Core Stability of Minimum Coloring Games

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Abstract. In cooperative game theory, a characterization of games with stable cores is known as one of the most notorious open problems. We study this problem for a special case of the minimum coloring games, introduced by Deng, Ibaraki & Nagamochi, which arises from a cost allocation problem when the players are involved in conflict. In this paper, we show that the minimum coloring game on a perfect graph has a stable core if and only if every vertex of the graph belongs to a maximum clique. We also consider the problem on the core largeness, the extendability, and the exactness of minimum coloring games.

1 Introduction

One of the scopes of cooperative game theory is to establish the criterion of how to distribute a given revenue or cost among the agents in a fair manner when they work in cooperation. Since the effect of cooperation is usually non-linear and non-additive, the proportional division might not be considered fair. Several criteria, called solutions, are proposed by many researchers. When game theory was founded, von Neumann & Morgenstern [24] proposed a solution called a stable set, which turned out to be very useful for the analysis of a lot of bargaining situations but also turned out to be too difficult to reveal some fundamental properties. Much easier to investigate is the core, due to Gillies [11]. So, people are interested in when the core and the stable set coincide, namely when the core is stable. This question is known as one of the most notorious problems. So far, there are some necessary or sufficient conditions known (see, e.g., [23]), but they are far from a characterization of cooperative games with stable cores. From the computational point of view, the problem around stable sets is also eccentric. Deng & Papadimitriou [8] pointed out that determining the existence of a stable set for a given cooperative game is not known to be computable, and it is still unsolved.

Since combinatorial optimization problems can be found in several real-world situations, naturally they also raise some revenue/cost allocation problems. A *combinatorial optimization game* is a cooperative game which arises from a combinatorial optimization problem. There are many kinds of combinatorial optimization games proposed and studied, according to the underlying combinatorial optimization problems. However, as far as the core stability is concerned, almost nothing is studied. The only exception is a work by Solymosi & Raghavan [22] on assignment games.

In this paper, we study core stability of minimum coloring games introduced by Deng, Ibaraki & Nagamochi [6], which arise from cost allocation problems when the

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agents are involved in conflict [18]. The reason that we restrict to perfect graphs is that it is NP-complete to decide whether a given graph yields a minimum coloring game with a nonempty core [6] (meaning that there seems no good characterization of minimum coloring games with nonempty cores) and that a graph G is perfect if and only if the minimum coloring game on G is totally balanced [7], where the total balancedness is a quite nice property. We prove that the minimum coloring game on a perfect graph has a stable core if and only if every vertex belongs to a maximum clique. We also consider the problem on the extendability, the largeness, and the exactness of cores, which are concepts related to core stability. We prove that they are equivalent for the minimum coloring game on a perfect graph, and also equivalent to that every clique is contained in a maximum clique.

Armed with our characterizations, we also study algorithmic aspects of these properties. First we give a polynomial-time algorithm to determine whether a given perfect graph yields a minimum coloring game with stable core or not. On the other hand, we prove that it is hard (or coNP-complete, technically speacking) to determine whether a given perfect graph yields a minimum coloring game which is extendable, exact or with large core. To the best of our knowledge, this is the first computational intractability result for extendability, exactness and core largeness of cooperative games.

2 Preliminaries

2.1 Notation

Throughout the paper, for a vector $\boldsymbol{x} \in \mathbb{R}^N$ and $S \subseteq N$, we write $\boldsymbol{x}(S) := \sum\{x_i \mid i \in S\}$. When $S = \emptyset$, set $\boldsymbol{x}(S) := 0$. For a subset $S \subseteq N$ of a finite set N, the *characteristic vector* of S is a vector $\mathbb{1}_S \in \{0,1\}^N$ defined as $(\mathbb{1}_S)_i = 1$ if $i \in S$ and $(\mathbb{1}_S)_i = 0$ otherwise. Note that for $S, T \subseteq N$ it holds that $\mathbb{1}_S(T) = \sum\{(\mathbb{1}_S)_i \mid i \in T\} = |S \cap T|$. We use the notation $A \subset B$ to mean that "A is a proper subset of B."

2.2 Graphs

A *graph* G is a pair $G = (V, E)$ of a finite set V, called the set of *vertices*, and a set $E \subseteq \binom{V}{2}$ of 2-element subsets of V, called the set of *edges*. For $U \subseteq V$, the subgraph of G induced by U is denoted by $G[U]$, where the vertices of $G[U]$ are the elements of U and the edges of $G[U]$ are the edges of G which are also 2-element subsets of U. The *complement* of $G = (V, E)$ is a graph with vertex set V and edge set the complement of E. A *clique* is a vertex subset inducing a graph with every pair being an edge (such a graph is called *complete*). A clique is *maximal* if none of its proper supersets is a clique. A clique is *maximum* if it has a maximum size among all cliques. The size of a maximum clique of G is denoted by $\omega(G)$. An *independent set* is a vertex subset inducing a graph with no edge. A *coloring* of $G = (V, E)$ is a map $c : V \to \mathbb{N}$ such that $c(u) \neq c(v)$ for every $\{u, v\} \in E$. A *minimum coloring* of G is a coloring with minimum possible $|c(V)|$. The *chromatic number* of G is $|c(V)|$ of a minimum coloring c of G and denoted by $\chi(G)$. Conventionally, the chromatic number of a graph with no vertex is defined to be zero. A graph $G = (V, E)$ is *perfect* if $\omega(G[U]) = \chi(G[U])$ for every $U \subseteq V$. A prominent example of non-perfect graphs is a cycle of length five.

2.3 Cooperative games

A *cooperative game* (or simply a *game*) is a pair (N, γ) of a nonempty finite set N and a function $\gamma : 2^N \to \mathrm{I\!R}$ satisfying $\gamma(\emptyset) = 0$. An element of N is called a *player* of the game, and γ is called the *characteristic function* of the game. Furthermore, each subset $S \subseteq N$ is called a *coalition*. Literally, for $S \subseteq N$ the value $\gamma(S)$ is interpreted as the total profit (or the total cost) for the players in S when they work in cooperation. In particular, $\gamma(N)$ represents the total profit (or cost) for the whole players when they all agree on working together. When γ represents a profit, we call the game a *profit game*. On the other hand, when γ represents a cost, we call the game a *cost game*. (Thus, the terms "profit game" and "cost game" are not mathematically determined. They are just determined by the interpretation of a game.) In this paper, we will mainly consider a certain class of cost games.

One of the aims of cooperative game theory is to provide a concept of "fairness," namely, how to allocate the total cost (or profit) $\gamma(N)$ to each player in a "fair" manner when we take all the $\gamma(S)$'s into account. Now, we concentrate on cost games, and define some cost allocations which are considered fair in cooperative game theory. Formally, a cost allocation is defined as a preimputation in the terminology of cooperative game theory. A *preimputation* of a cost game (N, γ) is a vector $\boldsymbol{x} \in \mathrm{I\!R}^N$ satisfying $\boldsymbol{x}(N) = \gamma(N)$. Each component x_i expresses how much the player $i \in N$ should owe according to the cost allocation $\boldsymbol{x}$.

Let (N, γ) be a cost game. A vector $\boldsymbol{x} \in \mathrm{I\!R}^N$ is called an *imputation* if $\boldsymbol{x}$ satisfies the following conditions: $\boldsymbol{x}$ is a preimputation of (N, γ) and $x_i \leq \gamma(\{i\})$ for every $i \in N$. The set of all imputations of (N, γ) is denoted by $\mathsf{Imp}(N, \gamma)$. A vector $\boldsymbol{x} \in \mathrm{I\!R}^N$ is called a *core allocation* if $\boldsymbol{x}$ satisfies the following conditions: $\boldsymbol{x}$ is an imputation of (N, γ) and $\boldsymbol{x}(S) \leq \gamma(S)$ for all $S \subseteq N$. The set of all core allocations of (N, γ) is called the *core* of (N, γ) and denote by $\mathsf{Core}(N, \gamma)$. The core was introduced by Gillies [11].

Note that $\mathsf{Core}(N, \gamma) \subseteq \mathsf{Imp}(N, \gamma)$ and both can be empty. Therefore, a cost game with a nonempty core is especially interesting, and such a cost game is called *balanced*. Moreover, we call a cost game *totally balanced* if each of the subgames is balanced. (Here, a *subgame* of a cost game (N, γ) is a cost game $(T, \gamma^{(T)})$ for some nonempty $T \subseteq N$ defined as $\gamma^{(T)}(S) = \gamma(S)$ for each $S \subseteq T$.) Naturally, a totally balanced game is also balanced. A special subclass of the totally balanced games consists of submodular games (Shapley [20]), where a cost game (N, γ) is called *submodular* (or *concave*) if it satisfies $\gamma(S)+\gamma(T) \geq \gamma(S \cup T)+\gamma(S \cap T)$ for all $S, T \subseteq N$. Therefore, we have a chain of implications "submodularity $\Rightarrow$ total balancedness $\Rightarrow$ balancedness," which are fundamental in cooperative game theory.

Let (N, γ) be a balanced cost game. The core $\mathsf{Core}(N, \gamma)$ is called *stable* if for every $\boldsymbol{y} \in \mathsf{Imp}(N, \gamma) \setminus \mathsf{Core}(N, \gamma)$ there exist a core allocation $\boldsymbol{x} \in \mathsf{Core}(N, \gamma)$ and a nonempty coalition $S \subset N$ such that $\boldsymbol{x}(S) = \gamma(S)$ and $x_i < y_i$ for each $i \in S$. (The concept of stability is due to von Neumann & Morgenstern [24].) The core $\mathsf{Core}(N, \gamma)$ is called *large* if for every $\boldsymbol{y} \in \mathrm{I\!R}^N$ satisfying that $\boldsymbol{y}(S) \leq \gamma(S)$ for all $S \subseteq N$ there exists $\boldsymbol{x} \in \mathsf{Core}(N, \gamma)$ such that $\boldsymbol{y} \leq \boldsymbol{x}$. (The largeness was introduced by Sharkey [21].) The game (N, γ) is *extendable* if for every nonempty $S \subseteq N$ and every $\boldsymbol{y} \in \mathsf{Core}(S, \gamma^{(S)})$ there exists $\boldsymbol{x} \in \mathsf{Core}(N, \gamma)$ such that $x_i = y_i$ for all $i \in S$. (The

$$
\begin{tikzcd}
\text{submodularity} \arrow[d, Rightarrow] & & \text{exactness} \arrow[dr, Rightarrow] & \\
\text{large core} \wedge \text{total balancedness} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{extendability} \wedge \text{total balancedness} \arrow[ur, Rightarrow] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{stable core} \wedge \text{total balancedness} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{total balancedness} \arrow[d, Rightarrow] \\
\text{large core} \wedge \text{balancedness} \arrow[r, Rightarrow] & \text{extendability} \wedge \text{balancedness} \arrow[r, Rightarrow] & \text{stable core} \wedge \text{balancedness} \arrow[r, Rightarrow] & \text{balancedness}
\end{tikzcd}
$$

Fig. 1. Implication relationship. The symbol "$\wedge$" represents "and."

extendability was introduced by Kikuta & Shapley [13], and named by van Gellekom, Potters & Reijnierse [23].) The game (N, γ) is called *exact* if for every $S \subset N$ there exists $\boldsymbol{x} \in \mathsf{Core}(N, \gamma)$ such that $\boldsymbol{x}(S) = \gamma(S)$. (The exactness was first defined by Schmeidler [19].) Note that an exact game is always totally balanced.

Here, we summarize the known relationships among these classes of games. See also Fig. 1. Sharkey [21] showed that if a game is submodular then it has a large core. Kikuta & Shapley [13] showed that if a balanced game has a large core then it is extendable, and if a balanced game is extendable then it has a stable core. Sharkey [21] showed that if a totally balanced game has a large core then it is exact. Biswas, Parthasarathy, Potters & Voorneveld [1] pointed out that he actually proved that extendability implies exactness. The reverse directions in Fig. 1 do not hold in general. (Some of them are explained by van Gellekom, Potters and Reijnierse [23].)

3 Minimum coloring games

Let $G = (V, E)$ be a graph. The *minimum coloring game* on G is a cost game (V, χ_G) where $\chi_G : 2^V \to \mathbb{R}$ is defined as $\chi_G(S) := \chi(G[S])$ for all $S \subseteq V$. Furthermore, we always assume that $V \neq \emptyset$ when we consider the minimum coloring game, so that the minimum coloring game meets the definition of a cooperative game.

Let us first make some easy observations.

Observation 1. *Let $G = (V, E)$ be a graph and (V, χ_G) be the minimum coloring game on G.*

(a) For every $S \subseteq T \subseteq V$, it holds that $\chi_G(S) \leq \chi_G(T)$.
(b) For every nonempty independent set $I \subseteq V$ of G it holds that $\chi_G(I) = 1$. In particular, $\chi_G(\{v\}) = 1$ for each $v \in V$.
(c) If $\boldsymbol{x} \in \mathsf{Core}(V, \chi_G)$, then it holds that $0 \leq x_v \leq 1$ for every $v \in V$.

Proof. (a) Since $S \subseteq T$, we have $\chi(G[S]) \leq \chi(G[T])$. The claim follows from the definition of χ_G.
(b) For a nonempty independent set I, we have $\chi(G[I]) = 1$.
(c) Let $\boldsymbol{x} \in \mathsf{Core}(V, \chi_G)$. By the definition of the core and the part (b), we have that $x_v \leq \chi_G(\{v\}) = 1$. Suppose that $x_v < 0$ for contradiction. Then, it holds that $\chi_G(V) < \chi_G(V) - x_v$. Furthermore, by part (a) we have $\chi_G(V \setminus \{v\}) \leq \chi_G(V)$,

and also we have $\boldsymbol{x}(V) = \chi_G(V)$ since $\boldsymbol{x} \in \mathsf{Core}(V, \chi_G)$. Therefore, we obtain $\chi_G(V \setminus \{v\}) \leq \chi_G(V) < \chi_G(V) - x_v = \boldsymbol{x}(V) - x_v = \boldsymbol{x}(V \setminus \{v\})$. This is a contradiction to $\boldsymbol{x} \in \mathsf{Core}(V, \chi_G)$. □

Deng, Nagamochi & Ibaraki [6] proved that it is NP-complete to decide whether the minimum coloring game on a given graph is balanced. Subsequently, Deng, Ibaraki, Nagamochi & Zang [7] showed that the minimum coloring game on a graph G is totally balanced if and only if G is perfect. So the decision problem on the total balancedness of a minimum coloring game is as hard as recognizing perfect graphs, which was found to be solved in polynomial time [2, 4]. Furthermore, Okamoto [17] showed that the minimum coloring game on a graph G is submodular if and only if G is complete multipartite. So we can decide whether a given graph yields a submodular minimum coloring game in polynomial time. The following proposition due to Okamoto [18] characterizes the core of the minimum coloring game on a perfect graph. This will be used nicely in a later investigation.

Proposition 1 (Okamoto [18]). *Let $G = (V, E)$ be a perfect graph. Then, the core of the minimum coloring game (V, χ_G) is the convex hull of the characteristic vectors of maximum cliques of G.*

4 Results

4.1 Core stability

The following theorem characterizes totally balanced minimum coloring games with stable cores.

Theorem 2. *Let $G = (V, E)$ be a perfect graph. Then, the minimum coloring game (V, χ_G) has a stable core if and only if every vertex $v \in V$ belongs to a maximum clique of G.*

First we prove the only-if part of the theorem. The proof uses the following lemma.

Lemma 1. *Let $G = (V, E)$ be a graph such that the minimum coloring game (V, χ_G) is balanced. If (V, χ_G) has a stable core, then for every $v \in V$ there exists a core allocation $\boldsymbol{x} \in \mathsf{Core}(V, \chi_G)$ such that $x_v \neq 0$.*

Proof. Assume that $\mathsf{Core}(V, \chi_G)$ is stable, and suppose, for the contradiction, there exists a vertex $v \in V$ such that

$$x_v = 0 \quad \text{for all } \boldsymbol{x} \in \mathsf{Core}(V, \chi_G). \tag{1}$$

(Particularly $V \neq \emptyset$.) Take such a vertex v. Let $\hat{\boldsymbol{x}} \in \mathsf{Core}(V, \chi_G)$ be an arbitrary core allocation. Since $V \neq \emptyset$, it holds that $\chi_G(V) > 0$. So, there exists $w \in V$ such that $\hat{x}_w > 0$. Now, define $\boldsymbol{y} \in \mathbb{R}^V$ as

$$y_u := \begin{cases} \hat{x}_u & \text{if } u \notin \{v, w\}, \\ \hat{x}_w & \text{if } u = v, \\ 0 & \text{if } u = w. \end{cases}$$

Namely, $\boldsymbol{y}$ is obtained from $\hat{\boldsymbol{x}}$ by interchanging the v-th component and the w-th component. Then, $\boldsymbol{y}$ is an imputation of (V, χ_G). Since $y_v = \hat{x}_w > 0$, due to (1), we can see that $\boldsymbol{y}$ is not a core allocation. Hence, $\boldsymbol{y} \in \mathsf{Imp}(V, \chi_G) \setminus \mathsf{Core}(V, \chi_G)$.

Since $\mathsf{Core}(V, \chi_G)$ is stable, there exist a nonempty set $S \subset V$ and a core allocation $\overline{\boldsymbol{x}} \in \mathsf{Core}(V, \chi_G)$ such that $\overline{\boldsymbol{x}}(S) = \chi_G(S)$ and $\overline{x}_u < y_u$ for every $u \in S$. Now we cliam that $S \setminus \{v\} \neq \emptyset$. To show this, suppose not, i.e., $S \setminus \{v\} = \emptyset$. Since $S \neq \emptyset$, we have that $S = \{v\}$. Then, it follows that

$$\begin{aligned}
\chi_G(\{v\}) &= \overline{x}_v && (\text{since } \chi_G(S) = \overline{\boldsymbol{x}}(S)) \\
&< y_v && (\text{since } \overline{x}_u < y_u \text{ for every } u \in S) \\
&\leq \chi_G(\{v\}) && (\text{since } \boldsymbol{y} \in \mathsf{Imp}(V, \chi_G)).
\end{aligned}$$

This is a contradiction, hence the claim follows.

Going back to the proof of Lemma 1, we obtain

$$\begin{aligned}
\chi_G(S) &= \overline{\boldsymbol{x}}(S) \\
&= \overline{\boldsymbol{x}}(S \setminus \{v\}) && (\text{by (1)}) \\
&< \boldsymbol{y}(S \setminus \{v\}) && (\text{by the choice of } \overline{\boldsymbol{x}} \text{ and Claim above}) \\
&\leq \hat{\boldsymbol{x}}(S \setminus \{v\}) && (\text{by the construction of } \boldsymbol{y}) \\
&\leq \chi_G(S \setminus \{v\}) && (\text{since } \hat{\boldsymbol{x}} \in \mathsf{Core}(V, \chi_G)) \\
&\leq \chi_G(S) && (\text{by Observation 1(a)}).
\end{aligned}$$

This is a contradiction. □

Then, let us prove the only-if part of the theorem.

Proof (of the only-if part of Theorem 2). Assume that (V, χ_G) has a stable core. By Lemma 1, for every $v \in V$ there exists a core allocation $\boldsymbol{x} \in \mathsf{Core}(V, \chi_G)$ such that $x_v > 0$. On the other hand, by Proposition 1, $\boldsymbol{x}$ is a convex combination of the characteristic vectors of maximum cliques of G. Therefore, at least one maximum clique of G must contain v. □

In order to prove the if part, we need some more lemmas.

Lemma 2. *Let $G = (V, E)$ be a graph with $\chi(G) = \omega(G)$. Then, there exists a nonempty independent set $I \subseteq V$ such that $K \cap I \neq \emptyset$ for every maximum clique K of G.*

Proof. Consider a minimum coloring of G and take the vertices colored by an identical color. Denote by I the set of these vertices. By the construction, I is an independent set. On the other hand, in each maximum clique K of G all colors used to color G can be found since $\chi(G) = \omega(G) = |K|$. Namely, every maximum clique intersects I. Thus, I is a desired independent set. □

Here is another lemma.

Lemma 3. *Let $G = (V, E)$ be a perfect graph, and consider the minimum coloring game (V, χ_G). Then, for every $\boldsymbol{y} \in \mathsf{Imp}(V, \chi_G) \setminus \mathsf{Core}(V, \chi_G)$ there exists a nonempty independent set $I \subseteq V$ such that $\boldsymbol{y}(I) > \chi_G(I)$ and $y_v > 0$ for every $v \in I$.*

Proof. Fix $\boldsymbol{y} \in \mathsf{Imp}(V, \chi_G) \setminus \mathsf{Core}(V, \chi_G)$ arbitrary, and define $\mathcal{S} := \{S \subseteq V \mid \boldsymbol{y}(S) > \chi_G(S)$ and $y_v > 0$ for every $v \in S\}$.

First, note that $\mathcal{S} \neq \emptyset$. To see this, since $\boldsymbol{y} \in \mathsf{Imp}(V, \chi_G) \setminus \mathsf{Core}(V, \chi_G)$, there exists $T' \subseteq V$ such that $\boldsymbol{y}(T') > \chi_G(T')$. Let $T := T' \setminus \{v \in T' \mid y_v \leq 0\}$. Then, it holds that $\boldsymbol{y}(T) \geq \boldsymbol{y}(T') > \chi_G(T') \geq \chi_G(T)$. (The last inequality is due to $T \subseteq T'$ and Observation 1(a).) Since $y_v > 0$ for each $v \in T$, it follows that $T \in \mathcal{S}$. This implies that $\mathcal{S}$ is nonempty.

Choose $S \in \mathcal{S}$ of minimum size. Since G is perfect, we have that $\chi(G[S]) = \omega(G[S])$. By Lemma 2, there exists a nonempty independent set $I \subseteq S$ such that for every maximum clique K of $G[S]$ we have $K \cap I \neq \emptyset$. Now, we claim that $I \in \mathcal{S}$. (This proves the lemma.) First of all, since $I \subseteq S$ it holds that $y_v > 0$ for every $v \in I$. So it suffices to show that $\boldsymbol{y}(I) > \chi_G(I)$.

Since I intersects with every maximum clique of $G[S]$, we can see that $\omega(G[S \setminus I]) < \omega(G[S])$. Since G is perfect, this means that

$$\chi_G(S \setminus I) < \chi_G(S). \tag{2}$$

Since I is nonempty, we have $|S \setminus I| < |S|$. By the minimality of S, it holds that

$$\boldsymbol{y}(S \setminus I) \leq \chi_G(S \setminus I). \tag{3}$$

Now, we obtain the following.

$$\begin{aligned}
\boldsymbol{y}(I) &= \boldsymbol{y}(S) - \boldsymbol{y}(S \setminus I) && (I \subseteq S)\\
&> \chi_G(S) - \chi_G(S \setminus I) && (S \in \mathcal{S} \text{ and } (3))\\
&\geq 1 && ((2) \text{ and the integrality of } \chi_G)\\
&= \chi_G(I) && (\text{Observation 1(b)}).
\end{aligned}$$

This concludes the proof. □

Now, we are ready to prove the if part of Theorem 2.

Proof (of the if part of Theorem 2). Let $\boldsymbol{y} \in \mathsf{Imp}(V, \chi_G) \setminus \mathsf{Core}(V, \chi_G)$. Then, by Lemma 3, there exists a nonempty independent set $I \subseteq V$ such that $\boldsymbol{y}(I) > \chi_G(I) = 1$ and $y_v > 0$ for every $v \in I$. Denote by $\mathcal{K}$ the set of maximum cliques of G. To every vertex $v \in I$, we assign a maximum clique $K(v) \in \mathcal{K}$ such that $v \in K(v)$, and fix this assignment. By our assumption, this assignment is well-defined. Since I is an independent set, this assignment is injective.

For every $K \in \mathcal{K}$, let

$$\lambda_K := \begin{cases} \dfrac{y_v}{\boldsymbol{y}(I)} & \text{if } K = K(v) \text{ for some } v \in I \\ 0 & \text{otherwise.} \end{cases}$$

Since the assignment $v \mapsto K(v)$ is injective, the value λ_K is well-deined. Then, for each $K \in \mathcal{K}$, we have that $0 \leq \lambda_K \leq 1$ (since $y_v > 0$ for every $v \in I$ and $\boldsymbol{y}(I) > 1$ by the choice of I with Lemma 3, and $y_v \leq 1$ for every $v \in I$ by Observation 1(b) and the definition of an imputation). Furthermore, we can check that $\sum_{K\in\mathcal{K}} \lambda_K = 1$. Therefore, if we let $\boldsymbol{x} := \sum_{K\in\mathcal{K}} \lambda_K \mathbb{1}_K$, by Proposition 1, it holds that $\boldsymbol{x} \in \mathsf{Core}(V, \chi_G)$.

If $v \in I$ then $x_v = \lambda_{K(v)}$. This is because $v \notin K(u)$ for $u \in I \setminus \{v\}$. Therefore, if $v \in I$, then

$$x_v = \lambda_{K(v)} = \frac{y_v}{\boldsymbol{y}(I)} < y_v,$$

since $\boldsymbol{y}(I) > 1$. Furthermore, it holds that

$$\boldsymbol{x}(I) = \sum_{u\in I} x_u = \sum_{u\in I} \lambda_{K(u)} = 1 = \chi_G(I).$$

Thus, $\boldsymbol{x}$ is an appropriate core allocation and hence the core is stable. □

4.2 Exactness, extendability, and core largeness

We prove that exactness, extendability and core largeness are equivalent for minimum coloring games on perfect graphs. This is also characterized in terms of graphs, and summarized as the following theorem.

Theorem 3. *Let $G = (V, E)$ be a perfect graph. Then, the following conditions are equivalent.*

(1) The minimum coloring game (V, χ_G) is exact.
(2) The minimum coloring game (V, χ_G) is extendable.
(3) The core $\mathsf{Core}(V, \chi_G)$ is large.
(4) Every clique of G is contained in a maximum clique of G.

First remark that the implication "(3) ⇒ (2) ⇒ (1)" is true for any kinds of games [13]. It remains to prove "(1) ⇒ (4)" and "(4) ⇒ (3)."

Let us first prove "(1) ⇒ (4)."

Proof (of (1) ⇒ (4)). Let $G = (V, E)$ be a perfect graph such that (V, χ_G) is exact. Let S be a clique of G. Then, by exactness, there exists $\boldsymbol{x} \in \mathsf{Core}(V, \chi_G)$ such that $\boldsymbol{x}(S) = \chi_G(S) = |S|$. Denoting by $\mathcal{K}$ the set of maximum cliques of G, by Proposition 1, we can express $\boldsymbol{x}$ as

$$\boldsymbol{x} = \sum_{K\in\mathcal{K}} \lambda_K \mathbb{1}_K, \tag{4}$$

where $\lambda_K \geq 0$ for every $K \in \mathcal{K}$ and $\sum_{K\in\mathcal{K}} \lambda_K = 1$. Then, it holds that

$$\begin{aligned}
|S| = \boldsymbol{x}(S) &= \sum_{K\in\mathcal{K}} \lambda_K \mathbb{1}_K(S) && \text{(by (4))}\\
&= \sum_{K\in\mathcal{K}} \lambda_K |S \cap K| \\
&\leq \sum_{K\in\mathcal{K}} \lambda_K |S| && \text{(since } S \cap K \subseteq S\text{)}\\
&= |S| \sum_{K\in\mathcal{K}} \lambda_K = |S| && \text{(since } \textstyle\sum_{K\in\mathcal{K}} \lambda_K = 1\text{).}
\end{aligned}$$

So, the equality holds throughout the expressions, meaning that $S \cap K = S$ for each $K \in \mathcal{K}$ with $\lambda_K > 0$. Thus, S is contained in a maximum clique of G. □

To show "(4) ⇒ (3)," we use some more facts. The first one is due to van Gellekom, Potters & Reijnierse [23]. For a cost game (N, γ), let

$$L(N,\gamma) := \{\boldsymbol{y} \in \mathbb{R}^N \mid \boldsymbol{y}(S) \leq \gamma(S) \text{ for every } S \subseteq N\},$$

and call it the set of *lower vectors*.

Lemma 4 (van Gellekom, Potters & Reijnierse [23]). *Let (N, γ) be a balanced cost game. Then (N, γ) has a large core if and only if $\boldsymbol{y}(N) \geq \gamma(N)$ for all extreme points $\boldsymbol{y}$ of $L(N, \gamma)$.*

In order to apply Lemma 4 to our setting, we have to know the extreme points of $L(V, \chi_G)$ for a perfect graph G. The following lemma can be shown with a similar method to the proof of the weak perfect graph conjecture due to Lovász [16].

Lemma 5. *Let $G = (V, E)$ be a perfect graph. Then, each extreme point of $L(V, \chi_G)$ is the characteristic vector of a maximal clique of G.*

Armed with Lemmas 4 and 5, we are able to show "(4) ⇒ (3)."

Proof (of (4) ⇒ (3)). Let G be a perfect graph such that every clique is contained in a maximum clique of G. Choose an extreme point of $L(V, \chi_G)$. By Lemma 5, this is the characteristic vector of some maximal clique K of G. Namely, this extreme point is $\mathbb{1}_K$. By our assumption, K is a maximum clique of G. Therefore, it holds that $\mathbb{1}_K(V) = |K| = \omega(G) = \chi_G(V)$. Hence, by Lemma 4, the core is large. □

This completes the whole proof of Theorem 3.

5 Algorithmic aspects

In this section, using the theorems we have obtained already, we discuss the algorithmic issues for minimum coloring games. The first problem we consider is the following.

Problem: CORE STABILITY FOR PERFECT GRAPHS
Instance: A perfect graph $G = (V, E)$ **Question:** Does the minimum coloring game (V, χ_G) have a stable core?

Now, we describe an algorithm which shows the following theorem.

Theorem 4. *The problem* CORE STABILITY FOR PERFECT GRAPHS *can be solved in polynomial time.*

Proof. Consider the algorithm in Algorithm 1.

Let us prove that Algorithm 1 is correct. The first observation is that in each "foreach" loop we compute a clique K_v of maximum size which contains v. That is just because M is huge. Now, if $|K_v| < \omega(G)$, then we can see that a maximum clique containing v is not a maximum clique of G. Namely, v is not contained in any maximum

Algorithm 1: A polynomial-time algorithm for CORE STABILITY FOR PERFECT GRAPHS

Input: a perfect graph $G = (V, E)$.

Output: "Yes" if $(V, \chi(G))$ has a stable core; "No" otherwise.

1 $\omega(G) \leftarrow$ the weight of a maximum clique in G;
2 $M \leftarrow |V|$;
3 **foreach** *vertex* $v \in V$ **do**
4   Set a weight vector $\boldsymbol{w} \in \mathbb{R}^V$ as $w_v = M$ and $w_u = 1$ $(u \in V \setminus \{v\})$;
5     $\omega(G, \boldsymbol{w}) \leftarrow$ the maximum weight of a clique in G with respect to $\boldsymbol{w}$;
6   **if** $\omega(G, \boldsymbol{w}) - \omega(G) < M - 1$ **then**
7   **return** *"No"*;
    end
end
8 **return** *"Yes"*.

clique of G. Then, by Theorem 2, the game does not have a stable core. Therefore, we have to check that $|K_v| < \omega(G)$ if and only if $\omega(G, \boldsymbol{w}) - \omega(G) < M - 1$ (i.e., the condition in Line 6 is true). First of all, we can see that $|K_v| = \omega(G, \boldsymbol{w}) - M + 1$. So, we have that $|K_v| - \omega(G) = \omega(G, \boldsymbol{w}) - \omega(G) + M - 1$. Hence, $|K_v| < \omega(G)$ holds if and only if $\omega(G, \boldsymbol{w}) - \omega(G) < M - 1$. This completes the proof of the correctness.

Now, we discuss the running time of Algorithm 1. Computing a maximum weight clique in a perfect graph can be done in polynomial time [12]. So, Lines 1 and 5 can be executed in polynomial time. Line 2 is also fine. In the "foreach" loop, Line 4 can be done swiftly. The condition check in Line 6 is easy. The number of iterations of the foreach loop is at most $|V|$. Hence, the overall running time is polynomial in the size of input. □

Next, we discuss the following three problems.

Problem: EXTENDABILITY FOR PERFECT GRAPHS
Instance: A perfect graph $G = (V, E)$
Question: Is the minimum coloring game (V, χ_G) extendable?

Problem: EXACTNESS FOR PERFECT GRAPHS
Instance: A perfect graph $G = (V, E)$
Question: Is the minimum coloring game (V, χ_G) exact?

Problem: CORE LARGENESS FOR PERFECT GRAPHS
Instance: A perfect graph $G = (V, E)$
Question: Does the minimum coloring game (V, χ_G) have a large core?

Thanks to Theorem 3, these problems are equivalent to the following problem.

Problem: SIZE EQUALITY OF A MAXIMUM CLIQUE AND A MINIMUM MAXIMAL CLIQUE IN PERFECT GRAPHS
Instance: A perfect graph $G = (V, E)$
Question: Do a maximum clique and a minimum maximal clique in G have the same size?

This problem turns out to be coNP-complete.

Theorem 5. *The problem* SIZE EQUALITY OF A MAXIMUM CLIQUE AND A MINIMUM MAXIMAL CLIQUE IN PERFECT GRAPHS *is coNP-complete. Consequently,* EXTENDABILITY FOR PERFECT GRAPHS, EXACTNESS FOR PERFECT GRAPHS *and* CORE LARGENESS FOR PERFECT GRAPHS *are coNP-complete.*

Proof. The membership in coNP is immediate. The coNP-hardness follows from a result due to Zverovich [25]. □

Theorem 5 deals with perfect graphs in general. Now, let us discuss some special cases for which the problem can be solved in polynomial time. Observe that, due to Theorem 3, it suffices to compute a minimum maximal clique in a given perfect graph. If it is also a maximum clique in the graph, then all maximal cliques are maximum cliques. Then, the condition (4) in Theorem 3 holds. If not, then this maximal clique is not contained in a maximum clique, meaning that the condition (4) is violated. Namely, we consider the following optimization problem.

Problem: MINIMUM MAXIMAL CLIQUE
Instance: A graph G
Feasible solution: A maximal clique K of G
Objective: Minimize $|K|$.

There are some classes of perfect graphs for which we can solve MINIMUM MAXIMAL CLIQUE in polynomial time. They include the bipartite graphs (easy), the comparability graphs [15], the chordal graphs [10], and the complements of chordal graphs [9]. (See also an article by Kratsch [14].) For these classes of graphs, as we already observed, we can conclude the following.

Theorem 6. *Consider a class of perfect graphs for which* MINIMUM MAXIMAL CLIQUE *can be solved in polynomial time. For this class of graphs,* EXTENDABILITY FOR PERFECT GRAPHS, EXACTNESS FOR PERFECT GRAPHS *and* CORE LARGENESS FOR PERFECT GRAPHS *can be solved in polynomial time.*

6 Summary

We discussed the core stability problem for minimum coloring games, introduced by Deng, Ibaraki & Nagamochi [6], of perfect graphs. We obtained a good characterization for a minimum coloring game with stable core (Theorem 2), and this led us to a polynomial-time algorithm for the corresponding decision problem (Theorem 4). We also discussed the extendability, the exactness and the core largeness for minimum coloring games of perfect graphs, and characterized them in terms of a property of graphs (Theorem 3). With this characterization, we showed that it is coNP-complete to determine whether a given perfect graph yields the minimum coloring game which is extendable, exact, or with large core (Theorem 5). For some subclasses of perfect graphs, we know that there exists a polynomial-time algorithm for this problem (Theorem 6).

Little is known about core stability of cooperative games. This paper expanded the knowledge of this problem, and also gave rise to some algorithmic perspectives.

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