ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

The Minimum Weight Triangulation Problem with Few Inner Points

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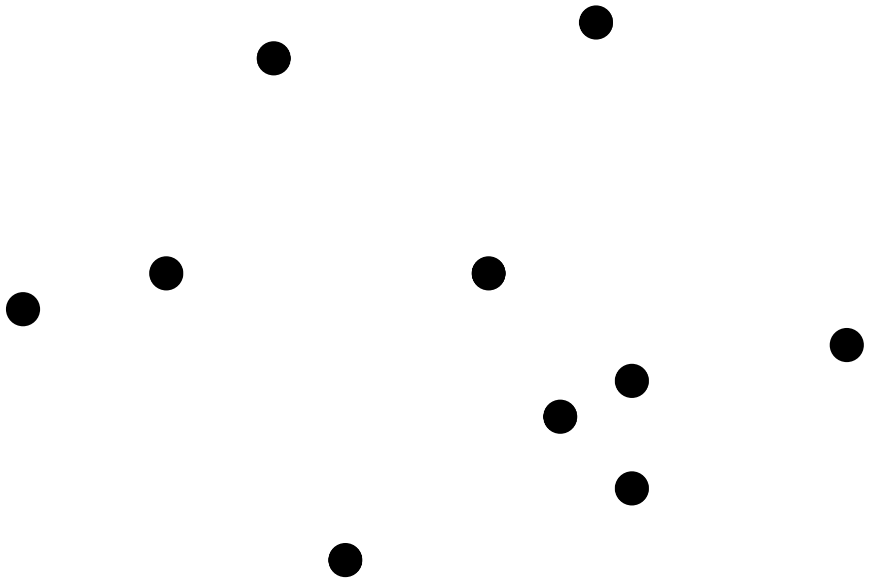
Radisson SAS Royal Hotel, Bergen, Norway

Triangulation

S a set of points in the plane, no three points lying on a line

A **triangulation** of S:



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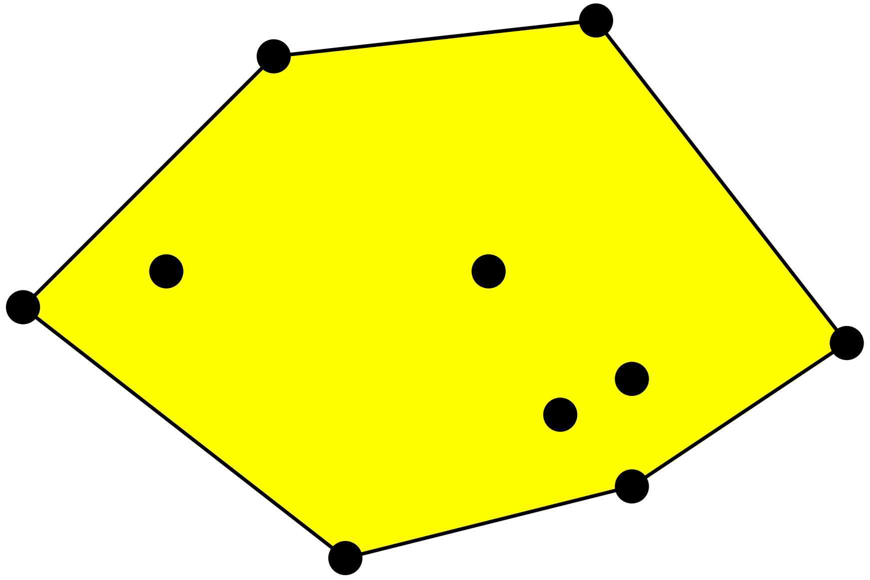

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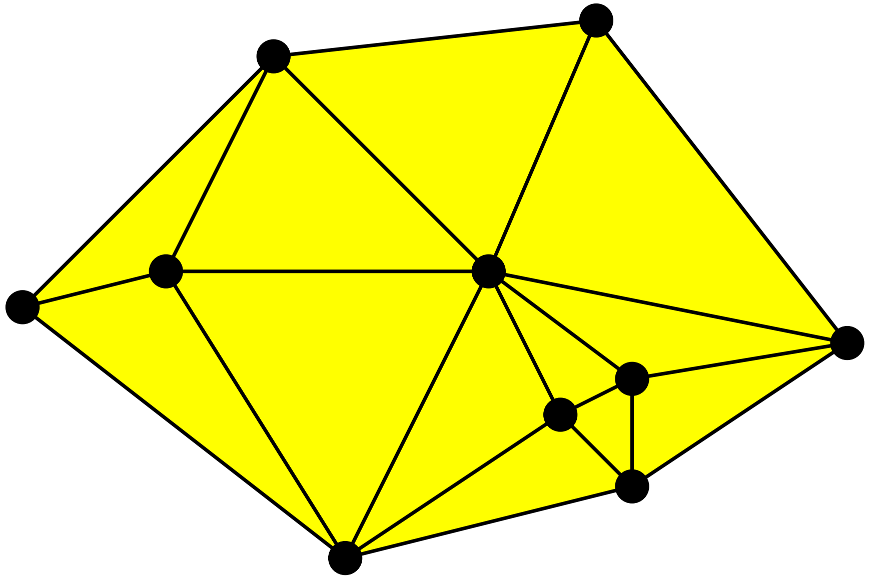

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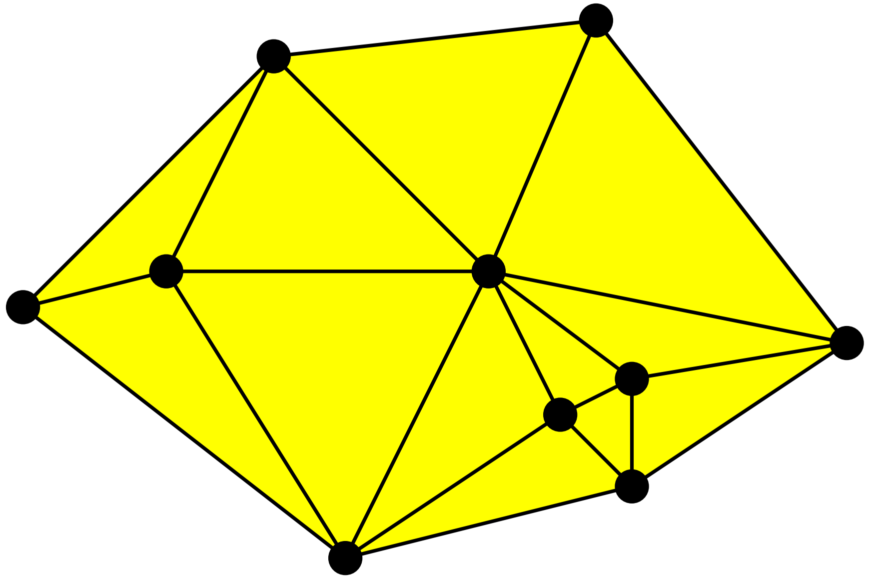

a subdivision of $\mathrm{conv}(S)$ into triangles s.t.
edges: straight line segments connecting points from S and
triangles: with no point from S in their interiors

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Minimum weight triangulation

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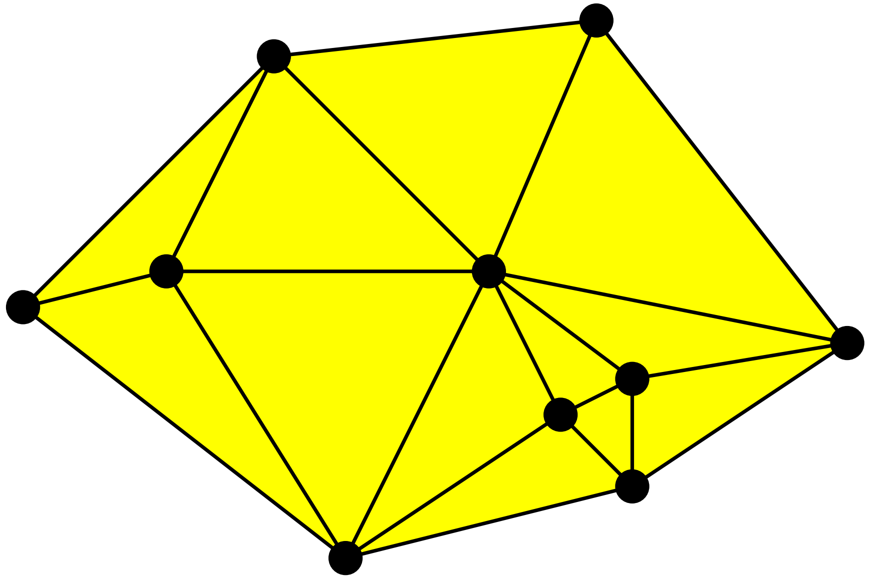
Weight of a triangulation of S:



sum of the lengths of the edges

Minimum weight triangulation

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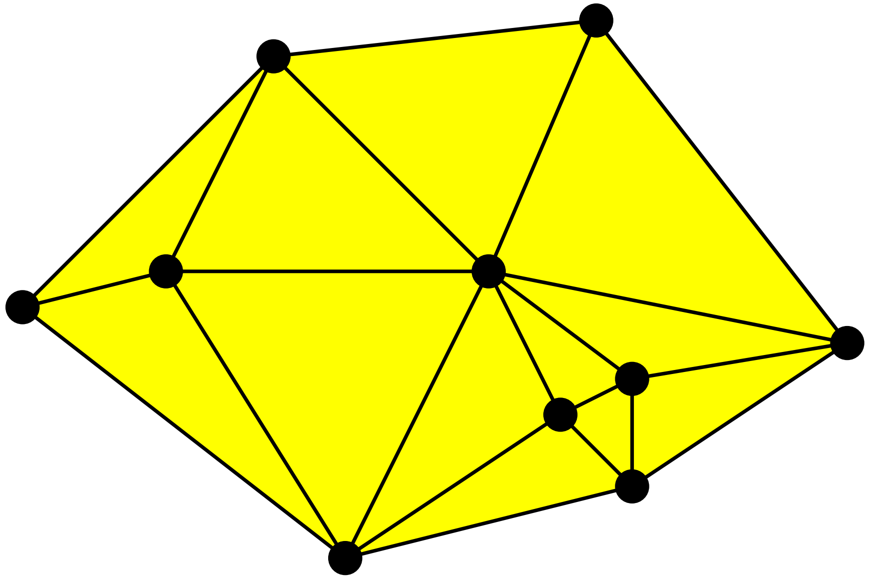
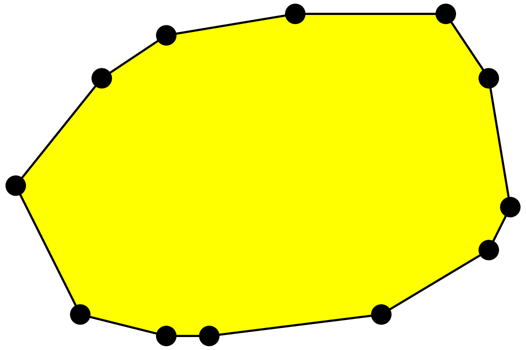


sum of the lengths of the edges

Minimum weight triangulation:
a triangulation with minimum weight

The minimum weight triangulation problem

- **In general**, not known to be
  - solvable in polynomial time, or
  - NP-hard.
- **If the points are in convex position,** can be solved in $O(n^3)$ time

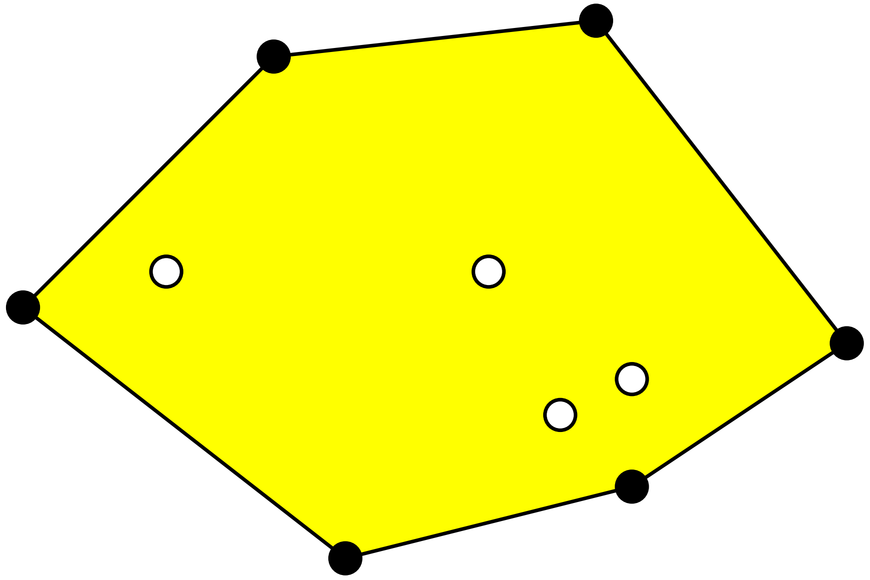


n the number of points

Distance-from-Triviality approach

Guo, Hüffner & Niedermeier (prev. talk)

“Distance from Triviality” = Number of inner points



The less inner points ⟷ The easier instance

Cf. Deĭneko, Hoffmann, Okamoto & Woeginger ’04:
TSP with few inner points

Result

Develop an algorithm to solve the minimum weight triangulation problem in $O(6^k n^5 \log n)$ time.

n	the total number of points
k	the number of inner points

Result

Develop an algorithm to solve the minimum weight triangulation problem in $O(6^k n^5 \log n)$ time.

n the total number of points
k the number of inner points

Consequences:

The minimum weight triangulation problem

- is **fixed parameter tractable** (FPT) w.r.t. the number of inner points.
- can be solved in **polynomial time if $k = O(\log n)$**.

Divide & Conquer and Dynamic Programming

Overview of the rest of my talk:

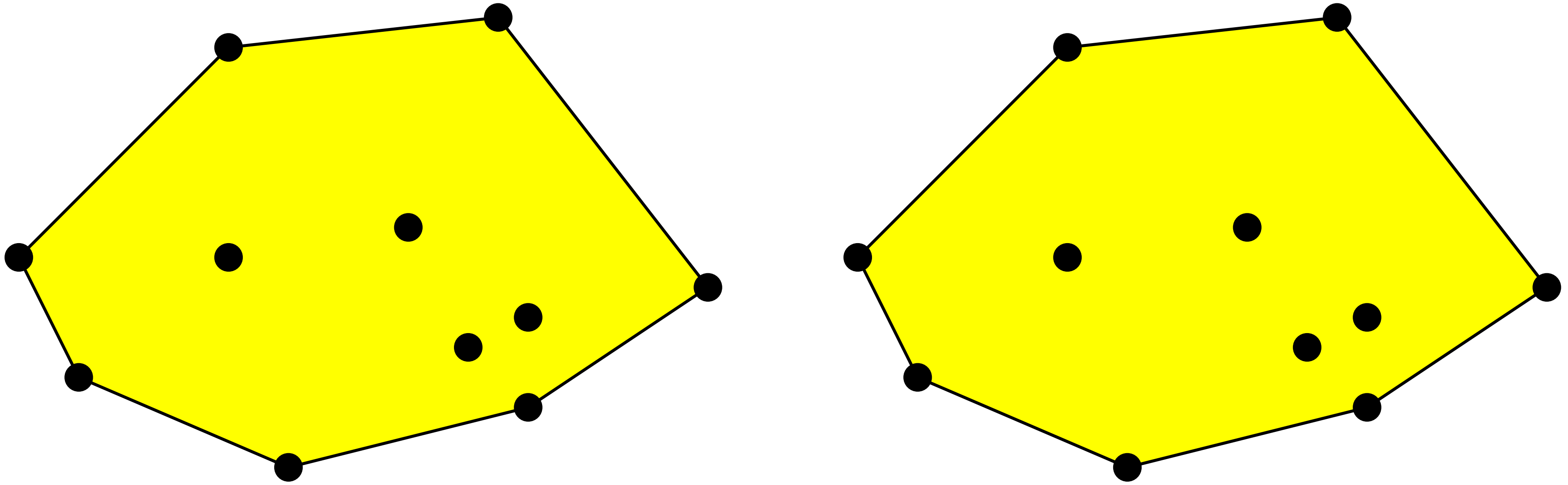
- Basic property
- Decompositions
- Dynamic Programming

Basic Property

S a point set, $\mathcal{T}$ a triangulation of S, p the leftmost pt of S

Obs 1 One of the following two happens.

- p forms a triangle together with its neighbors on the bd.
- $\exists$ an x-monotone inner path from p.



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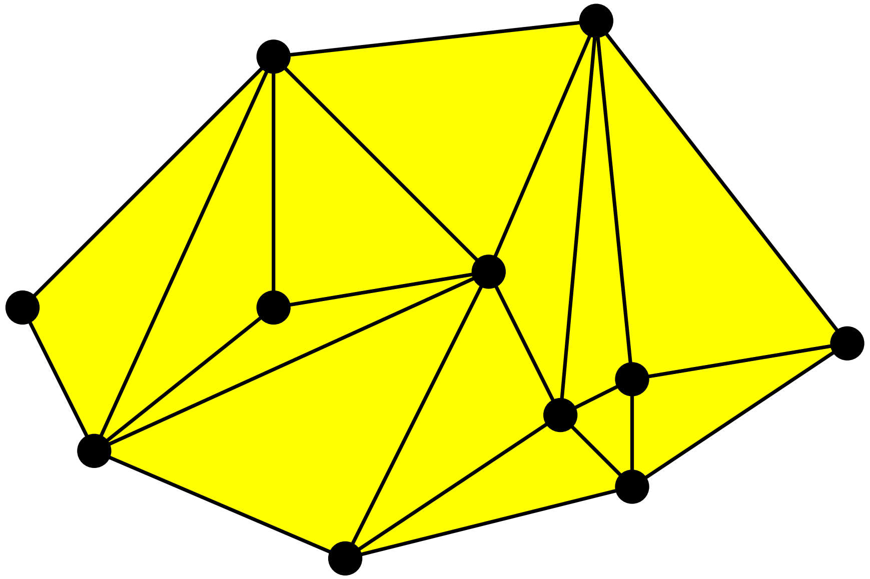

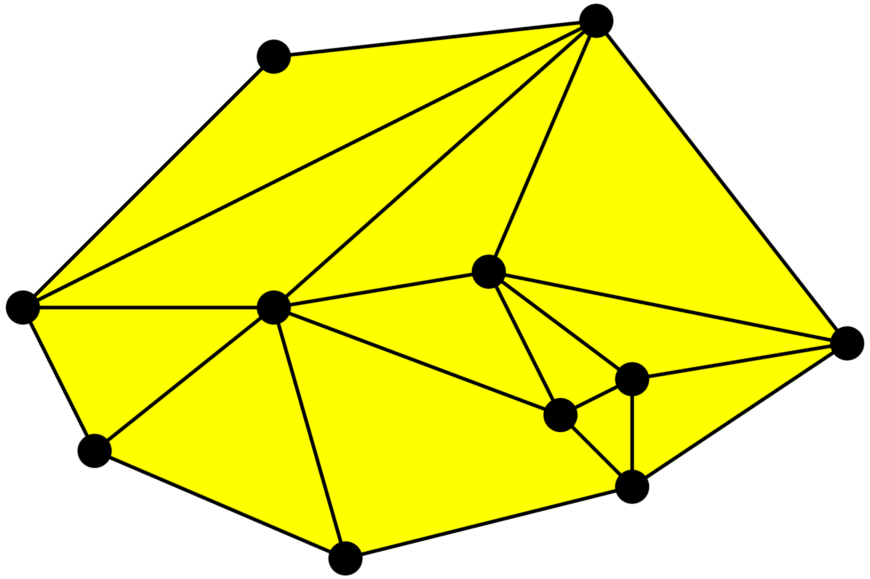

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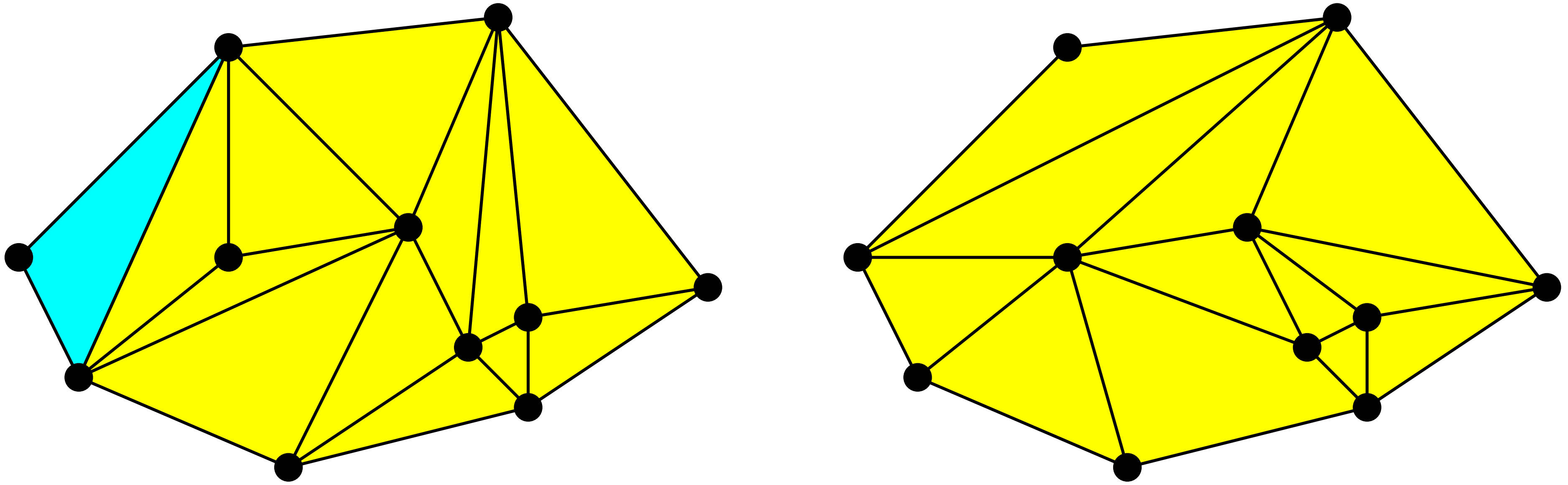

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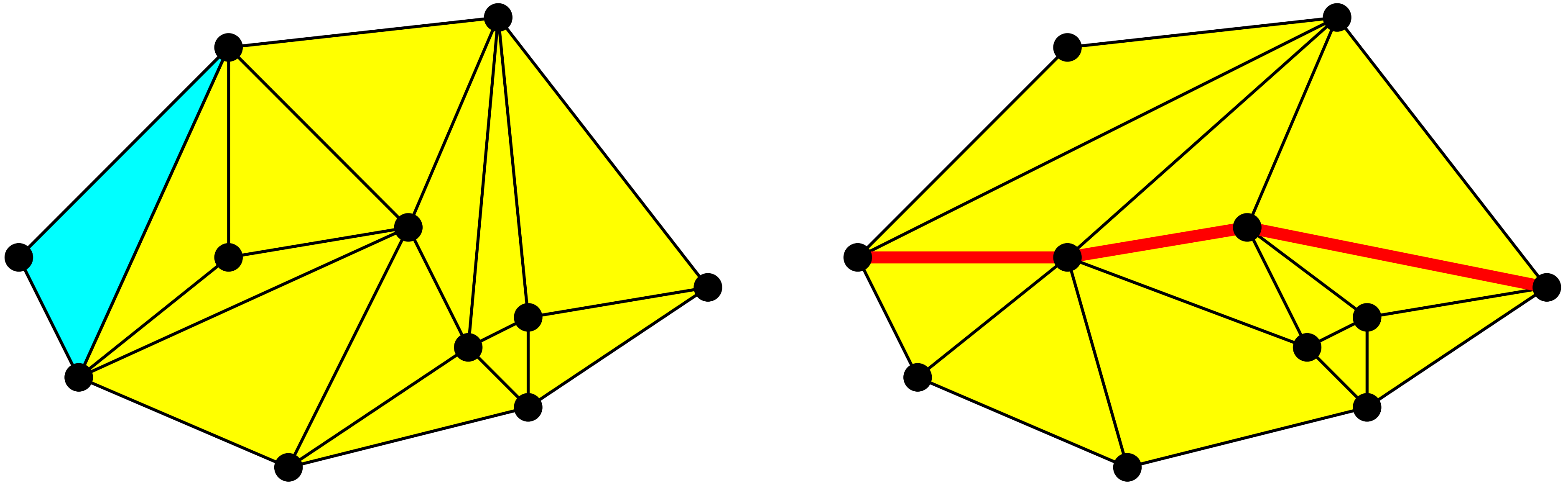

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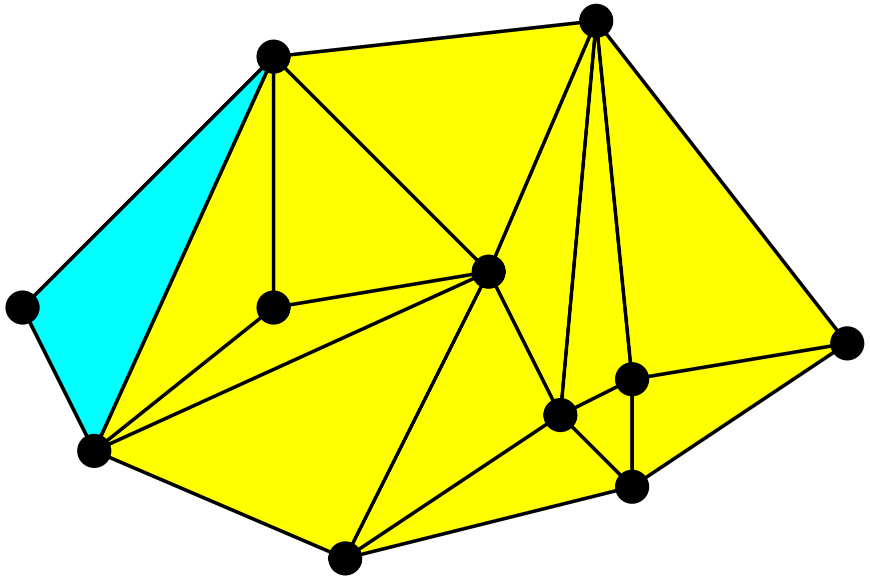

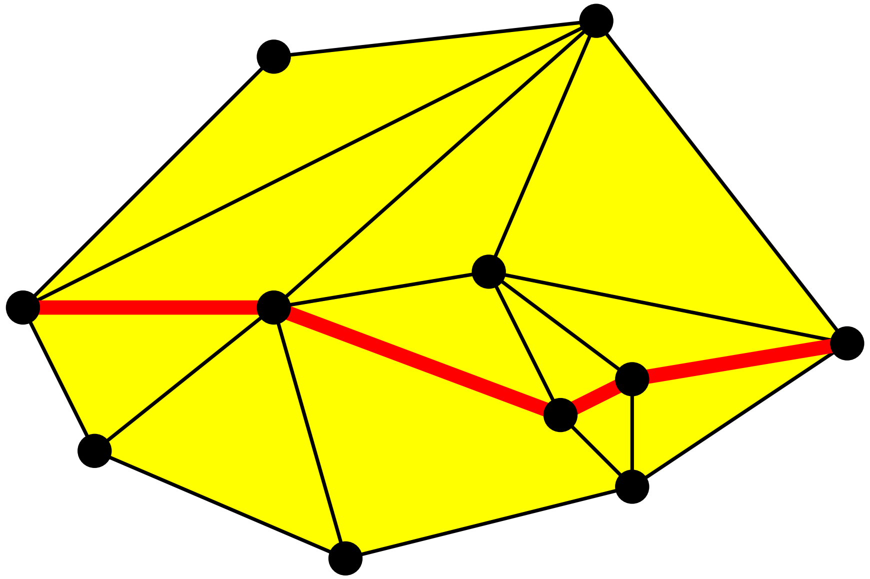

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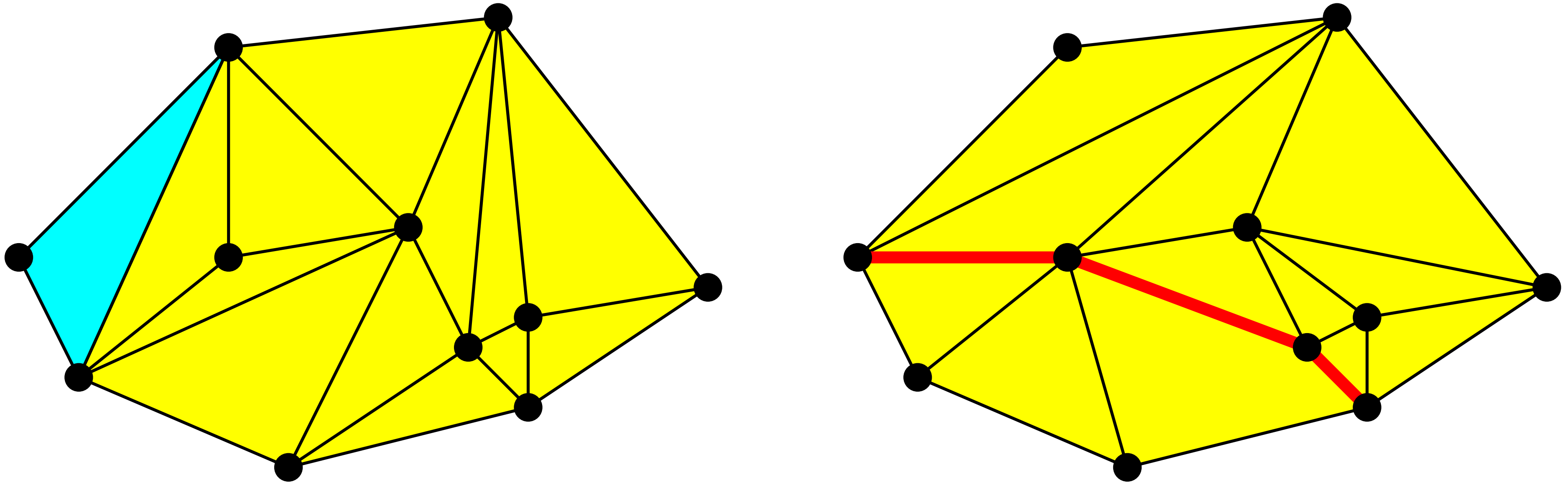

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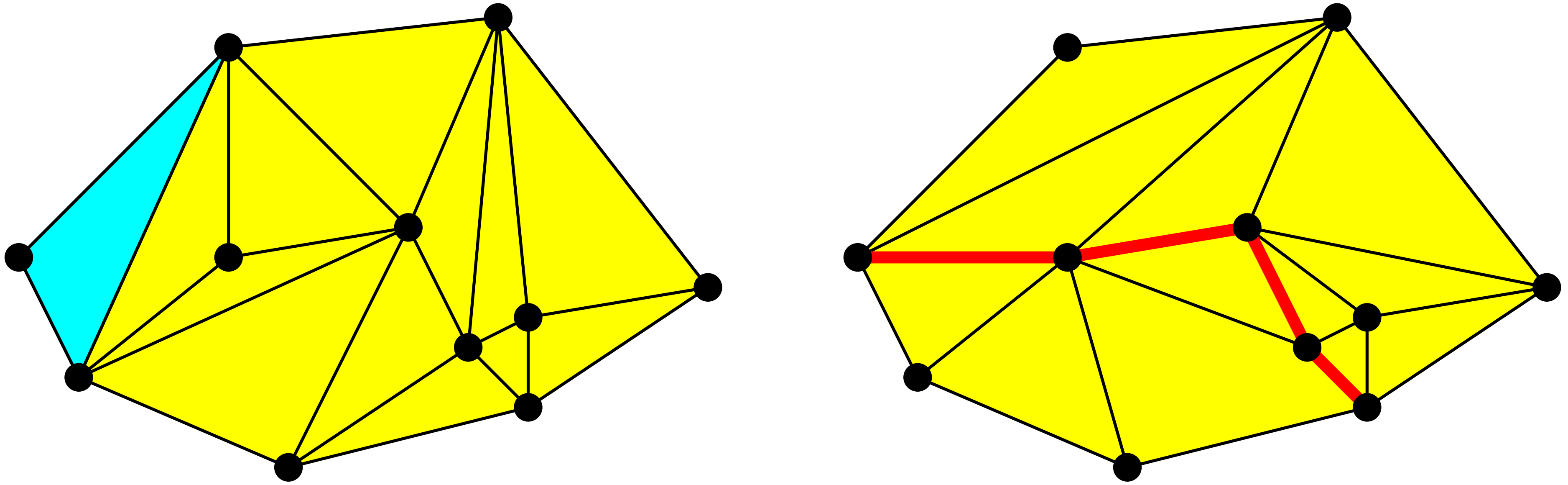

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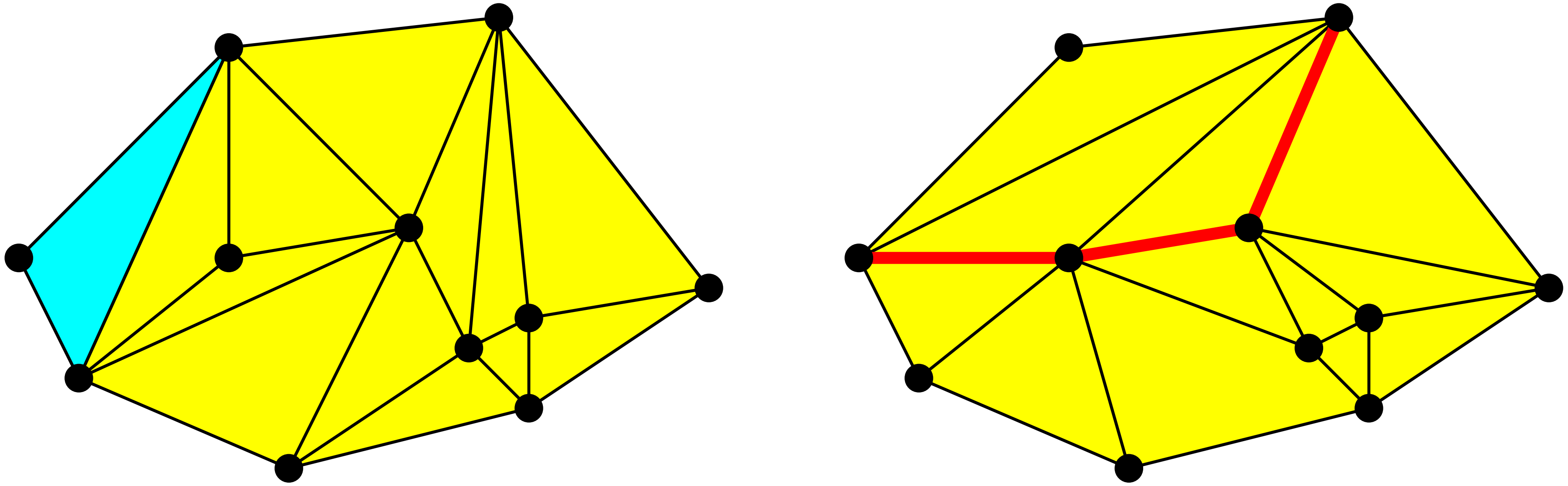

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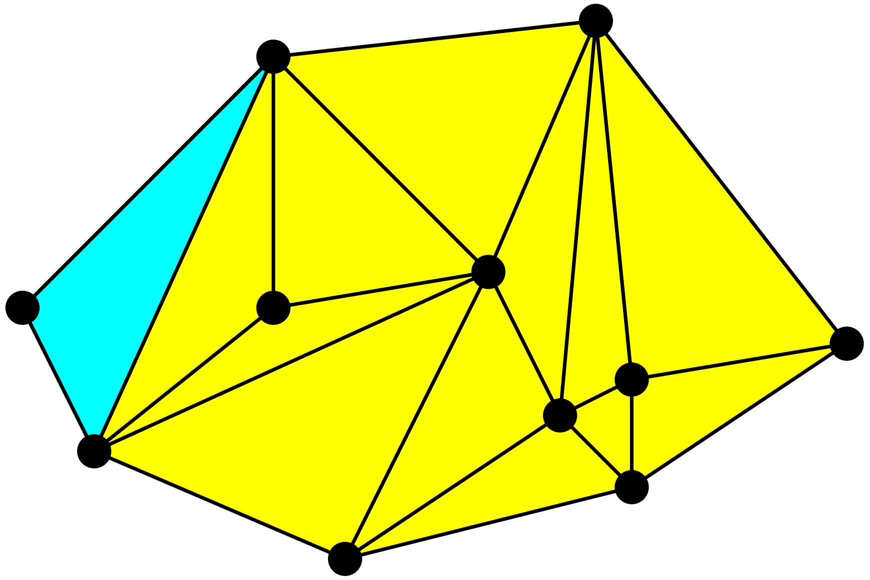
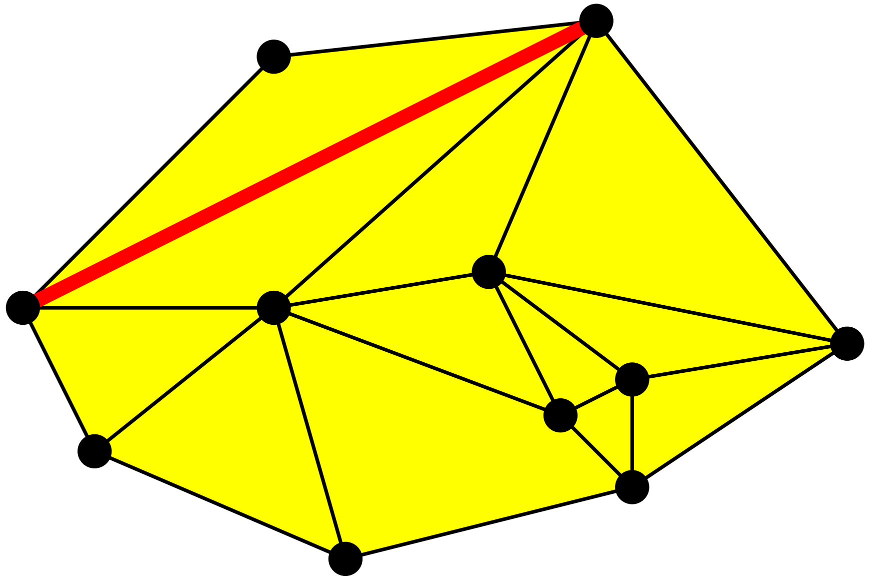




Strategy

S a point set, $\mathcal{T}$ a triangulation of S, p the leftmost pt of S

Obs 1 One of the following two happens.

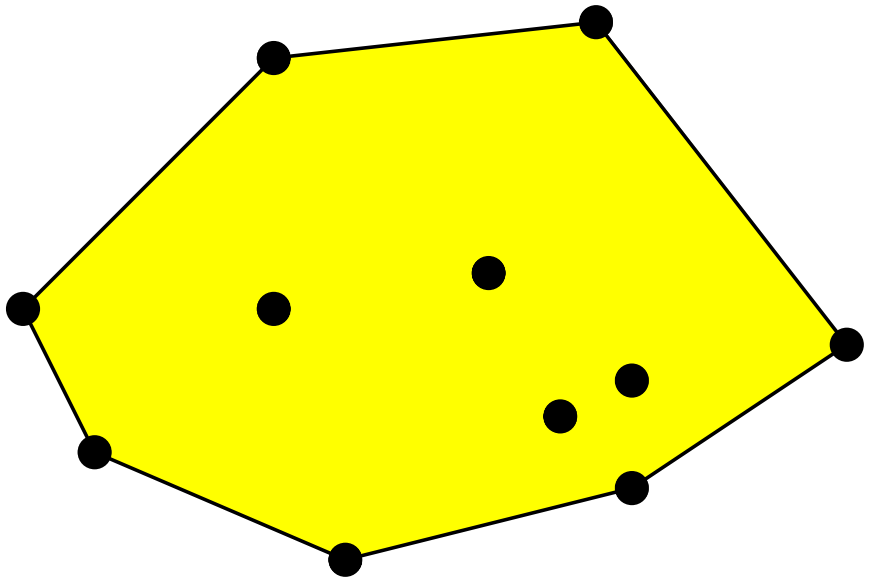
- ◆ p forms a triangle together with its neighbors on the bd.
- ◆ $\exists$ an x-monotone inner path from p.

In particular, this holds for min weight triangulations.

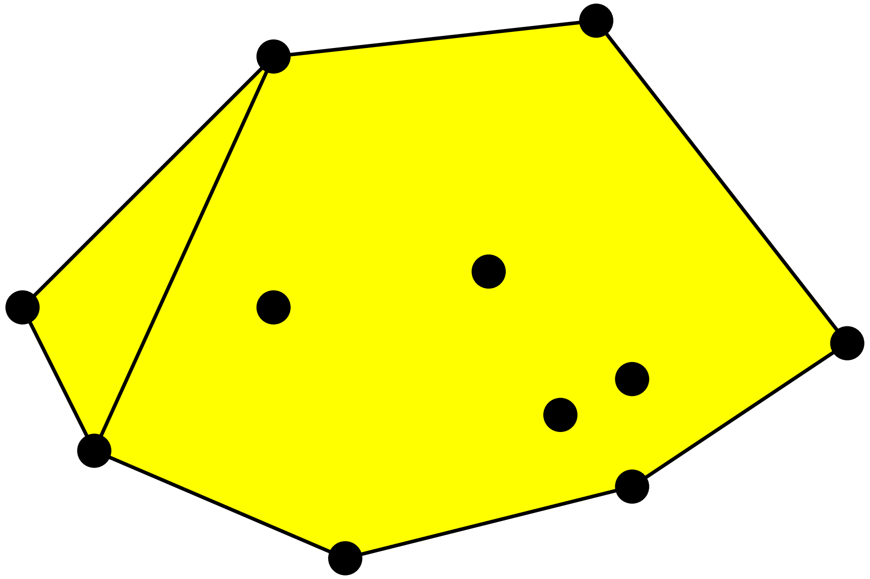
$$\Downarrow$$

Try all possible cases!!

Given a point set...



In a triangulation, this happens?



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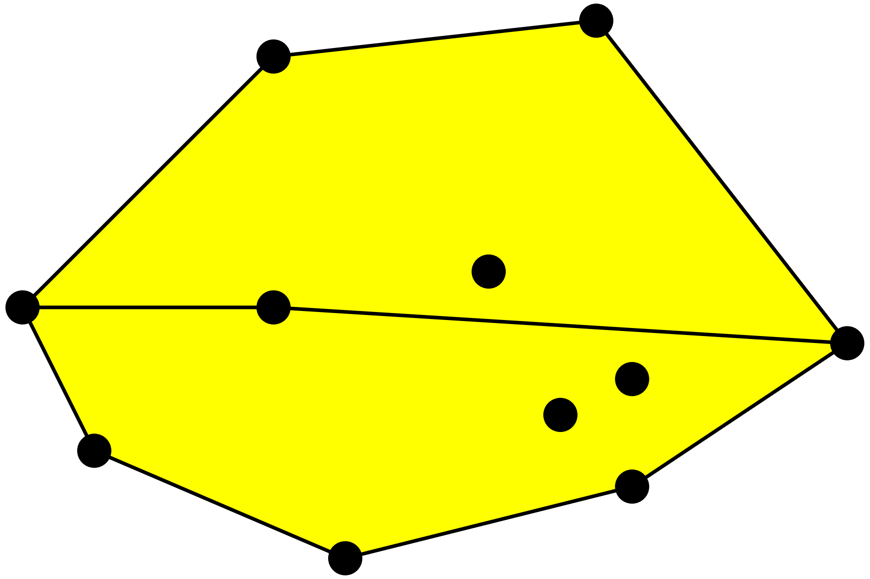

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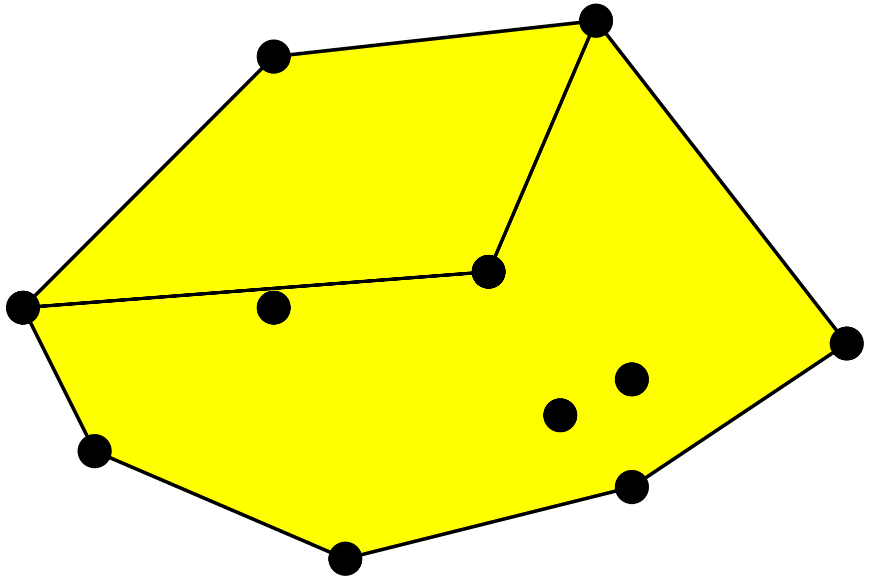

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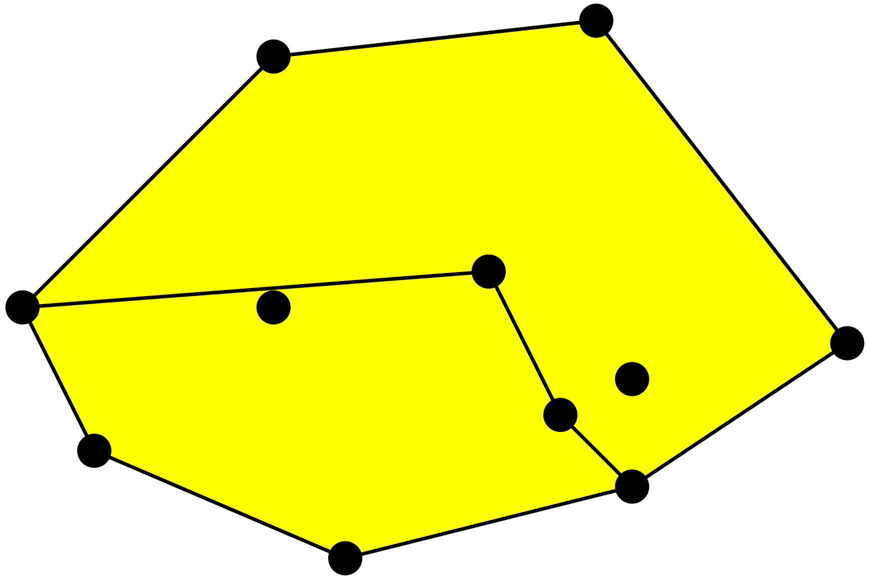
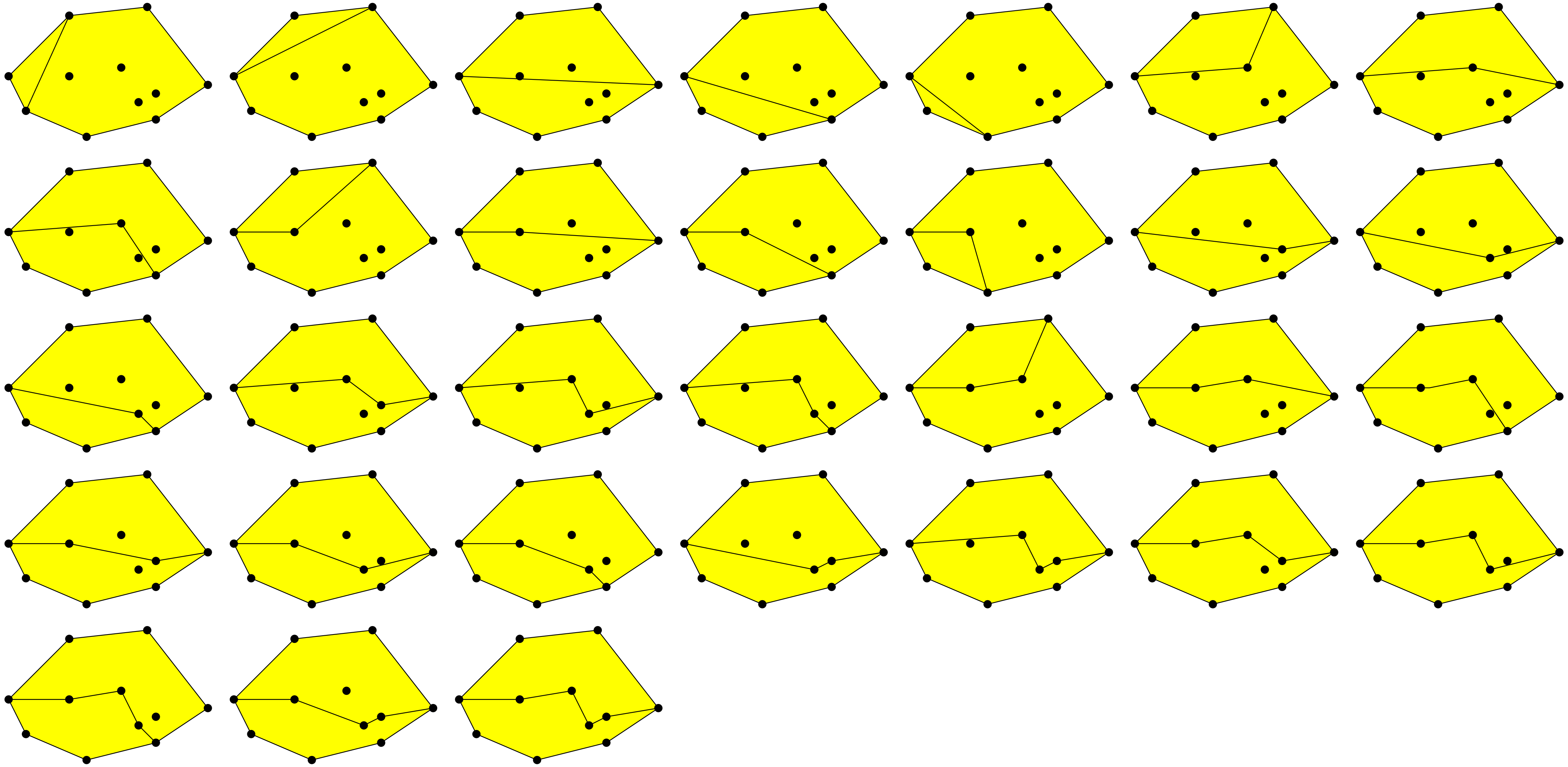


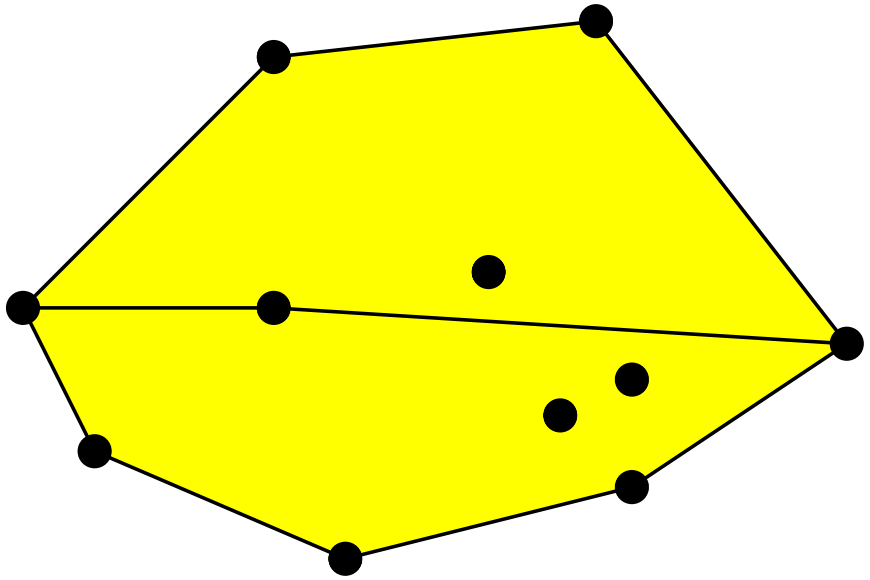
Try all possible cases!!



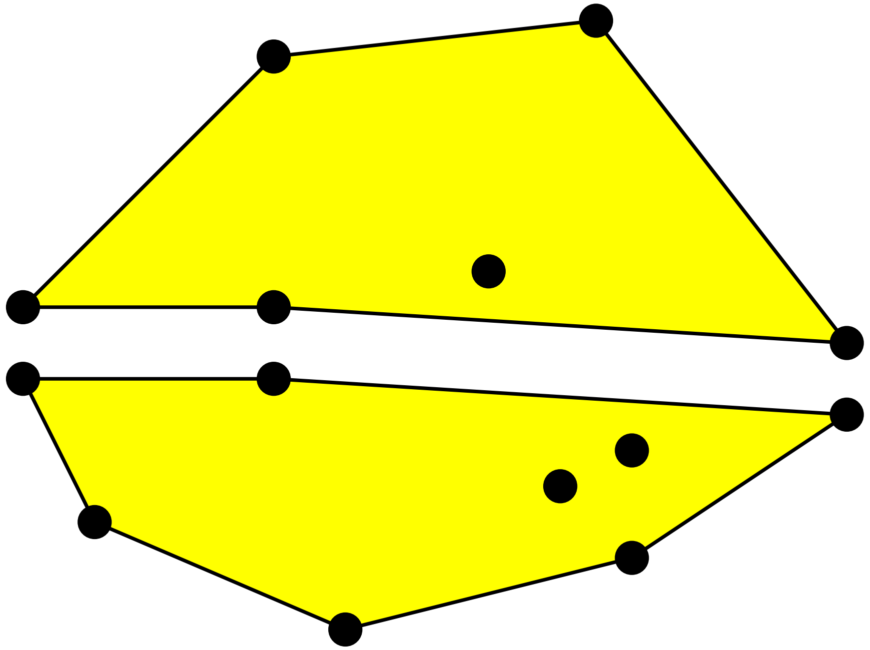
A min weight triangulation fits at least one of these cases.

 # possible cases $\leq 1 + 2^k n$.

Look at one of the cases...



Divide into two pieces!!



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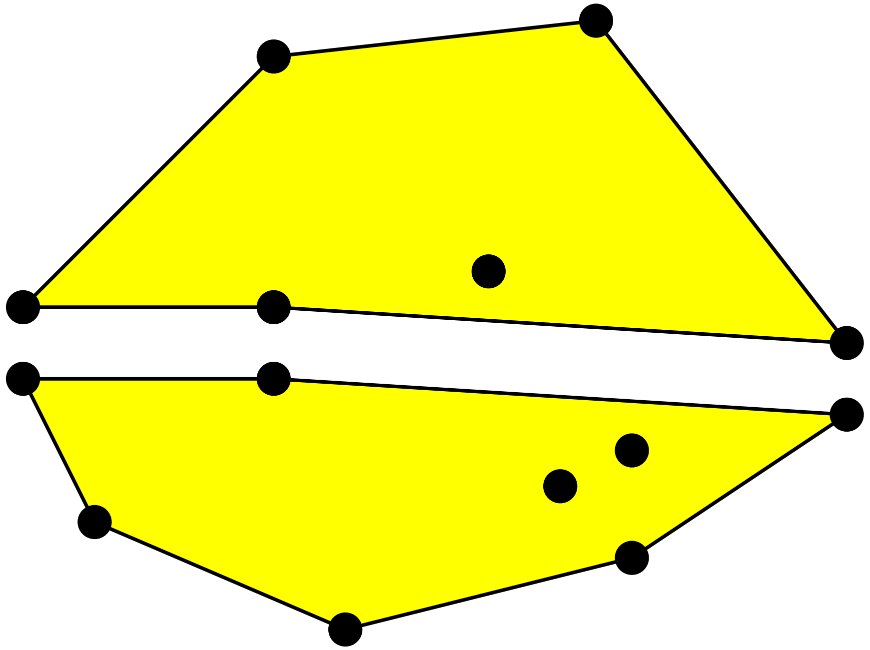
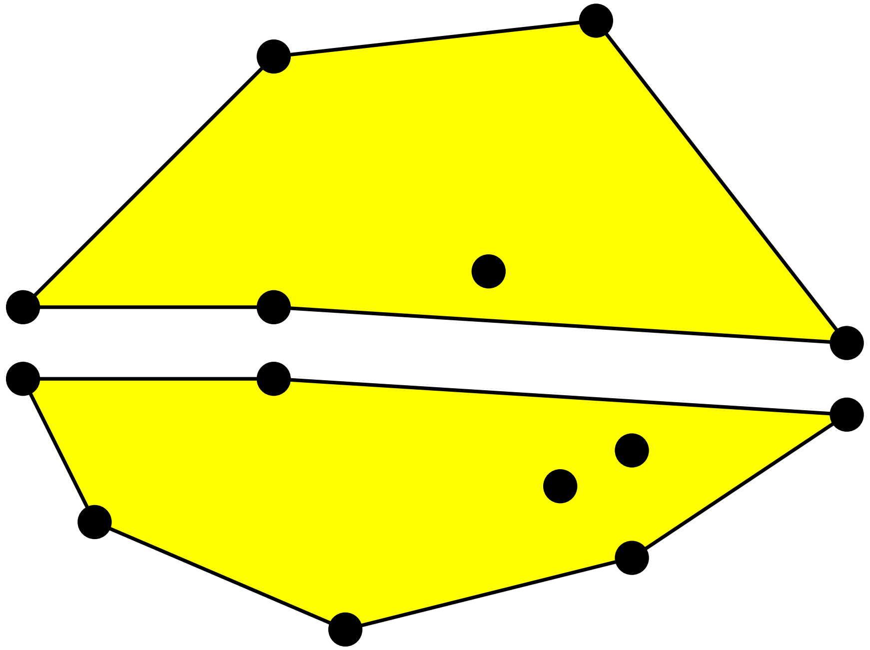


⇒ **Recursively solve on the smaller polygons!!**

Divide into two pieces!!



⇒ **Recursively solve on the smaller polygons!!**

Two Problems:

- Smaller polygons can be non-convex.
- Top-down recursion will not give FPT.

Two problems to resolve

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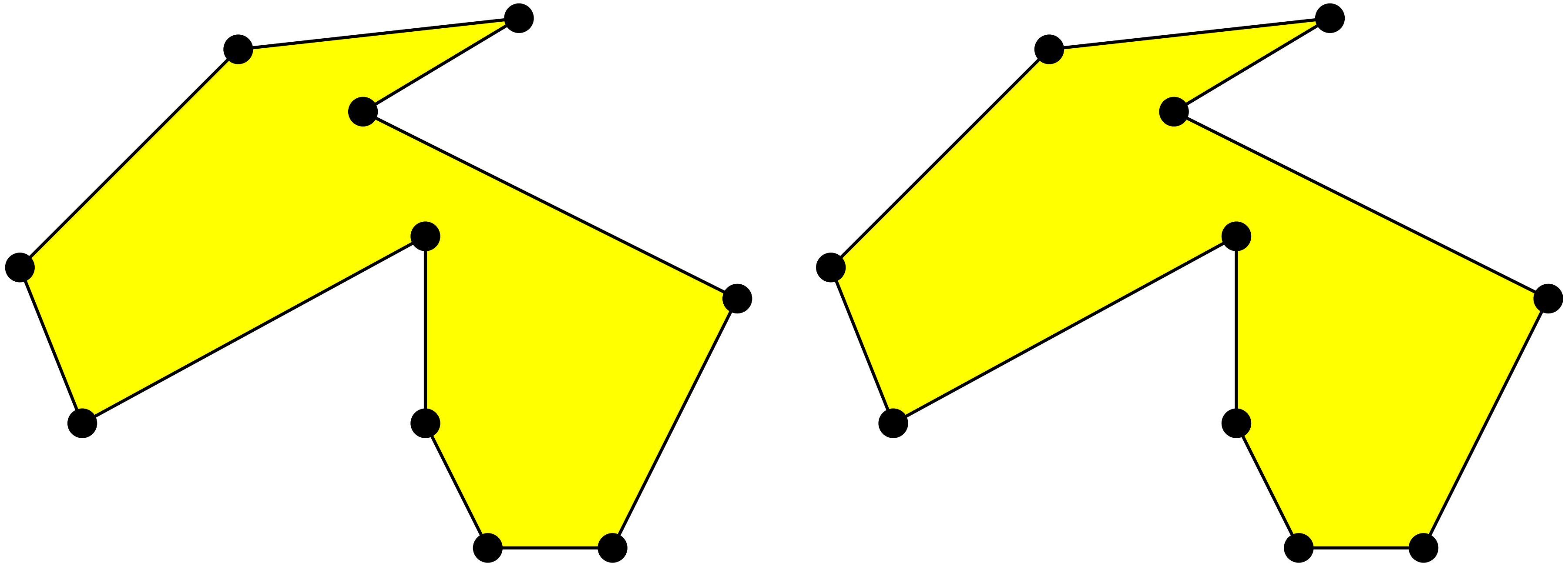
Two Problems:

- Smaller polygons can be non-convex.
  Generalize to simple polygons!!
- Top-down recursion will not give FPT.

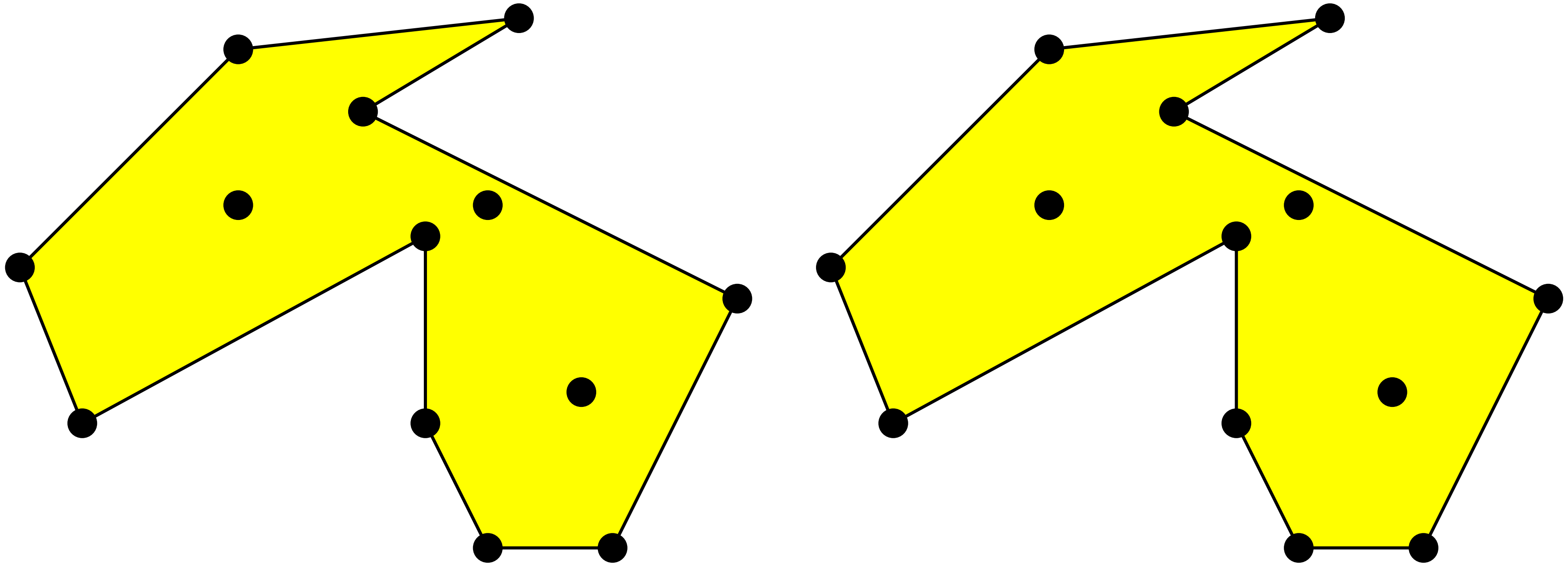
Obs 1 also holds for simple polygons

P a simple polygon



Obs 1 also holds for simple polygons

P a simple polygon, S a point set inside P,



Obs 1 also holds for simple polygons

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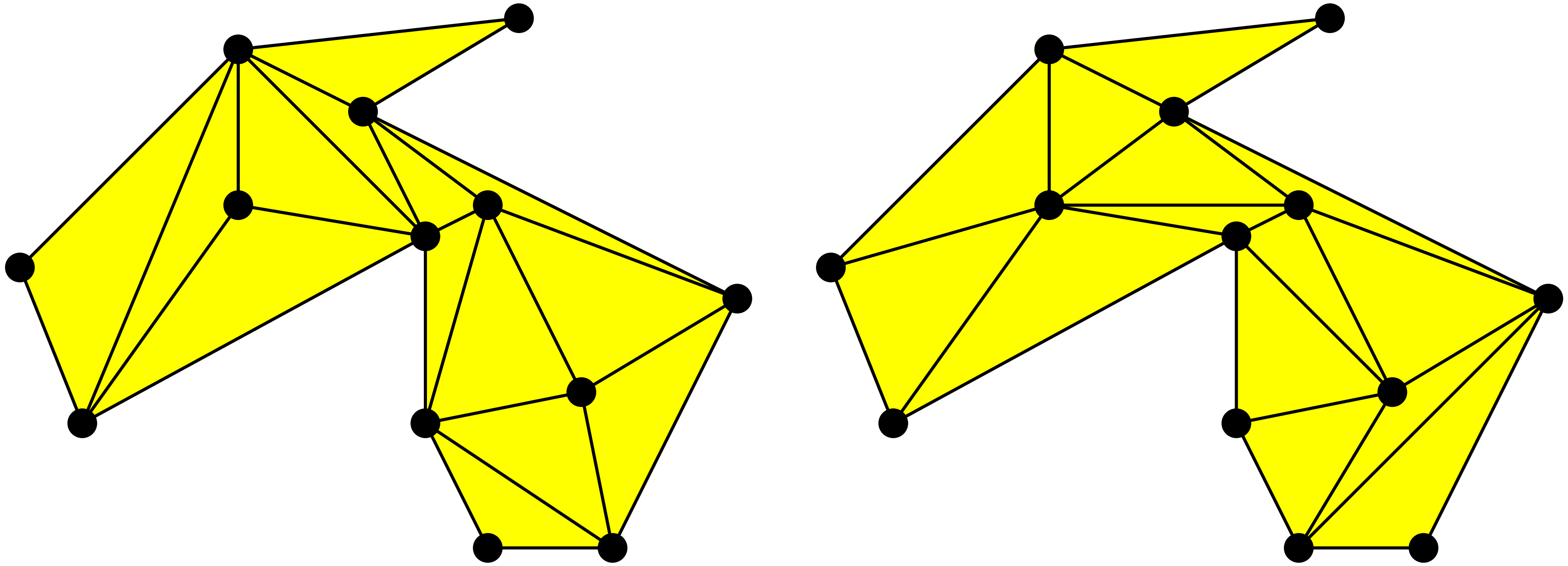

Obs 1 also holds for simple polygons

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Obs 1' One of the following two happens.

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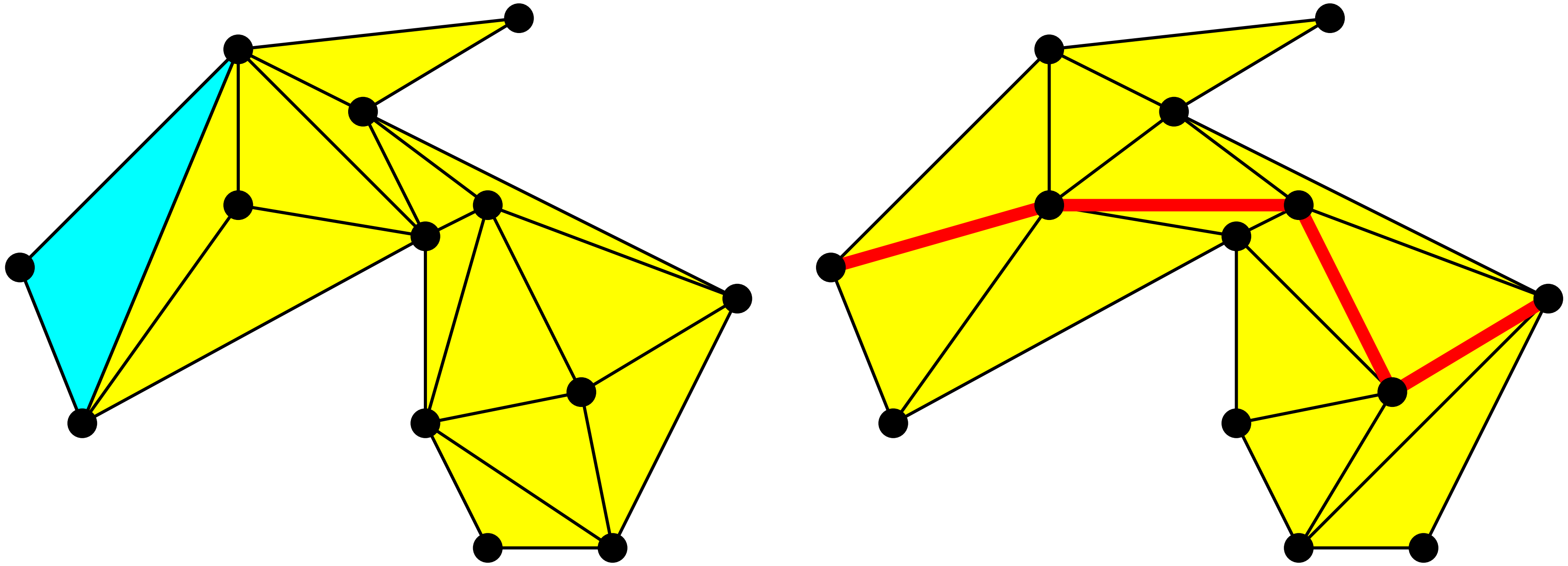

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Two Problems:

- Smaller polygons can be non-convex.
  Generalize to simple polygons!!
- Top-down recursion will not give FPT.
  Solve in a bottom-up manner by DP!!

Outline of the algorithm

(1) Enumerate all small polygons appearing in the subdivisions.

(2) Determine which polygons arise from which polygons

(3) Solve the recursion in a bottom-up manner by DP.

Two problems to resolve

Two Problems:

- Smaller polygons can be non-convex.
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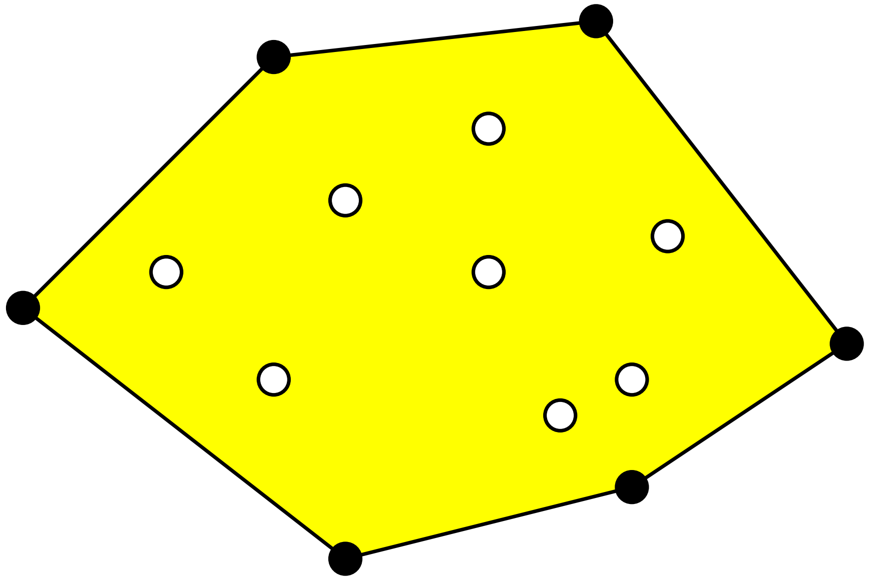
Two problems to resolve

Two Problems:

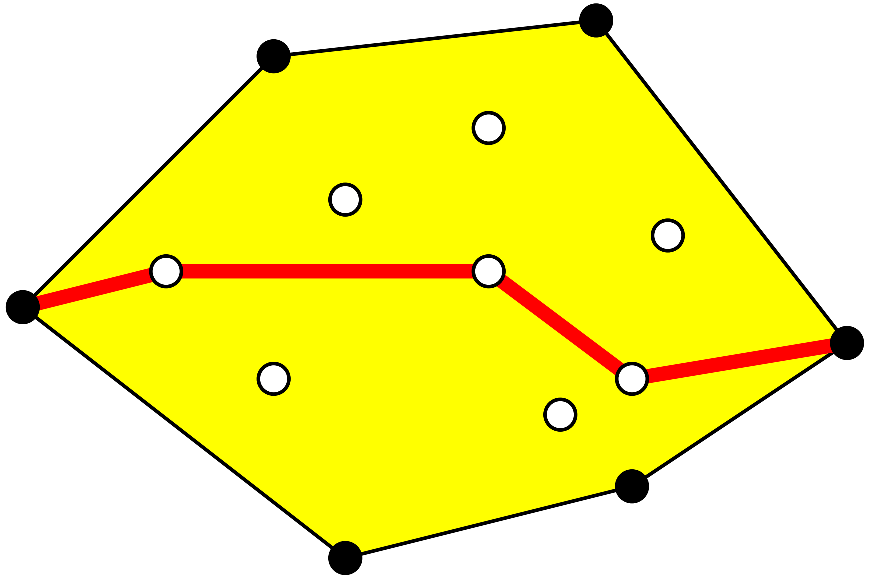
- Smaller polygons can be non-convex.
  Generalize to simple polygons!!
- Top-down recursion will not give FPT.
  Solve in a bottom-up manner by DP!!
  Need to control the types of polygons appearing in subdivisions.

Introduce three types of polygons ...

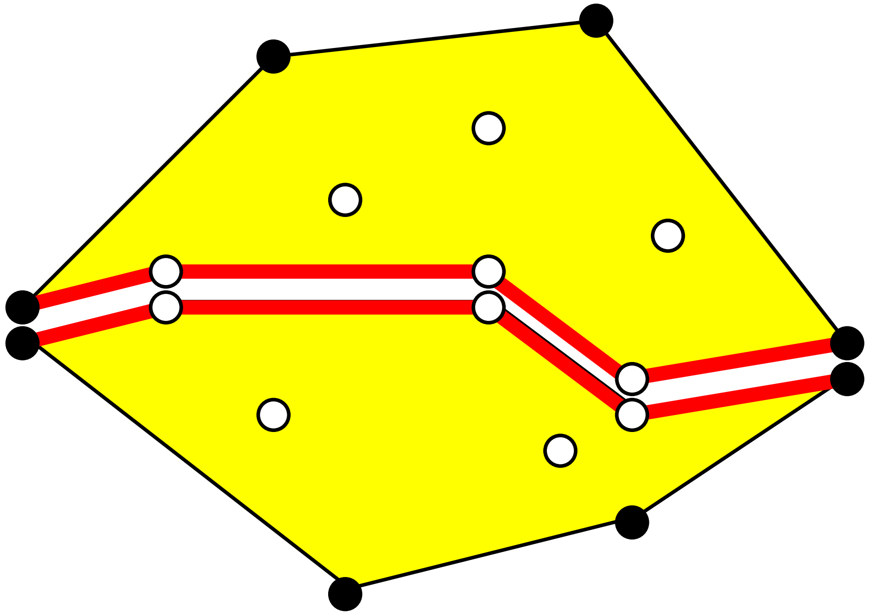
Again, given a point set...



Take an x-monotone inner path...



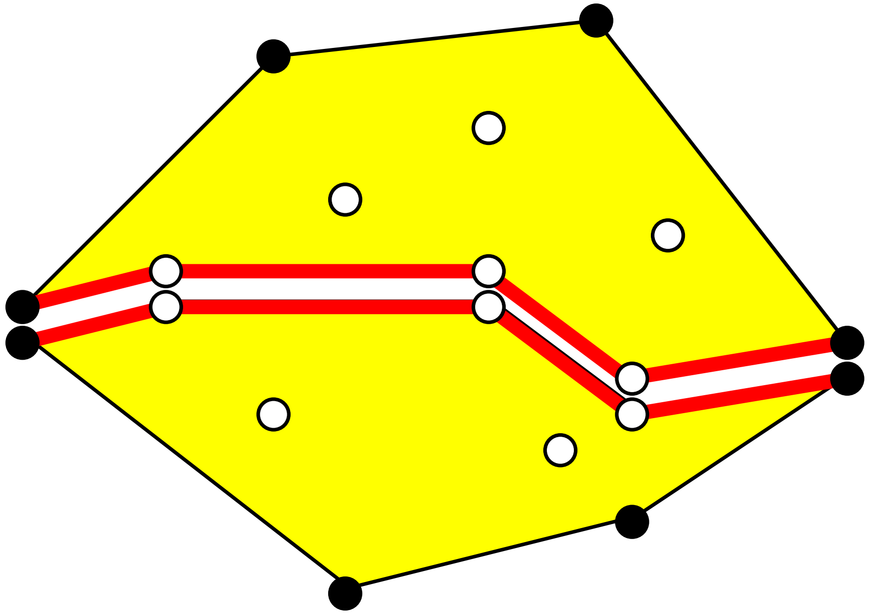
Then, subdivide



Type-1 polygons

A **type-1 polygon**:

the boundary consists of
one x-monotone inner path of the orig polygon and
one boundary chain of the orig polygon.



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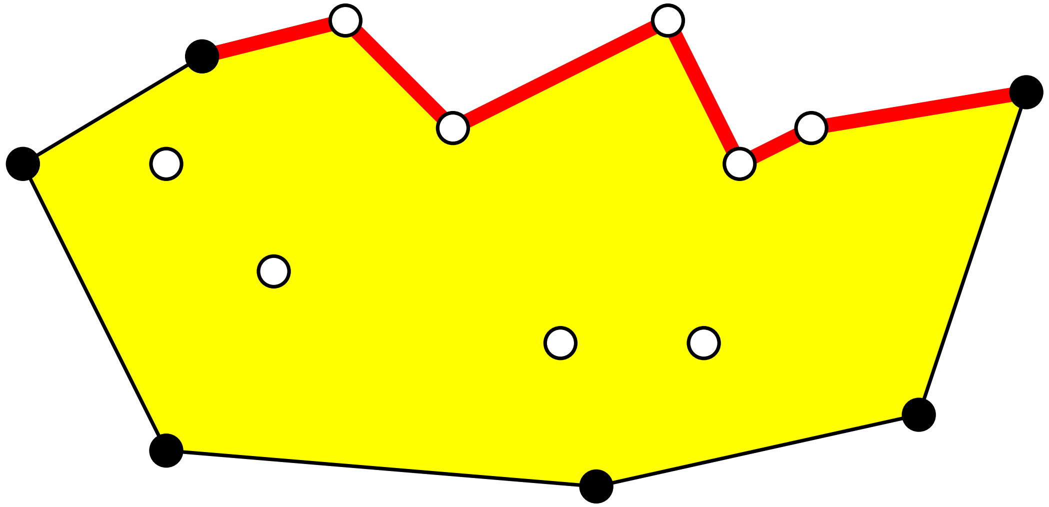

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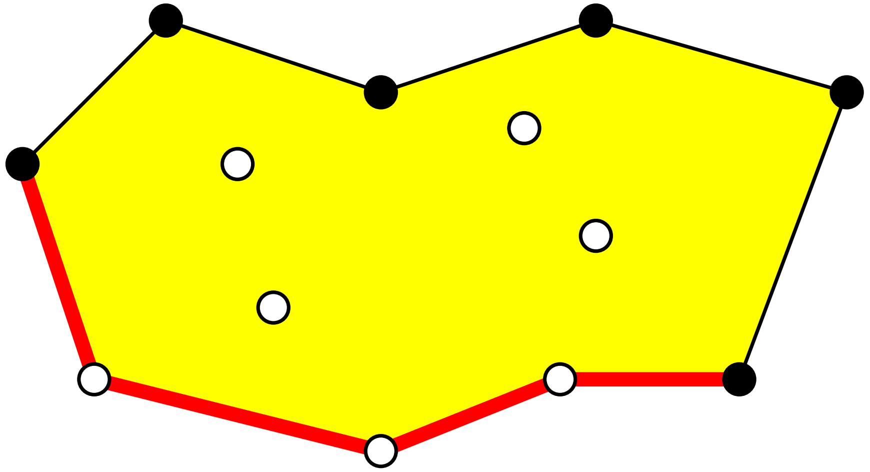
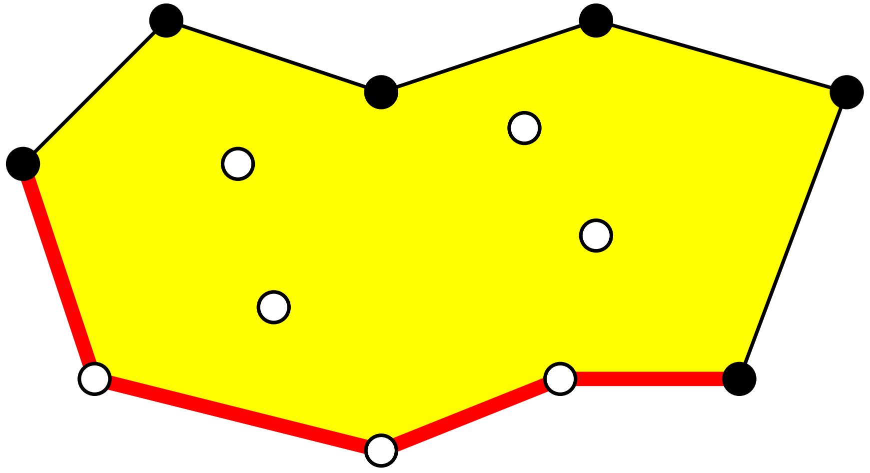
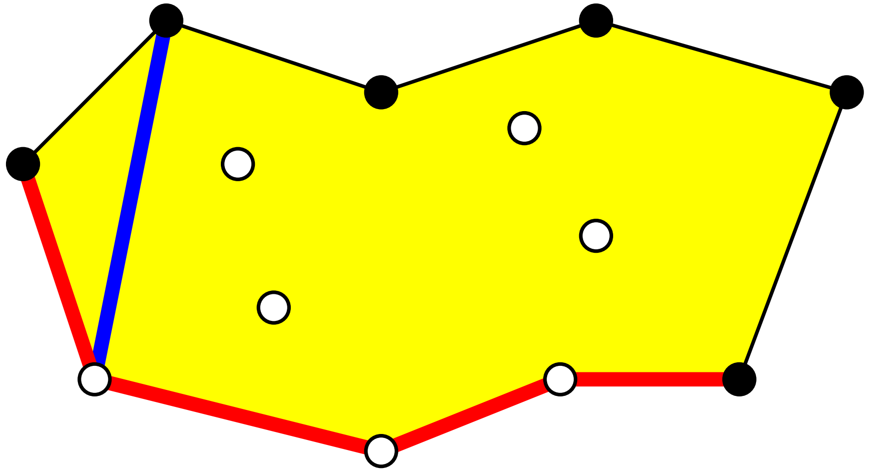


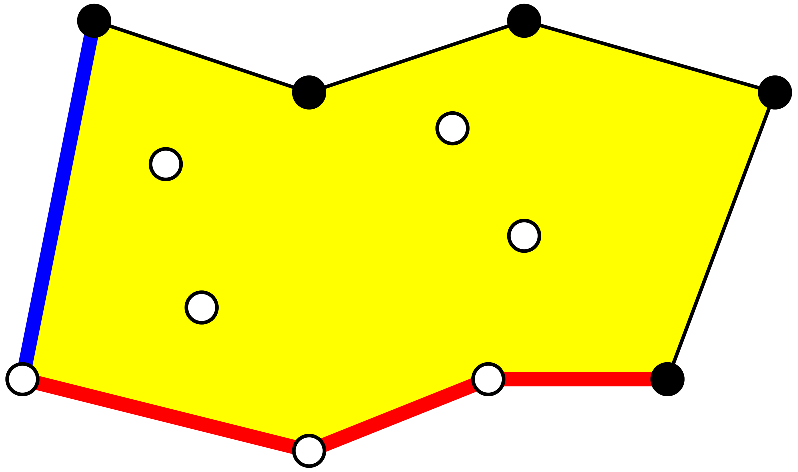
Take this polygon



Subdivide this way



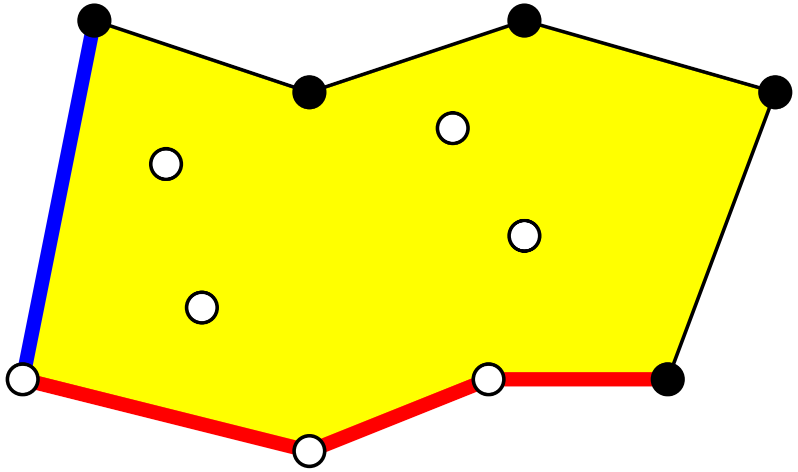
Look at this part



Type-2 polygons

A **type-2 polygon**:

the boundary consists of
two x-monotone inner paths of the orig polygon and
one boundary chain of the orig polygon.



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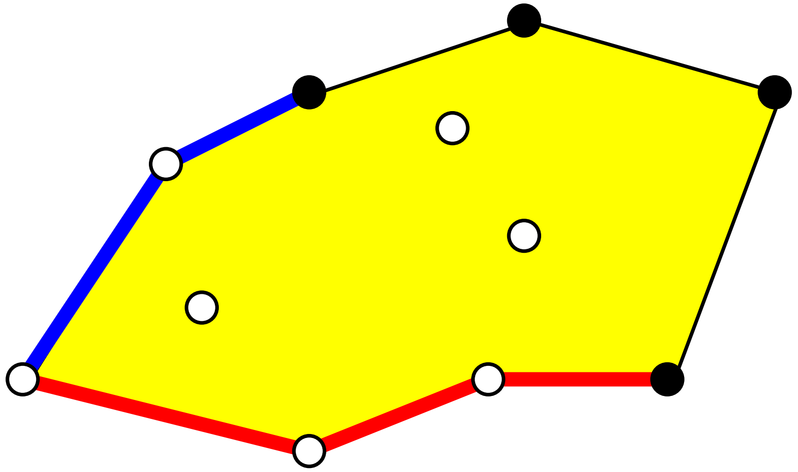
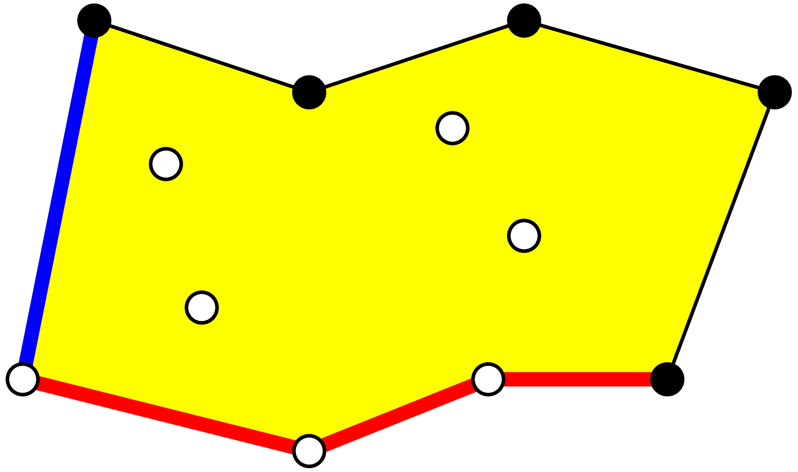
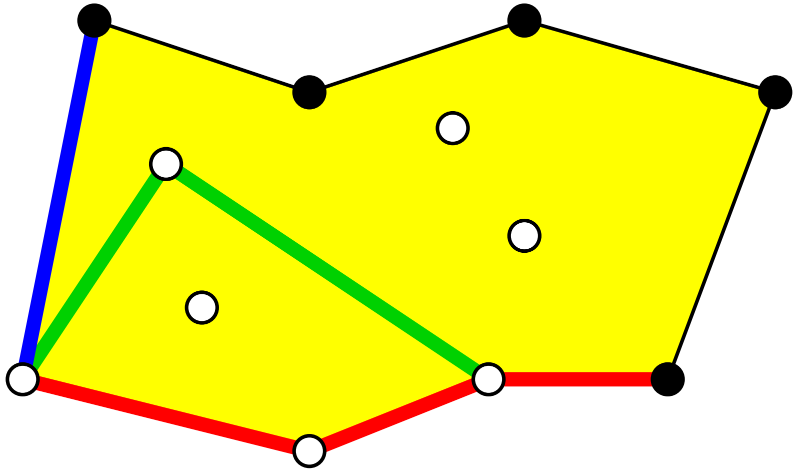


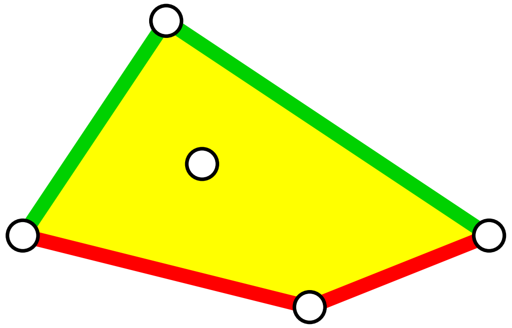
Look at this polygon



Take this inner path



Then subdivide



Type-3 polygons

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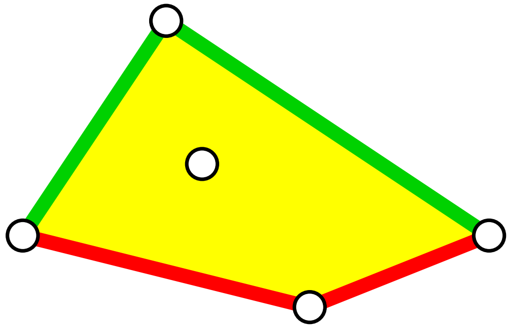
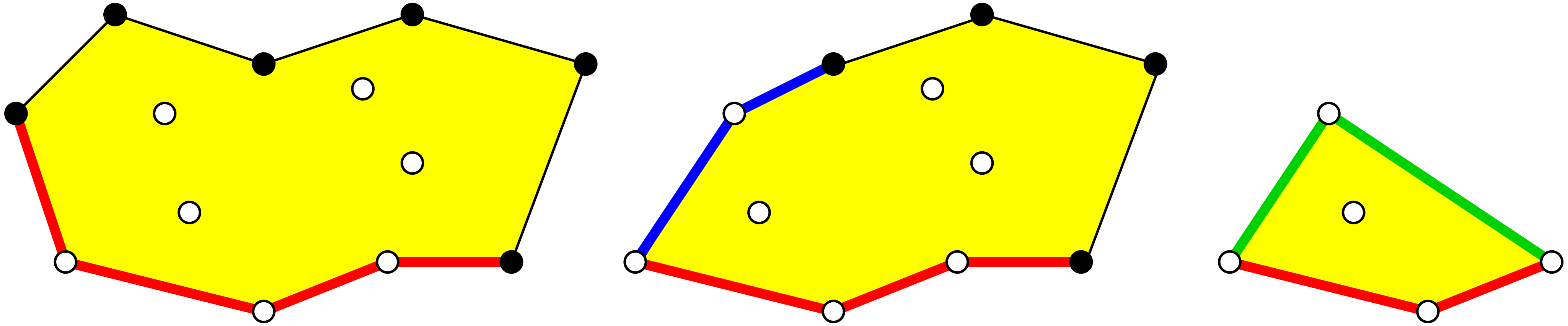


Important lemma

By **subdividing polygons carefully**, we can show that

> **Small polygons appearing in the subdivisions are limited to type-1, 2, 3 polygons.**



Outline of the algorithm

(1) Enumerate all small polygons appearing in the subdivisions.

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(3) Solve the recursion in a bottom-up manner by DP.

Outline of the algorithm

(1) **Enumerate all small polygons appearing in the subdivisions.**

- # of small polygons we enumerate $= O(3^k n^3)$.
- Can be done in $O(3^k n^4 \log n)$ time.

(2) **Determine which polygons arise from which polygons**

(3) **Solve the recursion in a bottom-up manner by DP.**

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(2) **Determine which polygons arise from which polygons**

- Can be done in $O(6^k n^5 \log n)$ time.

(3) **Solve the recursion in a bottom-up manner by DP.**

Outline of the algorithm

(1) **Enumerate all small polygons appearing in the subdivisions.**

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In total: **runs in $O(6^k n^5 \log n)$ time.**

Result (again)

The minimum weight triangulation problem can be solved in $O(6^k n^5 \log n)$ time.

n	the total number of points
k	the number of inner points

Consequences:

The minimum weight triangulation problem

- is **fixed parameter tractable** (FPT) w.r.t. the number of inner points.
- can be solved in **polynomial time if $k = O(\log n)$**.

Variation

The maximum weight triangulation problem can be solved in $O(6^k n^5 \log n)$ time.

n	the total number of points
k	the number of inner points

Consequences:

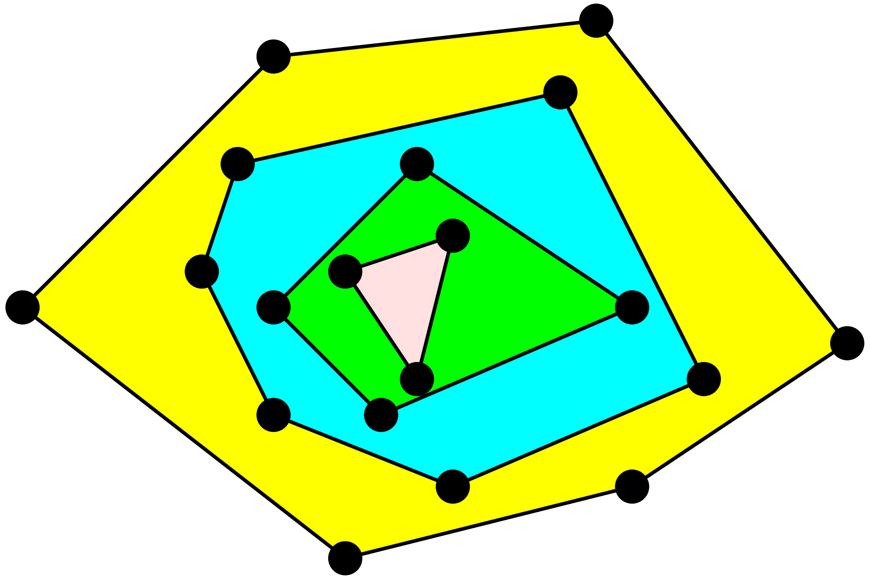
The maximum weight triangulation problem

- is **fixed parameter tractable** (FPT) w.r.t. the number of inner points.
- can be solved in **polynomial time** **if $k = O(\log n)$**.

Related past work

- **Not known to be NP-hard**
- **Approximation algorithms**
  - $O(1)$-approximation (Levcopoulos & Krznaric '98)
- **Exact algorithms** without run-time analysis
  - Integer prog. (Kyoda, Imai, Takeuchi & Tajima '97)
  - “Paths of a triangulation” (Aichholzer '99)
- **Parameterization**
  - Nested convex hulls (Anagnostou & Corneil '93)

Nested convex hulls



k the number of “nests”

A min weight triangulation can be found in $O(n^{3k+1})$ time. (Anagnostou & Corneil ’93)

Thank you

Tusen takk.