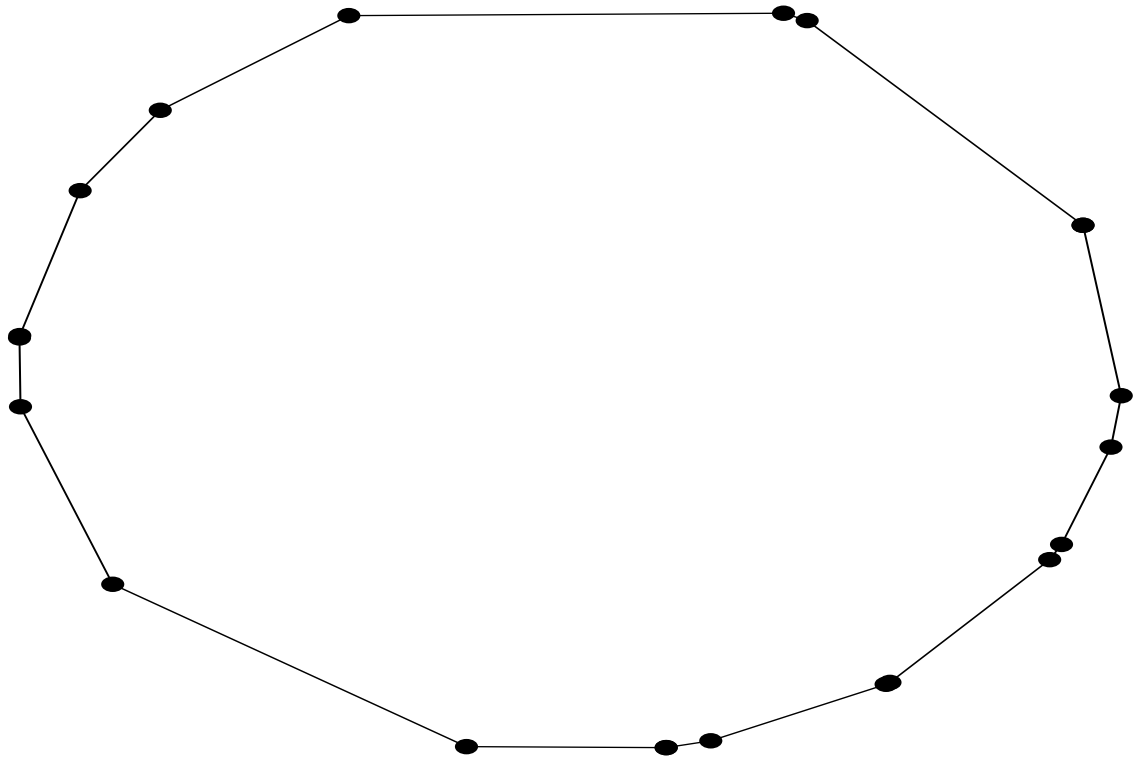
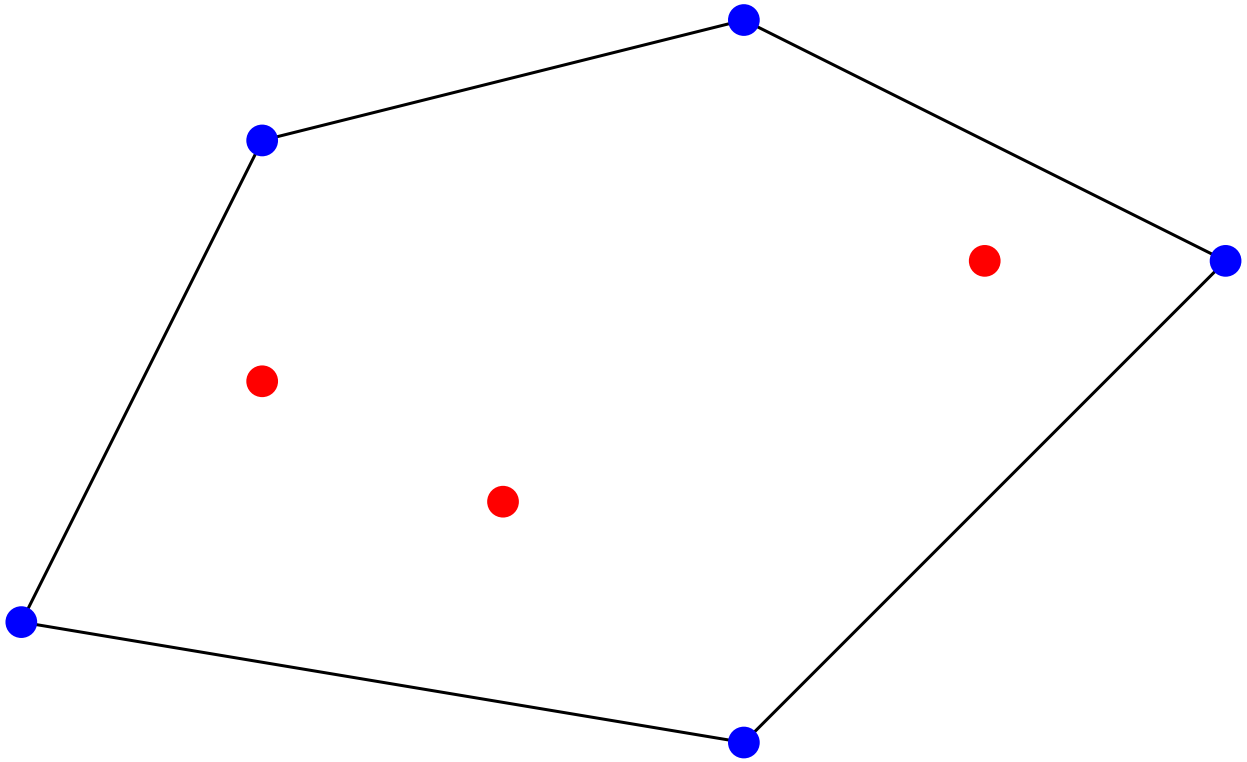


Observation

The **inner points** make the problem difficult.



Observation

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Motivation

How many inner points can we have in order to obtain a polynomial-time algorithm?

Result

We give two simple algorithms.

$n :=$ the total number of points
$k :=$ the number of inner points

- First algorithm runs in polynomial time when $k = O(1)$.
- Second algorithm runs in polynomial time when $k = O(\log n / \log \log n)$.

Open problem: Improve the bound!

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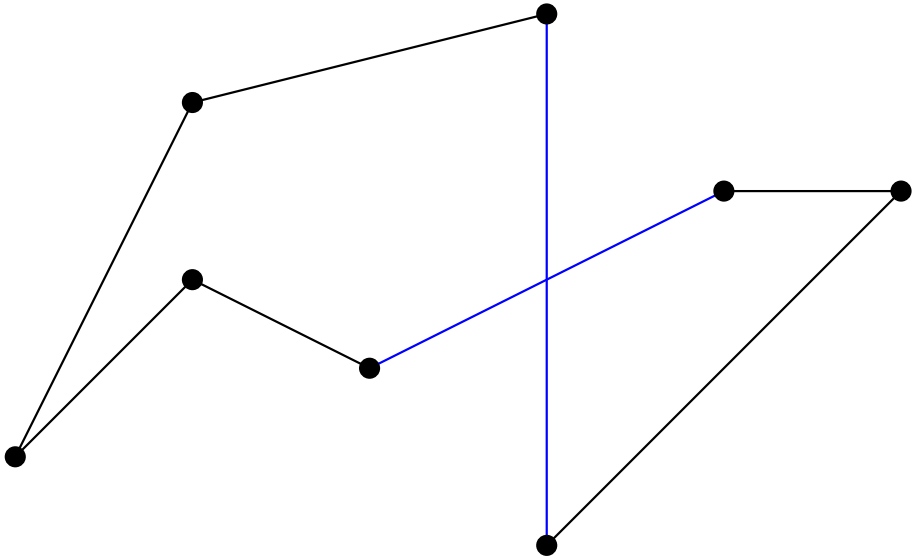
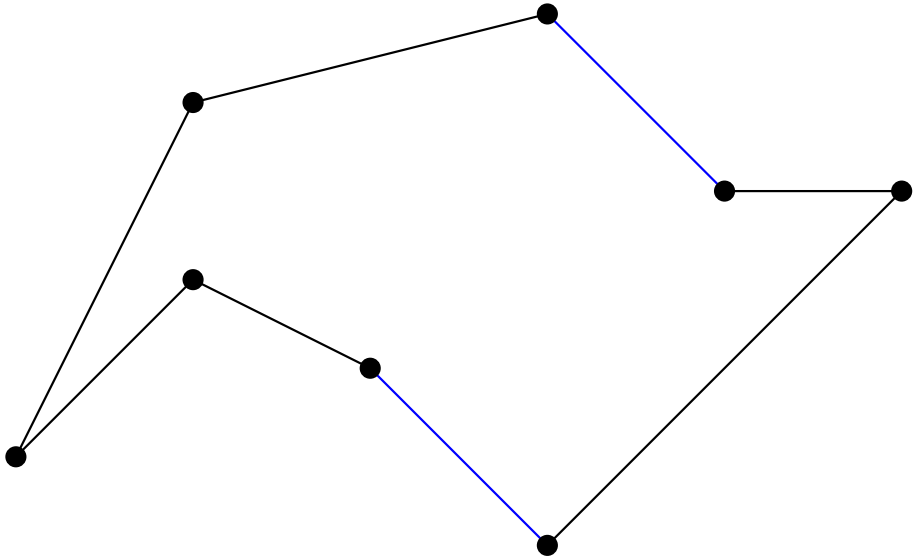
A useful fact

Fact (Flood '56)

An optimal tour has no self-crossing.

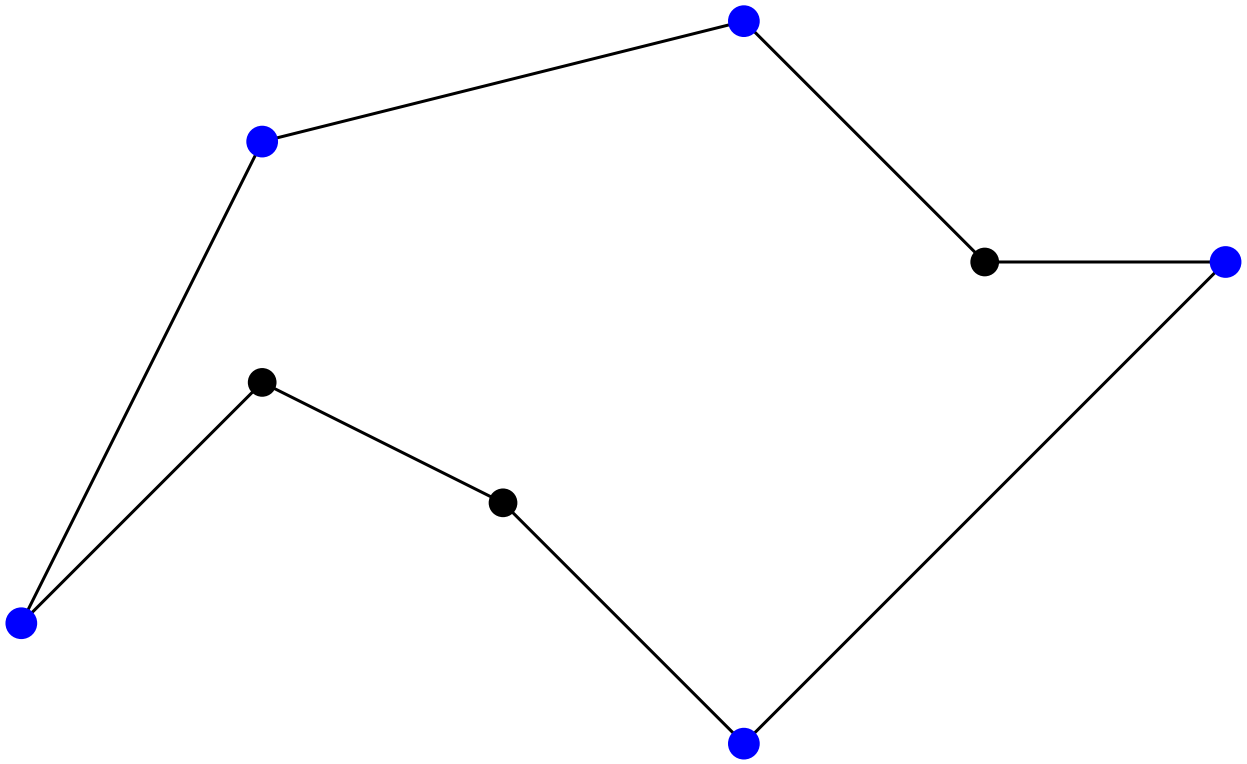
Proof





Corollary

An optimal tour visits the non-inner points in a cyclic order.



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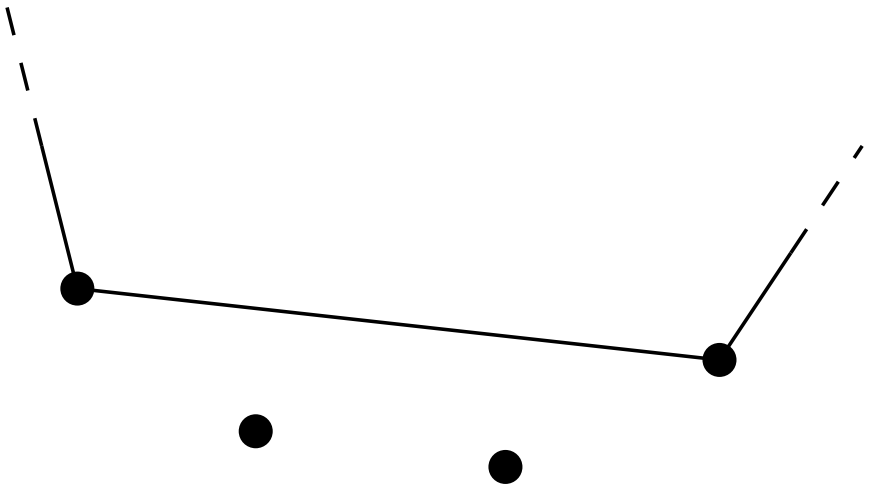


Suppose not.
Then $\exists$ a "skip."
Skipped points must be visited later,
which causes a self-crossing.
A contradiction.

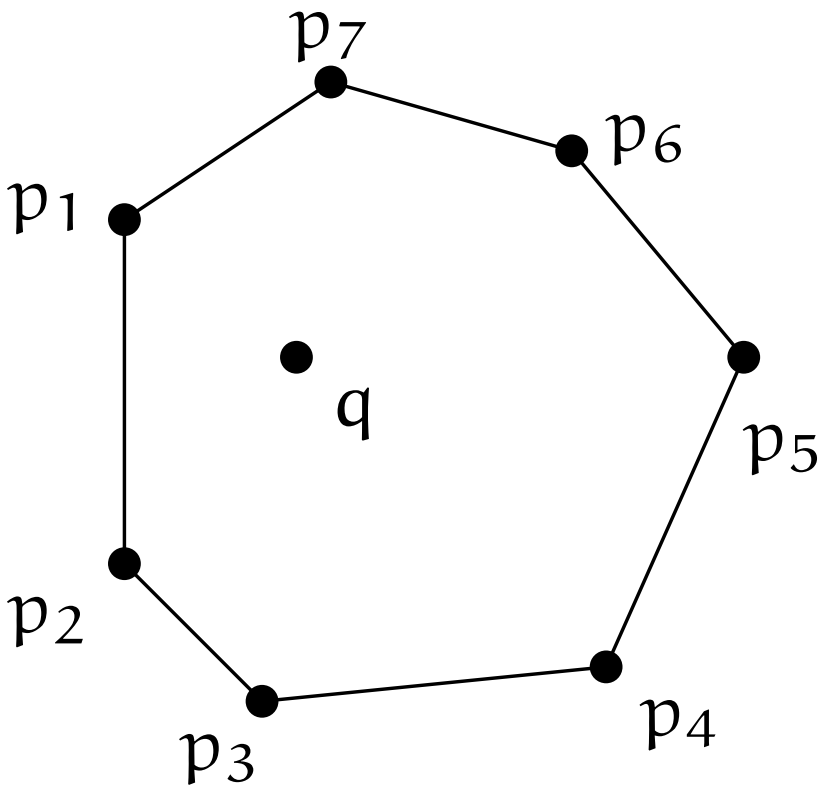
One inner point

Consider the case $k = 1$. ($k :=$ # of inner pts)

Inner point: q

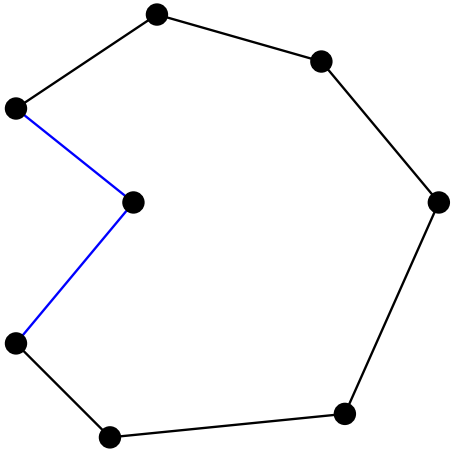
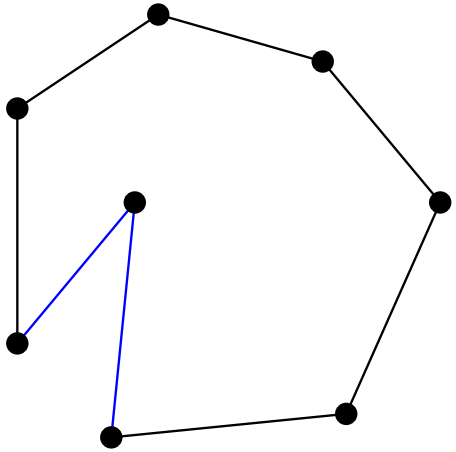
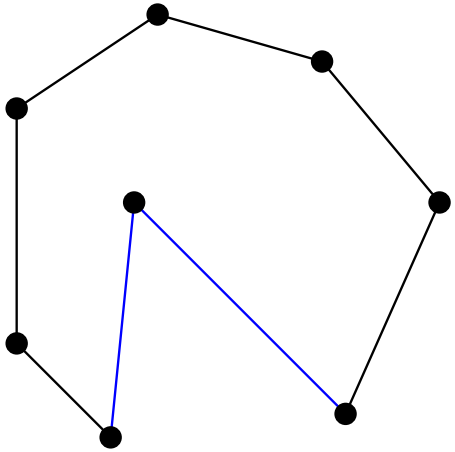
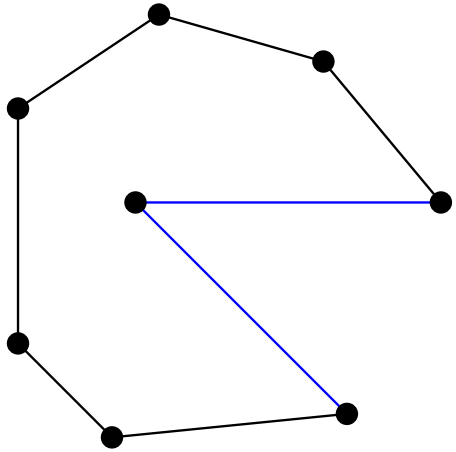
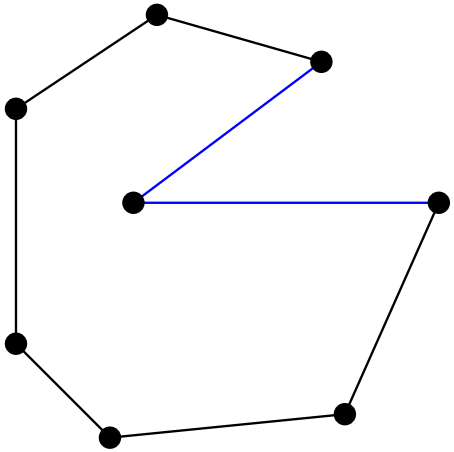
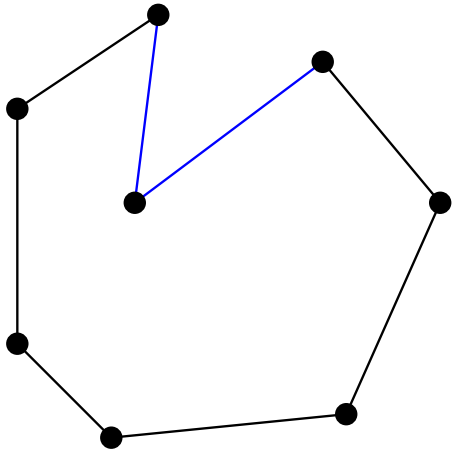
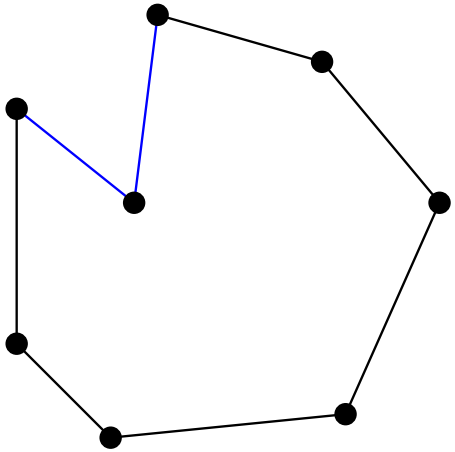
Non-inner points: $p_1, p_2, \ldots, p_{n-1}$

labeled according to a cyclic order



One inner point

\# of tours which "respect" the cycl. order = $n-1$.

Choose the best one.

First algorithm

(1) Distinguish the inner points and the non-inner points;

(2) Fix a cyclic order on the non-inner points;

(3) For each tour which respects the cyclic order

  (i) Compute the length of the tour;

(4) Choose the best one among them.

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Running time

There are k inner points.

- $\#$ of tours which "respect" the cycl. order $= O(k!n^k)$.
- They can be enumerated in $O(1)$ time per tour.
- The length of each tour can be computed in $O(n)$ time.

The running time $= \underbrace{O(n \log n)}_{\text{convex hull computation}} + O(k!n^{k+1})$.

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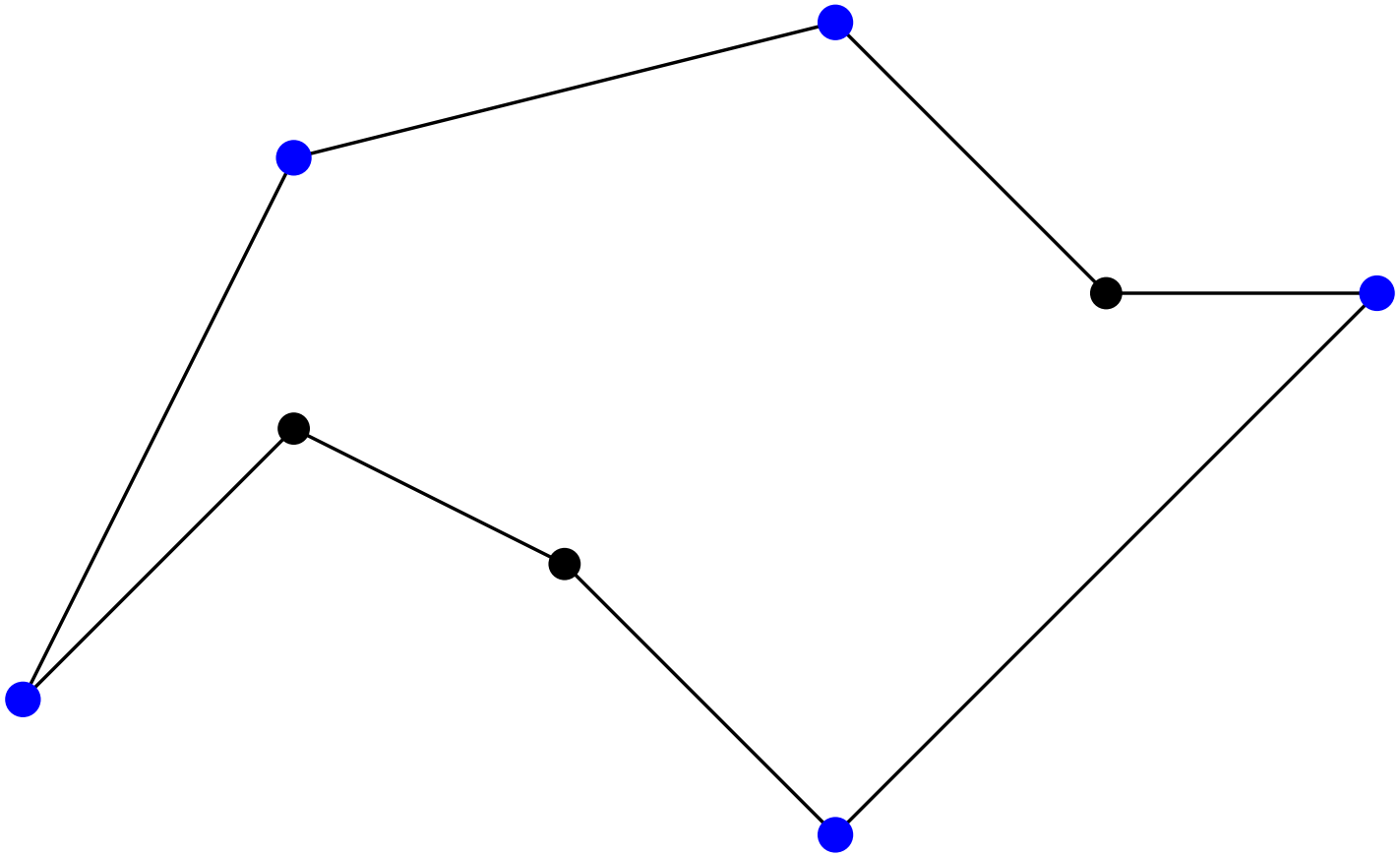
Idea for the second algorithm

Fact we already saw

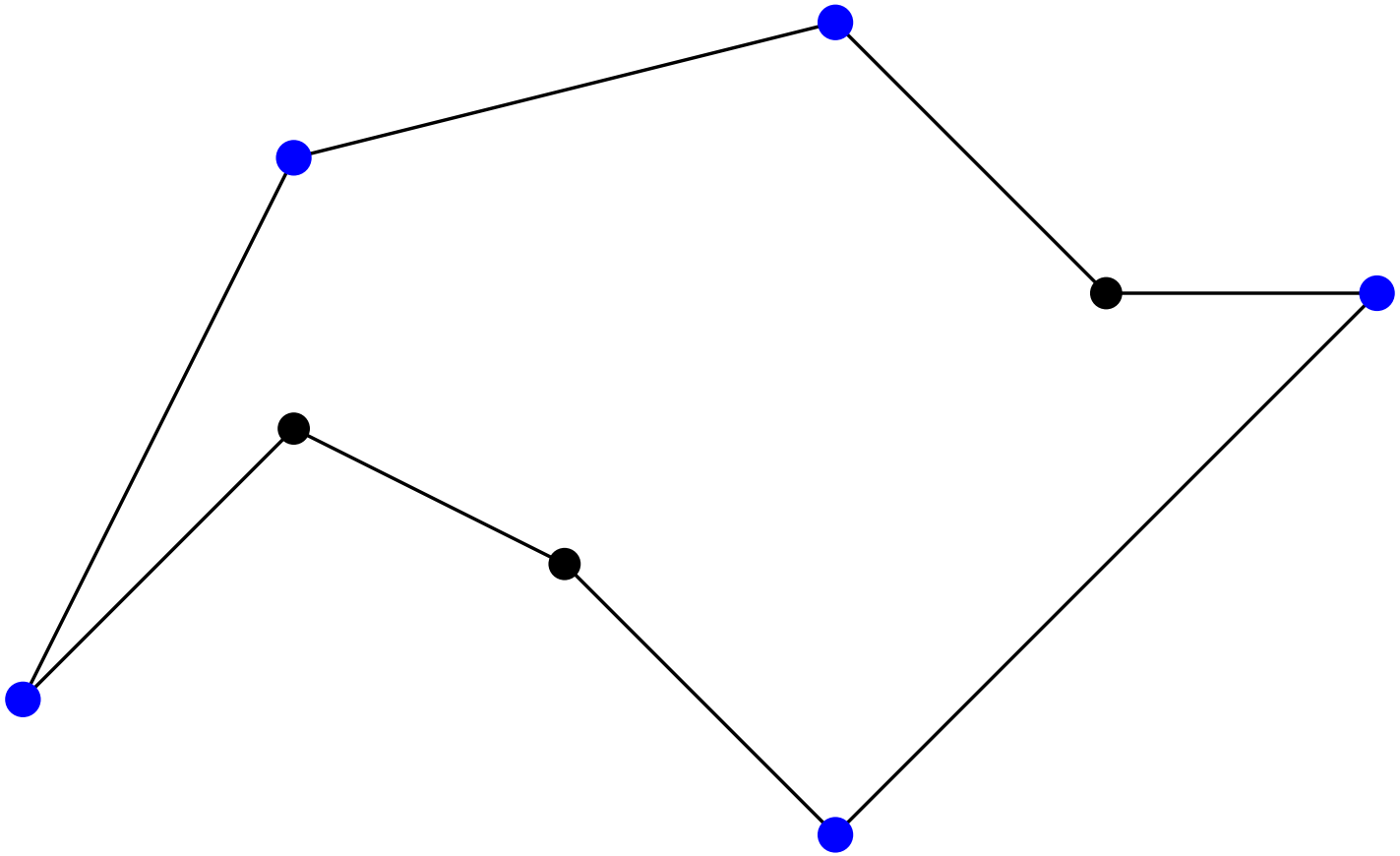
An optimal tour respects
a cyclic order on the non-inner points.



Idea for the second algorithm

Another fact

An optimal tour respects
some linear order on the inner points.



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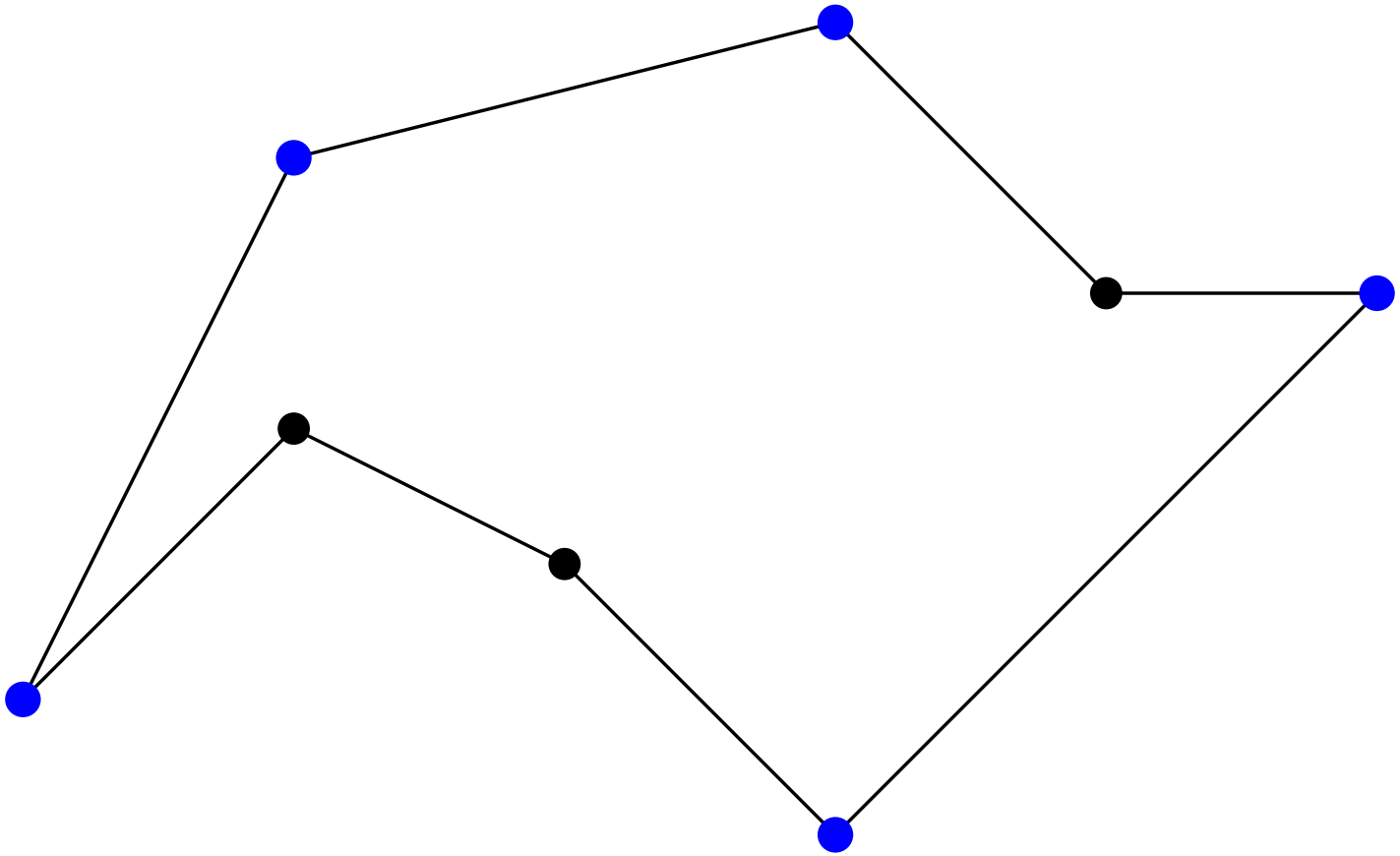
Idea

Try all linear orders on the inner points.



Outline of the second algorithm

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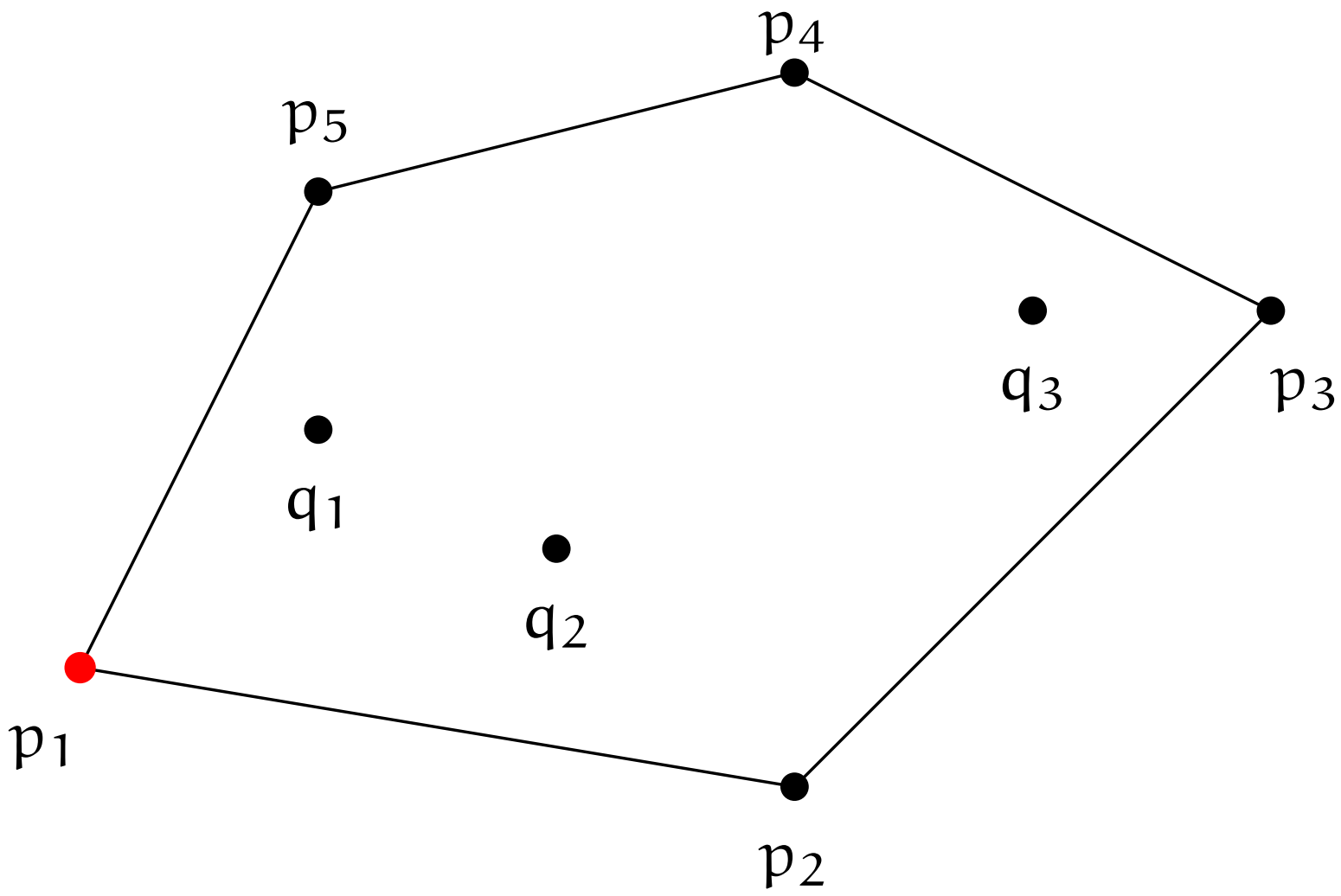

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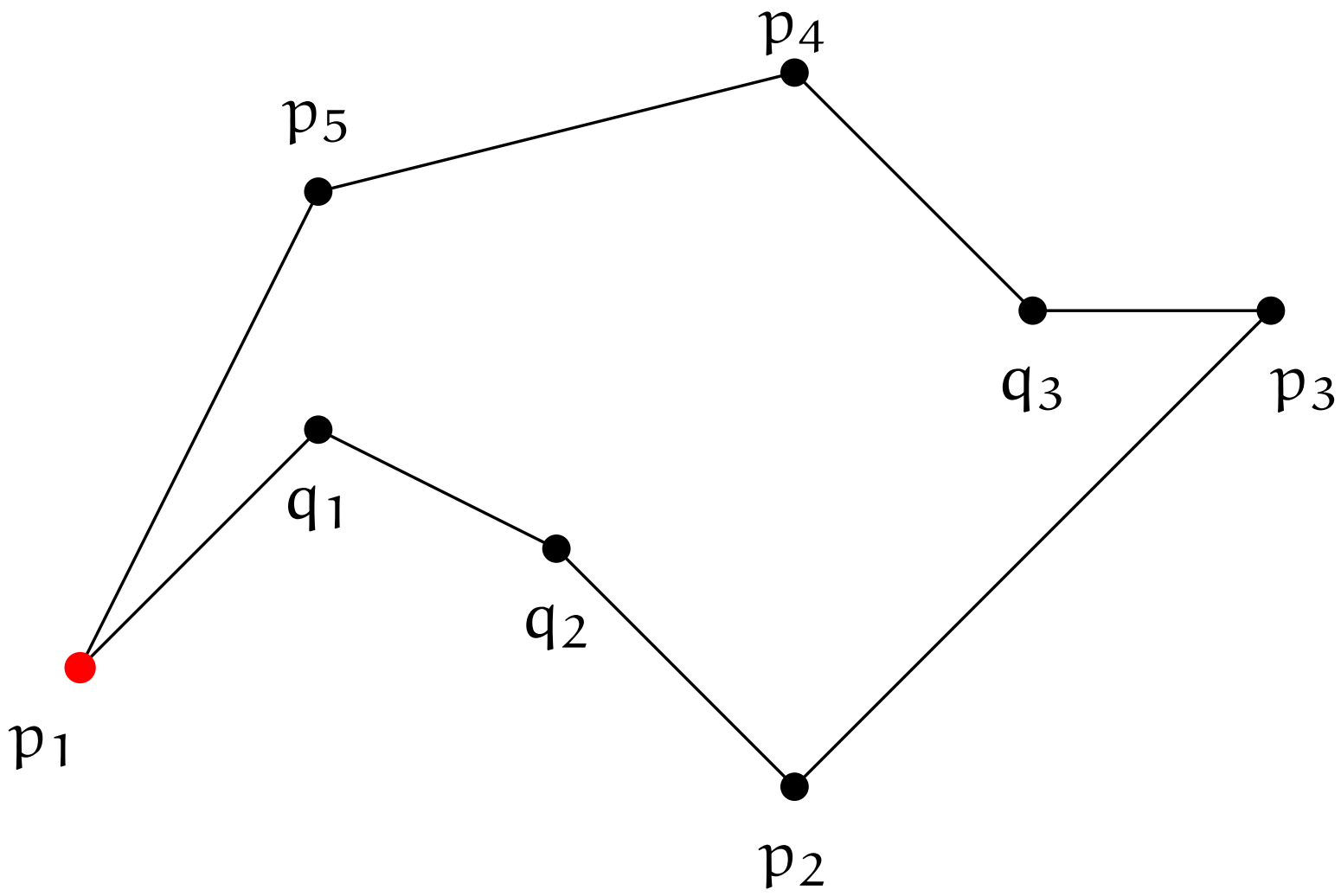

Optimal tour among those which respect the cyclic order and the order "1–2–3."

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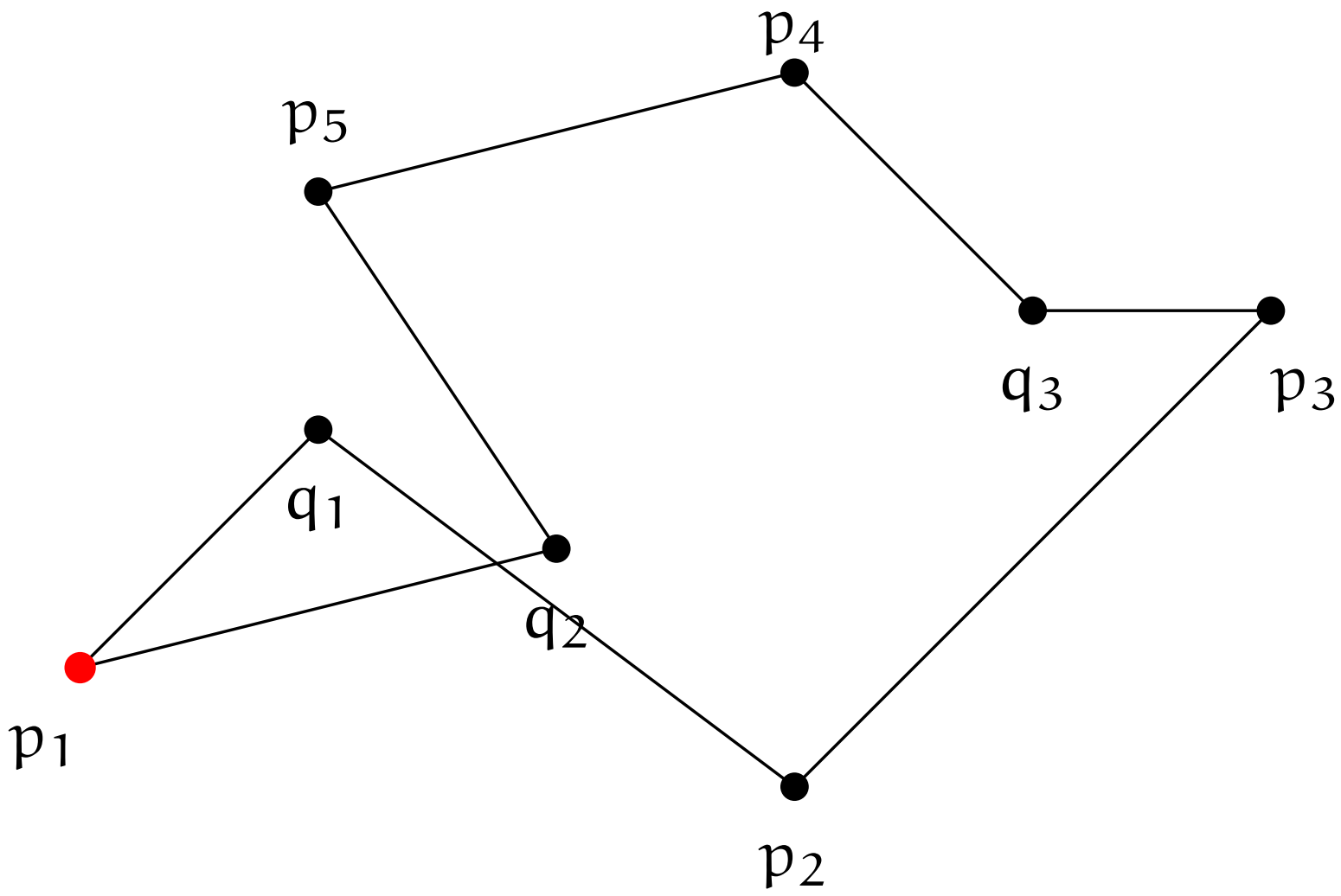

Optimal tour among those which respect the cyclic order and the order "1–3–2."

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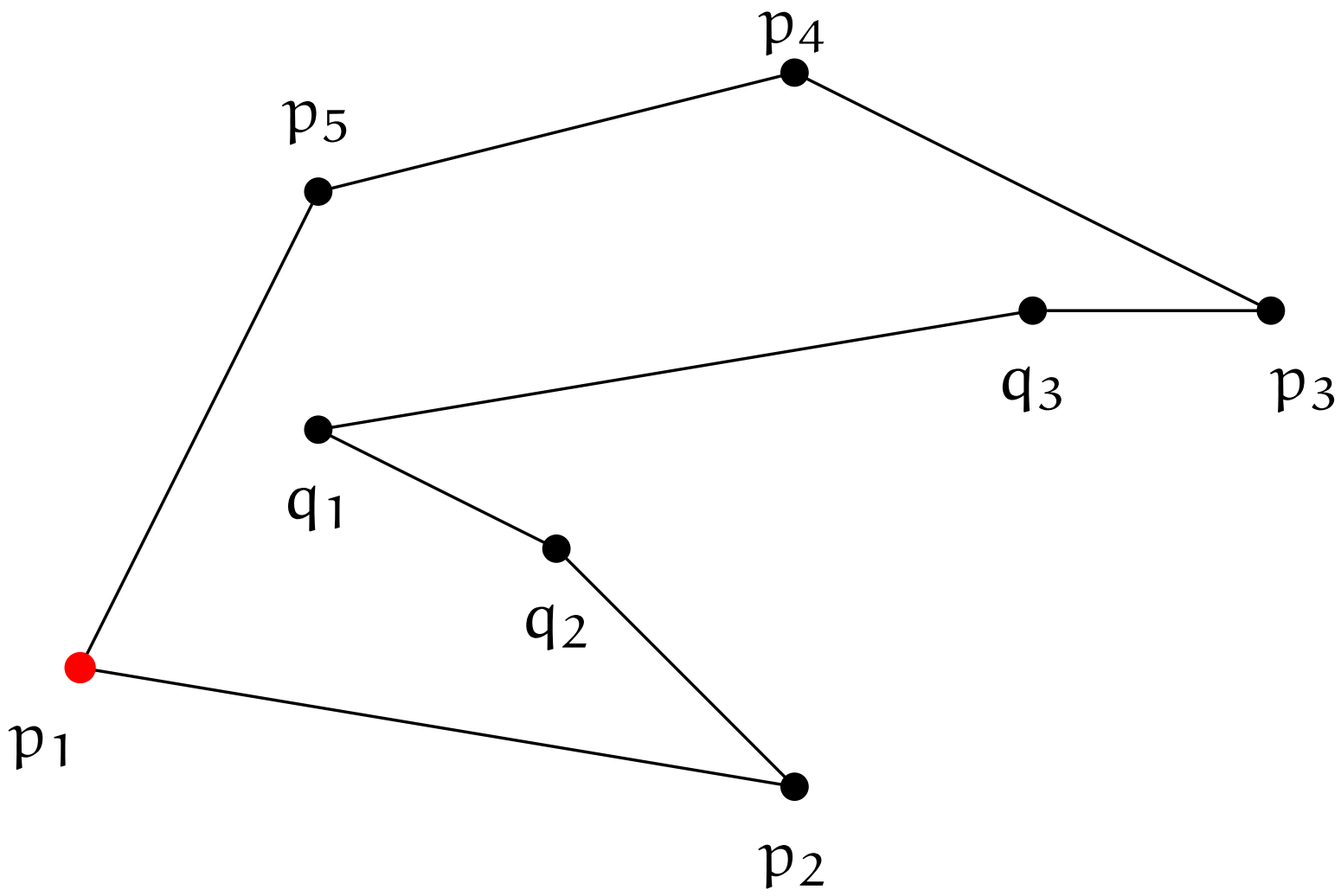
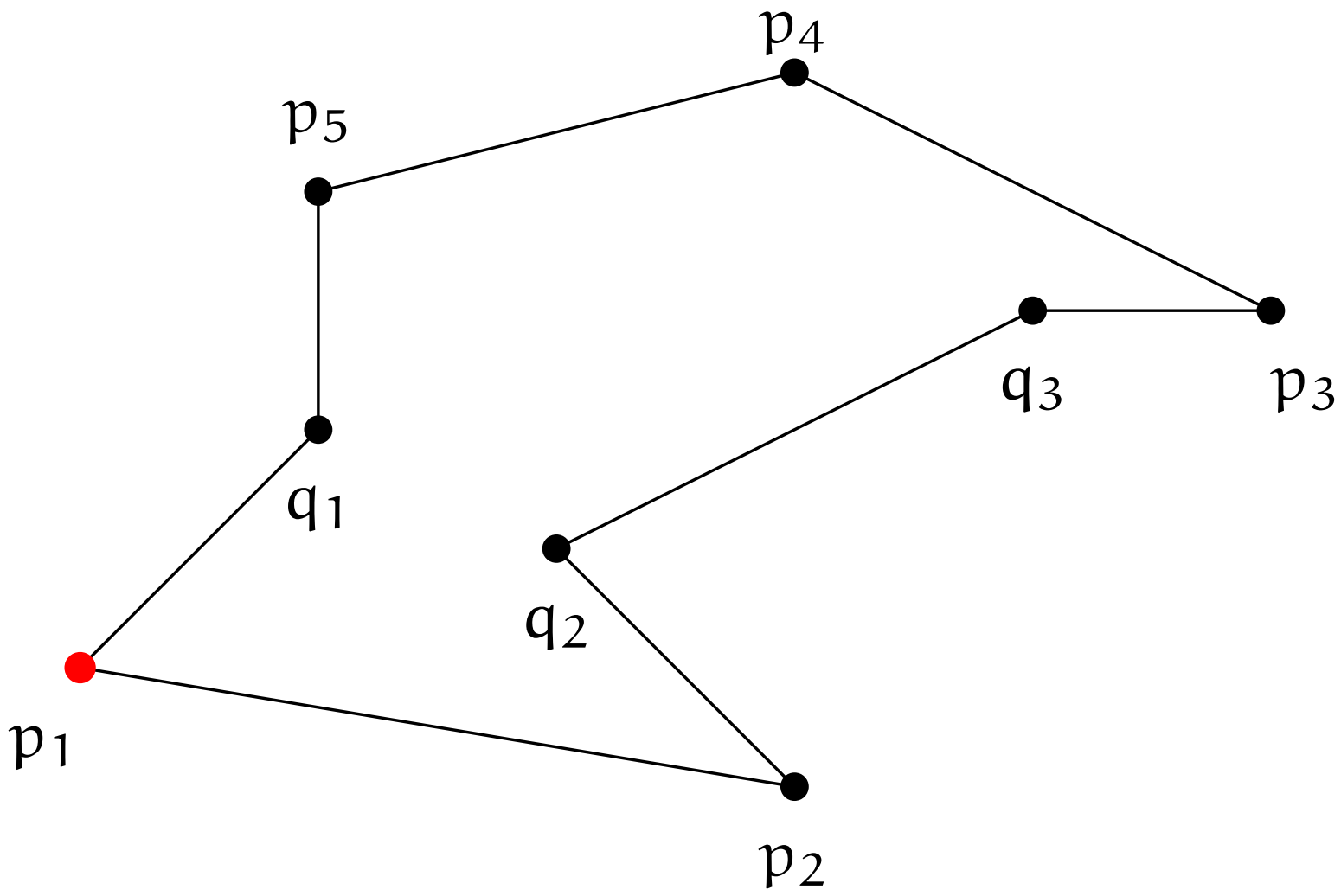


Optimal tour among those which respect the cyclic order and the order "2–1–3."

Example



Optimal tour among those which respect the cyclic order and the order "2–3–1."

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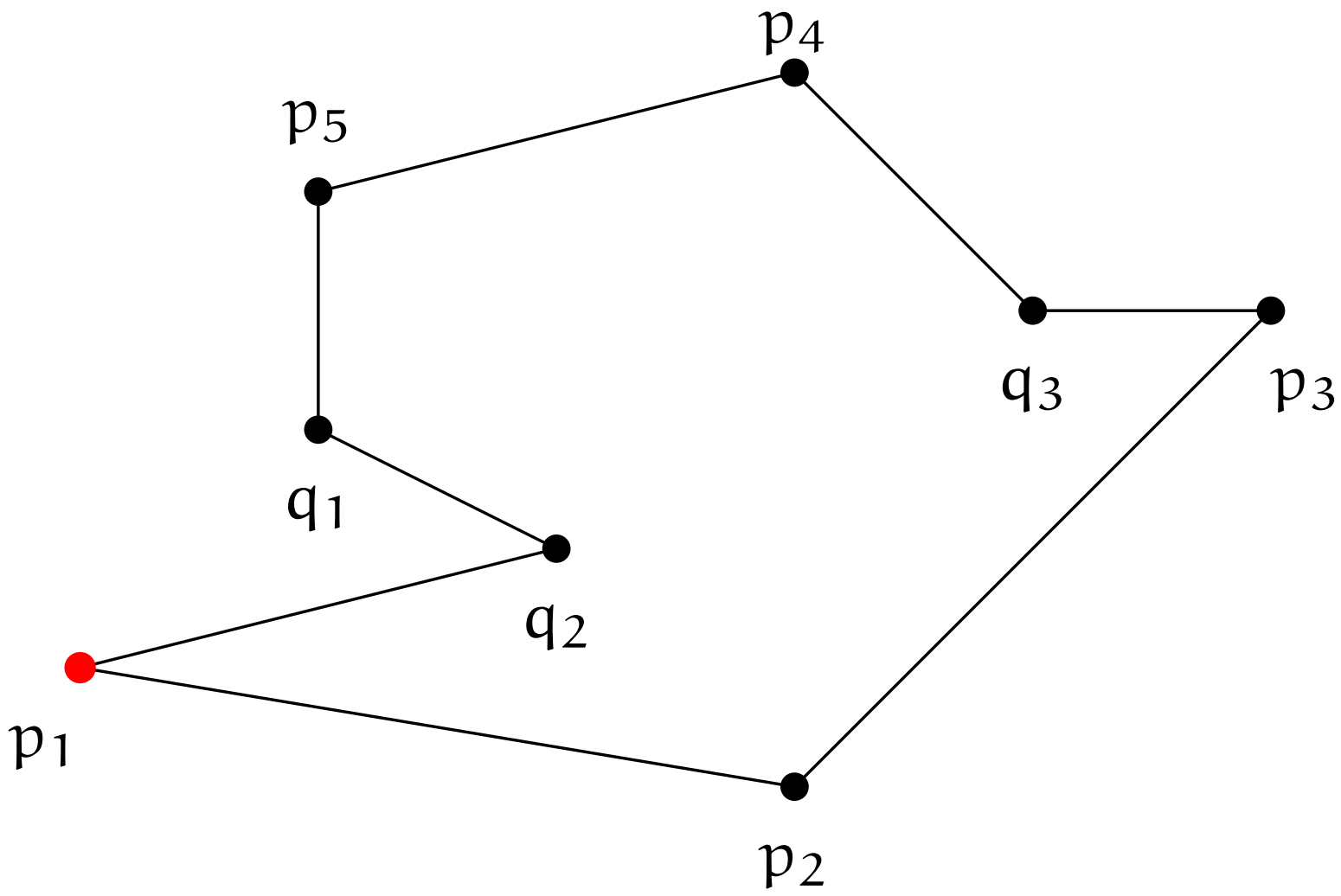

Optimal tour among those which respect the cyclic order and the order "3–1–2."

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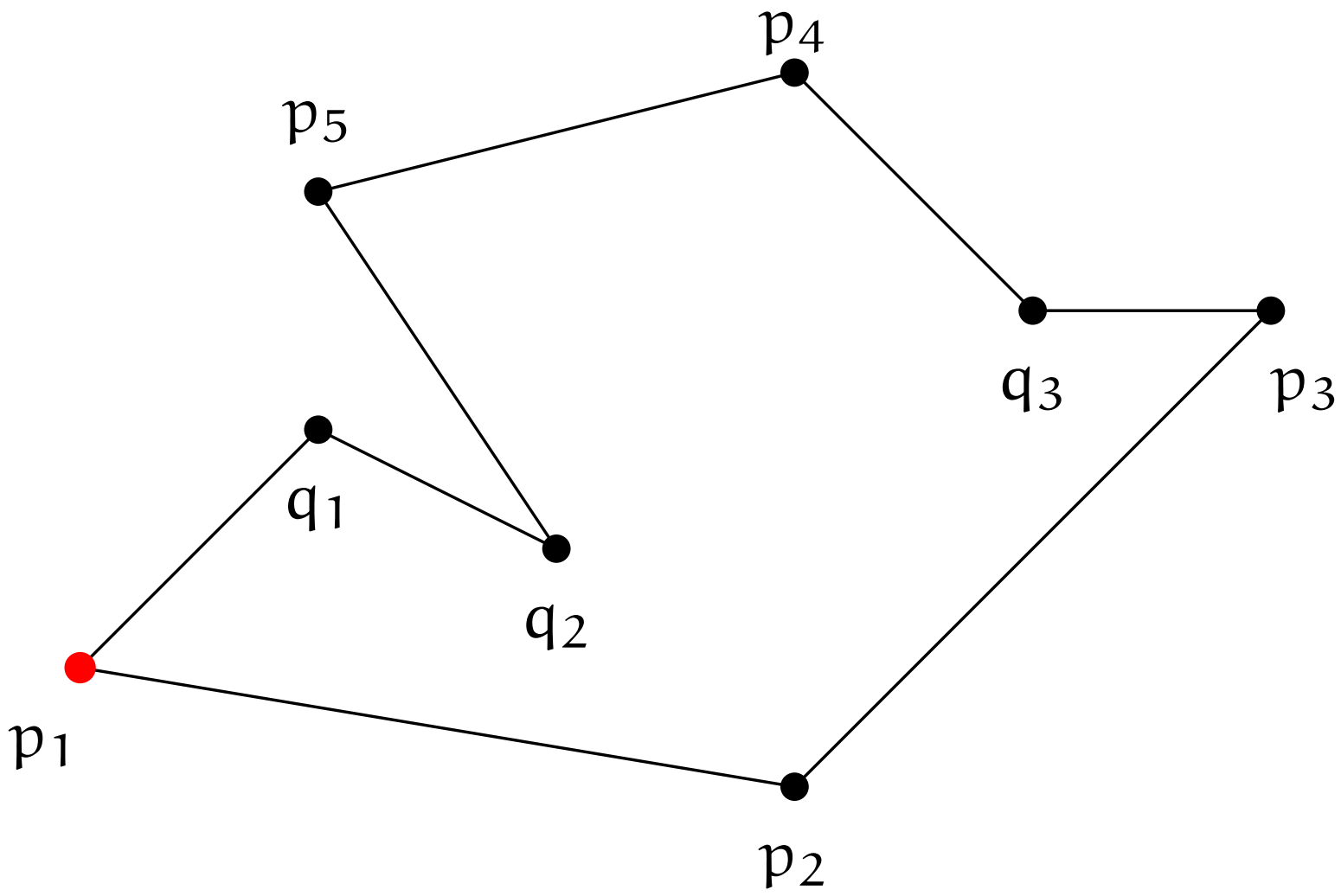
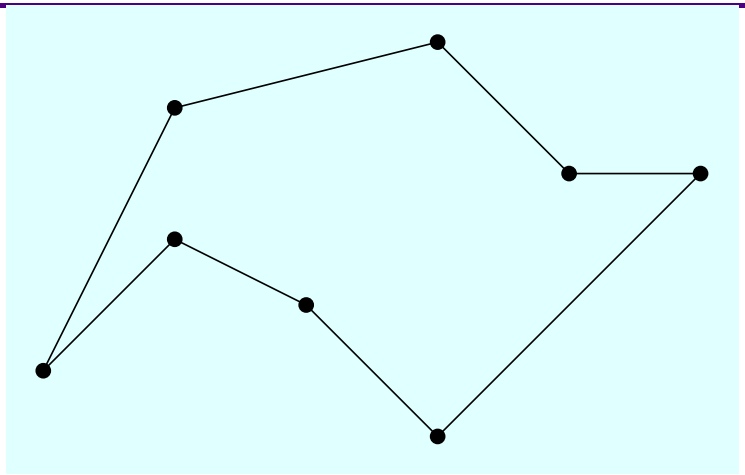
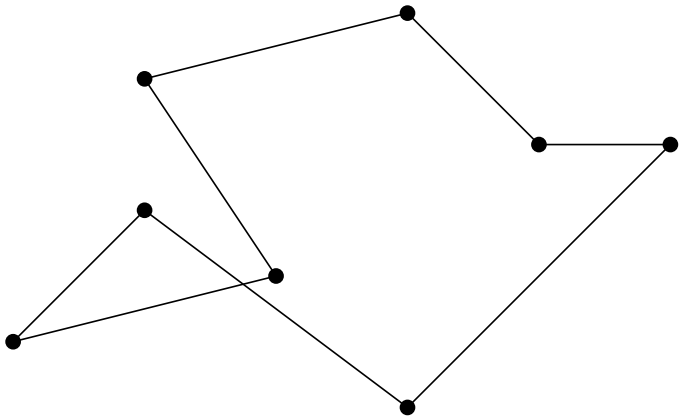
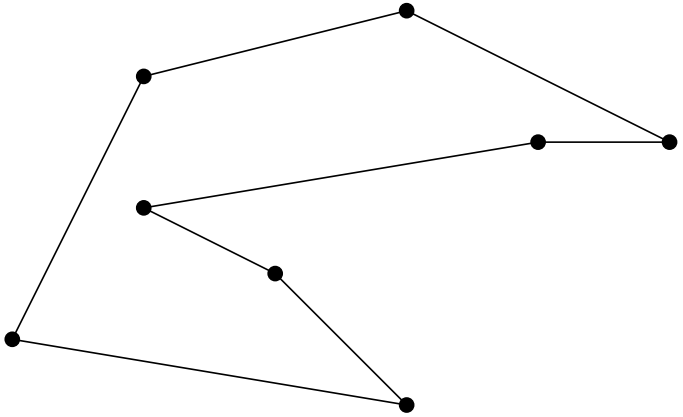


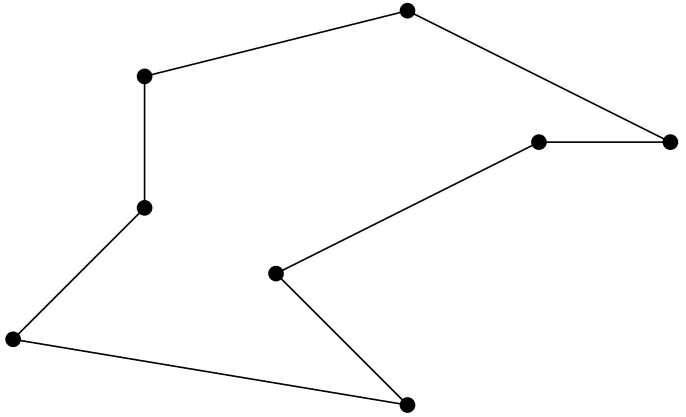
Optimal tour among those which respect the cyclic order and the order "3–2–1."

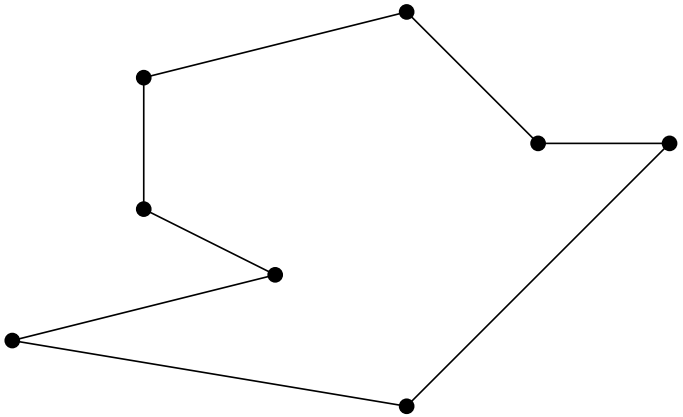
Choose the best one

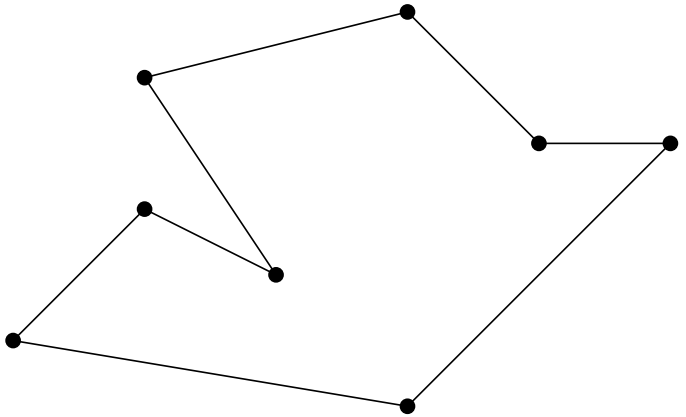












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Not yet clear: How to do Step (3i)??

Dynamic programming

$p_1, \ldots, p_{n-k}$ a cycl. order on the non-inner pts

$q_1, \ldots, q_k$ a linear order on the inner pts

$F(\underline{i}, j) :=$ the length of a shortest path
from p_1 to p_i
via $p_1, \ldots, p_i$ and $q_1, \ldots, q_j$
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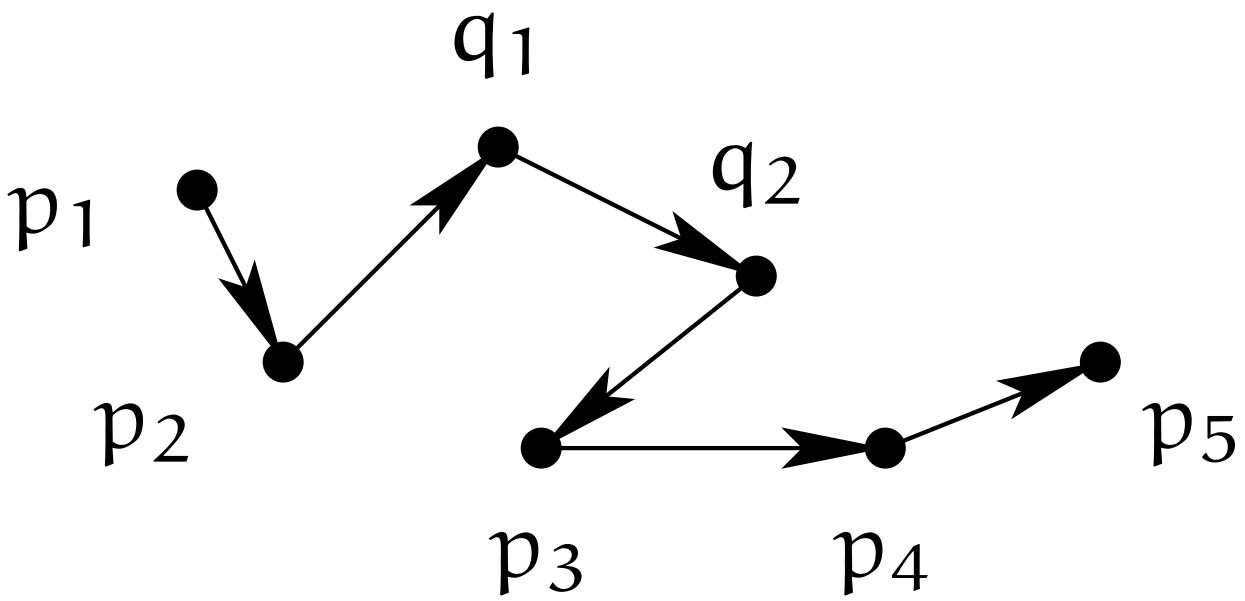

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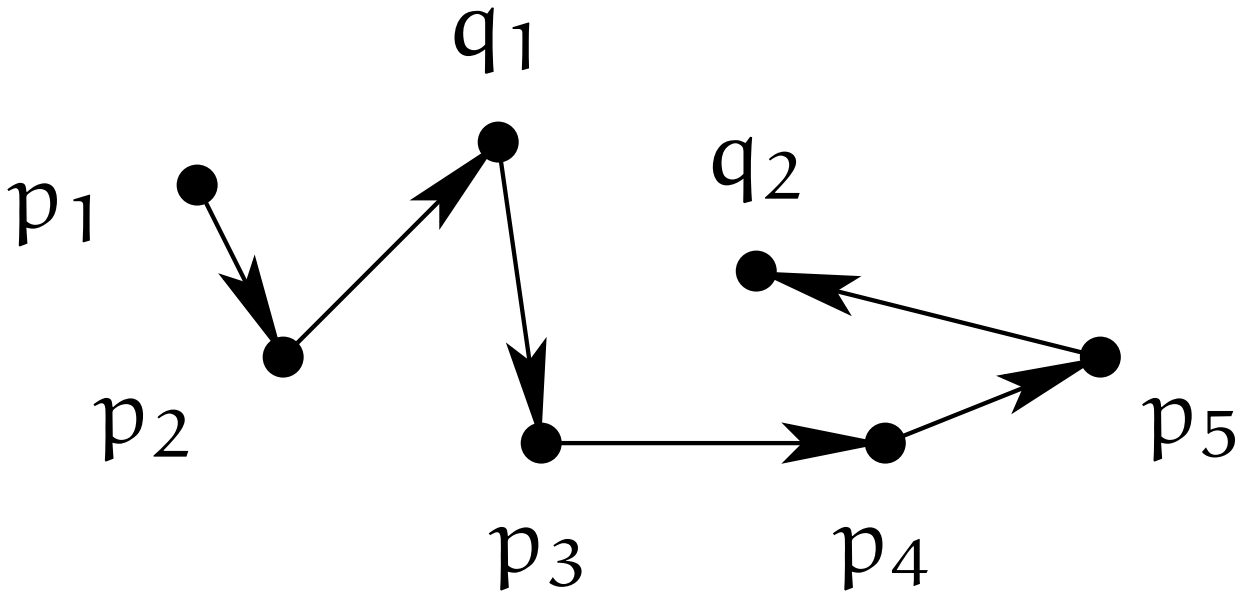
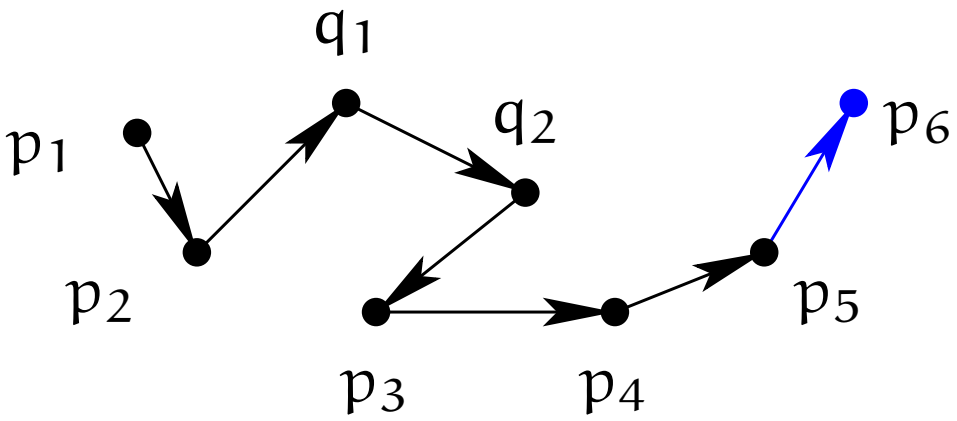


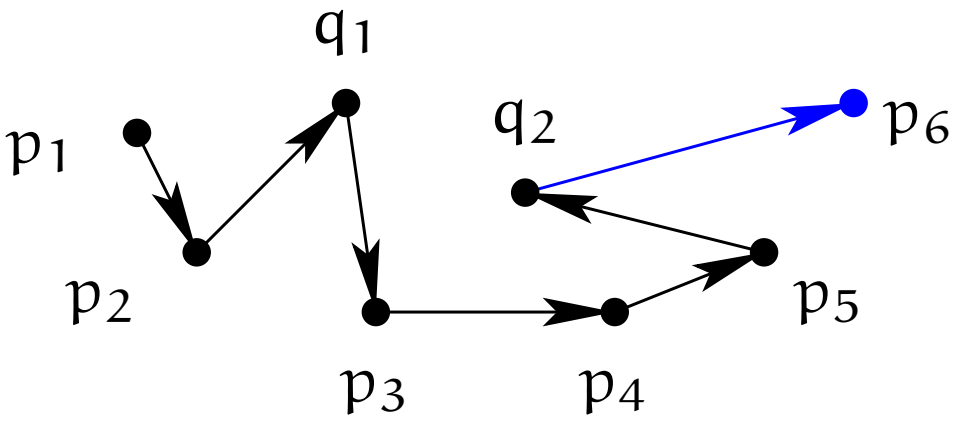
$(i = 5,\ j = 2)$

Main recurrence

It holds that

$$F(\underline{i+1}, j) = \text{minimum of } F(\underline{i}, j) + d(p_i, p_{i+1}) \text{ and } F(i, \underline{j}) + d(q_j, p_{i+1}).$$





$(i = 5,\ j = 2)$

Main recurrence

It holds that

$$
\begin{aligned}
F(\underline{i+1}, j) \;=\;& \text{minimum of} \\
& F(\underline{i}, j) + d(p_i, p_{i+1}) \text{ and} \\
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\end{aligned}
$$

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F(i, \underline{j+1}) \;=\;& \text{minimum of} \\
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Dynamic programming

- By the dynamic programming technique, $F(\underline{n-k}, k)$ and $F(n-k, \underline{k})$ can be computed in $O(kn)$ time.

- The length of a shortest tour which respects these two orders is

  the minimum of
  $F(\underline{n-k}, k) + d(p_{n-k}, p_1)$ and
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Outline of the second algorithm

(1) Distinguish the inner points and the non-inner points;

(2) Fix a cyclic order on the non-inner points;

(3) For each linear order on the inner points

  (i) Compute an optimal tour among those which respect these two orders;

(4) Choose the best one among them.

What remains: the analysis of the running time

Running time

There are k inner points.

- $\#$ of linear orders on k points $= k!$.
- They can be enumerated in $O(1)$ time per order.
- The length of an optimal tour which respects the two orders can be computed in $O(kn)$ time.

The running time $= \underbrace{O(n \log n)}_{\text{convex hull computation}} + O(k!kn)$.

When $k = O(\log n / \log \log n)$, this is poly. in n.

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Result

We gave two simple algorithms.

$n :=$ the total number of points
$k :=$ the number of inner points

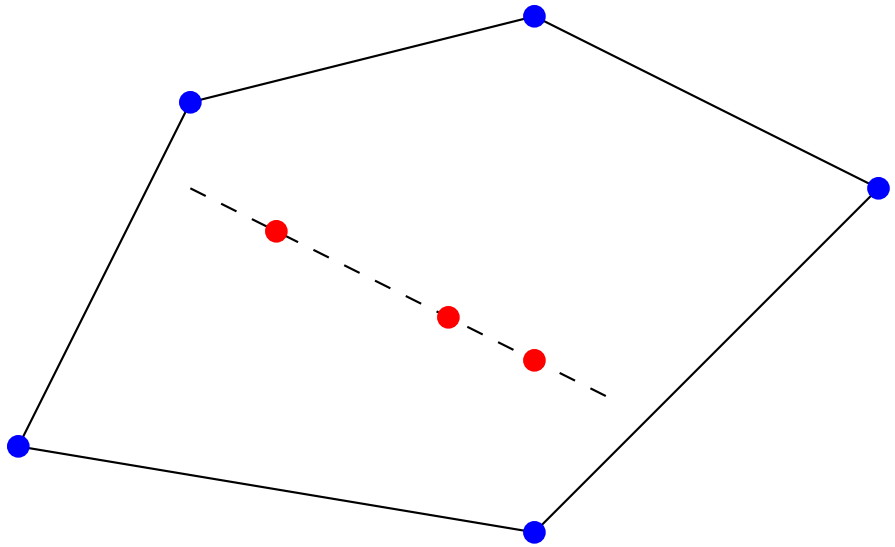
- First algorithm runs in $O(k!n^{k+1})$ time which is poly. when $k = O(1)$.
- Second algorithm runs in $O(k!kn)$ time which is poly. when $k = O(\log n / \log\log n)$.

Open problem: Improve the bound!

Related work

Thm (Deĭneko, van Dal & Rote '96)

2DTSP can be solved in $O(kn)$ time when the inner points lie on a line.



Our work { deals with the most general case.
still runs in linear time in n.

Variations

The same strategy works for other problems.

- The prize-collecting TSP
- The partial TSP

Result

The 2D versions of these problems
with k inner points
can be solved in polynomial time
when $k = O(\log n / \log \log n)$.