The forbidden minor characterization of line-search antimatroids of rooted digraphs

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Abstract

An antimatroid is an accessible union-closed family of subsets of a finite set. A number of classes of antimatroids are closed under taking minors such as point-search antimatroids of rooted (di)graphs, line-search antimatroids of rooted (di)graphs, shelling antimatroids of rooted trees, shelling antimatroids of posets, etc. The forbidden minor characterizations are known for point-search antimatroids of rooted (di)graphs, shelling antimatroids of rooted trees and shelling antimatroids of posets. In this paper, we give the forbidden minor characterization of line-search antimatroids of rooted digraphs.

Key Words: Antimatroid, Forbidden minor, Line graph, Line-search antimatroid

1 Introduction

Various kinds of shelling procedures give rise to a class of combinatorial structures called antimatroids, which were introduced by Edelman [2] and Jamison-Walder [5]. Antimatroids can be seen as a combinatorial abstraction of convexity, while matroids can be seen as a combinatorial abstraction of linear independence. Antimatroids are related to matroids in that both can be defined by a apparently similar axioms. This close relationship between antimatroids and matroids provides a lot of interesting properties of antimatroids. For example, antimatroids can be characterized by a greedy algorithm like matroids [1]. Note that one of the authors has recently given a greedy-algorithmic characterization of non-simple antimatroids, which is an extension of antimatroids [9].

Both antimatroids and matroids are subclasses of greedoids introduced by Korte–Lovász [6]. See [8] for details and various examples of greedoids. In greedoid theory, some classes are characterized by their forbidden minors: local poset greedoids [7]; undirected branching greedoids [3, 13], and poset-shelling antimatroids and point-search antimatroids of rooted (di)graphs [10]. In this paper, we give the forbidden minor characterization for line-search antimatroids of rooted digraphs.

Note that there are still other antimatroids whose forbidden minor characterizations have not been known yet; for example, line-search antimatroids of rooted undirected graphs.

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2 Preliminaries

2.1 Antimatroids

Let E be a nonempty finite set, and let $\mathcal{F}$ be a family of subsets of E such that

$$\emptyset \in \mathcal{F}, E \in \mathcal{F}; \tag{1}$$

$$\text{if } X \in \mathcal{F} \setminus \{\emptyset\}, \text{ then there exists an } e \in X \text{ such that } X \setminus \{e\} \in \mathcal{F}; \tag{2}$$

$$\text{if } X, Y \in \mathcal{F}, \text{ then } X \cup Y \in \mathcal{F}. \tag{3}$$

Then we call $(E, \mathcal{F})$ an *antimatroid* on E. When there is no risk of confusion, we use $\mathcal{F}$ instead of $(E, \mathcal{F})$. Each element of $\mathcal{F}$ is called a *feasible set*.

For an antimatroid $\mathcal{F}$, a *minor* $\mathcal{F}[A, B]$ is defined as follows:

$$\mathcal{F}[A, B] = \{X \setminus A : X \in \mathcal{F}, A \subseteq X \subseteq B\}, \tag{4}$$

where $A, B \in \mathcal{F}$ and $A \subseteq B$. We can easily check that each minor of an antimatroid is also an antimatroid.

2.2 Point-search antimatroids of rooted digraphs

A *digraph* G is a pair (V, E) such that V is a nonempty finite set of *vertices*, and E is a subset of $\{(x, y) : x, y \in V, x \neq y\}$ called a set of *edges*. For simplicity, we write xy instead of (x, y). For an edge $xy \in E$, x is called the *tail*, and y is called the *head*.

A *path* P in $G = (V, E)$ is a sequence of vertices $x_1 x_2 \cdots x_m$ with $x_i x_{i+1} \in E$ for $i = 1, \ldots, m-1$. A path $P = x_1 \cdots x_m$ is also called a path from x_1 to x_m. For a path $P = x_1 \cdots x_m$, if there exists an edge $x_i x_j \in E$ $(i + 1 < j)$, then the edge $x_i x_j$ is called a *short-cut* of the path P. A path without repeated vertices is called *elementary*. An elementary path without any short-cuts is called *straight*.

A *rooted digraph* is a triple $G = (V, E, r)$ where $(V \cup \{r\}, E)$ is a digraph and r is a specified vertex called the *root* such that there exists a path from r to every vertex of V. A path from the root r is called a *rooted path*. A vertex v is called an *atom* if $rv \in E$.

For a rooted digraph $G = (V, E, r)$, we consider the following procedure: first we choose one of the atoms, say v; next we shrink v to the root. If we repeat this procedure until all vertices are shrunk to the root, then we will obtain a sequence of vertices selected by the above procedure of shrinking. If we gather all of these sequences, then they form an antimatroid. Formally, for a rooted digraph $G = (V, E, r)$, we define the *point-search antimatroid* $\mathfrak{PS}_D(G)$ as follows:

$$\begin{aligned}\mathfrak{PS}_D(G) = \{X \subseteq V : & \text{ every vertex } v \in X \text{ can be reached by} \\ & \text{a rooted path in the subgraph induced by } X \cup \{r\}\}.\end{aligned} \tag{5}$$

Note that the class of point-search antimatroids is closed under taking minors.

In a rooted digraph $G = (V, E, r)$, let $e = xy \in E$ be an edge of G. Suppose $P = ru_1 u_2 \cdots u_m$ to be a straight rooted path such that $u_{m-1} u_m = e$. Then we say that e *is supported by* P, or P *supports* e. If there is no path supporting e, then e is called a *redundant* edge. If a rooted digraph contains no redundant edge, then it is called an *irredundant* rooted digraph. Note that redundant edges have no use for defining point-search antimatroids. In particular, irredundant rooted digraphs have no edge whose head is the root r or an atom. For a rooted digraph G, define G_0 as the rooted digraph such that the redundant edges of G are deleted, then the point-search antimatroids of G and G_0 are the same. Therefore, without loss of generality, when we consider point-search antimatroids of rooted digraphs, we only have to handle irredundant ones.

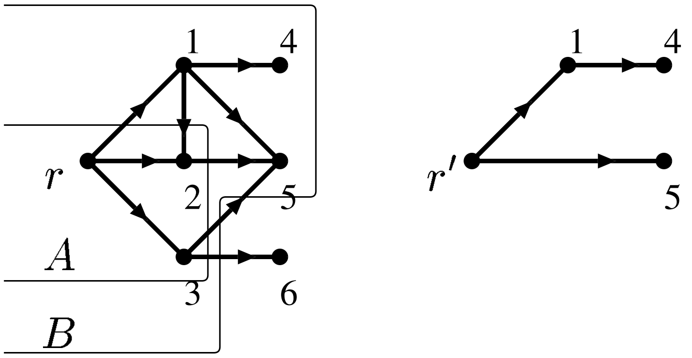

Figure 1: A rooted digraph and a rooted minor.

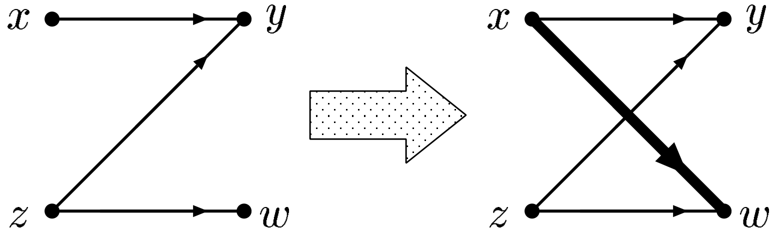

Figure 2: The Heuchenne condition.

Let $G = (V, E, r)$ be a rooted digraph, and $\mathfrak{PS}_D(G)$ be the point-search antimatroid of G. For $A, B \in \mathfrak{PS}_D(G)$ with $A \subseteq B$, remove $V \setminus B$ and the edges incident to $V \setminus B$ from G, shrink the vertices A to r. Then delete all the redundant edges from the resultant graph. This procedure gives us an irredundant rooted digraph, which we call a *rooted minor* and denote by $G[A, B]$. Figure 1 shows an example of rooted minors. Note that every rooted minor of an irredundant rooted digraph is also irredundant. Clearly, the point-search antimatroid of $G[A, B]$ is equal to the minor $\mathfrak{PS}_D(G)[A, B]$, namely $\mathfrak{PS}_D(G[A, B]) = \mathfrak{PS}_D(G)[A, B]$. Furthermore, suppose G' to be another irredundant rooted digraph. Then $\mathfrak{PS}_D(G)$ contains a minor isomorphic to $\mathfrak{PS}_D(G')$ if and only if there exists a rooted minor of G which is isomorphic to G'.

A *multi-digraph* H is a quadruple $(N, A; h, t)$ where N is a nonempty finite set of *nodes*, A is a finite set of *arcs*, and h, t are maps from A to N. For $a \in A$, $h(a) \in N$ is called the *head* of a, and $t(a) \in N$ is the *tail* of a. A digraph is a special case of multi-digraphs. A *path* in H is a sequence of arcs $a_1 \cdots a_k$ such that $h(a_i) = t(a_{i+1})$ for $i = 1, \dots, k-1$. If a path has no repeated arcs, it is called *simple*.

A multi-digraph $H = (N, A; h, t)$ defines a digraph $G = (A, E)$ by $E = \{(a, b) : a, b \in A, a \neq b, h(a) = t(b)\}$, which is called the *line graph* of H. A digraph G is a *line graph* if there exists some multi-digraph of which G is the line graph. Syslo [14] gives a polynomial-time algorithm which decides whether the given digraph is a line graph or not. The algorithm is based on the following characterization of line graphs [4, 11]:

Proposition 1. *Let $G = (V, E)$ be a digraph. G is a line graph if and only if for every $x, y, z, w \in V$, $(x, y), (z, y), (z, w) \in E$ imply $(x, w) \in E$, as shown in Figure 2.*

The condition of this proposition is called the *Heuchenne condition*, or the *H-condition*, for short.

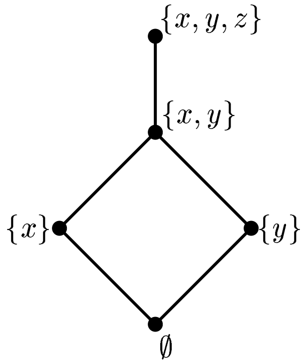

Figure 3: The forbidden minor D_5 of point-search antimatroids of rooted digraphs.

A *rooted multi-digraph* is a quintuple $(N, A, r; h, t)$ where $(N \cup \{r\}, A; h, t)$ is a multi-digraph and r is a specified node called a *root* such that for every arc there exists a simple path from r which contains it. A rooted multi-digraph $H = (N, A, r'; h, t)$ also gives its *rooted line graph* as follows: add a new node r'' and insert an arc $r''r'$ to H, and construct the line graph of this resultant multi-digraph, then we have a digraph G whose vertices are $A \cup \{r\}$ where r is a vertex corresponding to the arc $r''r'$. By assumption, it is obvious that there exists a rooted path to every vertex in G. Hence G is a rooted digraph.

3 The forbidden minor characterization of line-search antimatroids

In analogy to point-search antimatroids, we define the *line-search antimatroid* $\mathfrak{LS}_D(H)$ of a rooted multi-digraph $H = (N, A, r; h, t)$ as follows:

$$\begin{aligned}\mathfrak{LS}_D(H) = \{&X \subseteq A : \text{every arc } a \in X \text{ is contained in a simple} \\ &\text{path from } r \text{ on the subgraph induced by } X\}.\end{aligned} \tag{6}$$

Note that line-search antimatroids of rooted multi-digraphs are also closed under taking their minors.

Let G be the rooted line graph of a rooted multi-digraph H. Then the line-search antimatroid of H coincides with the point-search antimatroid of G. Therefore, the class of point-search antimatroids of rooted digraphs includes that of line-search antimatroids of rooted multi-digraphs. It is easily checked that there is a one-to-one correspondence between line-search antimatroids of rooted multi-digraphs and irredundant rooted digraphs which satisfy the H-condition.

Point-search antimatroids of rooted digraphs are characterized by the forbidden minor [10]:

Proposition 2. *$\mathcal{F}$ is the point-search antimatroid of a rooted digraph if and only if $\mathcal{F}$ does not contain a minor isomorphic to $D_5 = \{\emptyset, \{x\}, \{y\}, \{x, y\}, \{x, y, z\}\}$, as shown in Figure 3.*

Hence, in order to characterize line-search antimatroids of rooted digraphs, we only need to characterize point-search antimatroids of irredundant rooted digraphs which violate the H-condition.

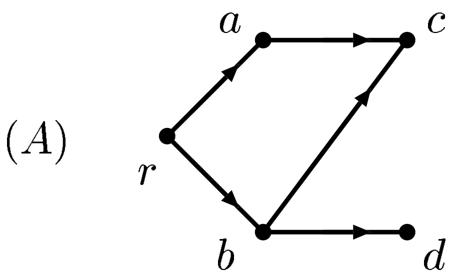

Figure 4: The rooted digraph A which violates the H-condition.

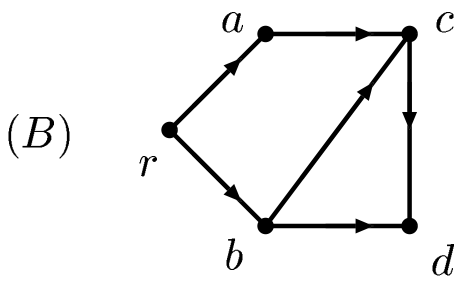

Figure 5: The rooted digraph B which violates the H-condition.

For example, the irredundant rooted digraph $A = (V(A), E(A), r)$ defined as

$$\begin{aligned} V(A) &= \{a, b, c, d\}, & (7)\\ E(A) &= \{(r, a), (r, b), (a, c), (b, c), (b, d)\}, & (8)\end{aligned}$$

which is shown in Figure 4 violates the H-condition.

Additionally, the following three kinds of irredundant rooted digraphs $B, C_{m,n}, D_{l,m,n}$ also violate the H-condition; $B = (V(B), E(B), r)$ is defined as

$$\begin{aligned} V(B) &= \{a, b, c, d\}, & (9)\\ E(B) &= \{(r, a), (r, b), (a, c), (b, c), (b, d), (c, d)\}, & (10)\end{aligned}$$

which is shown in Figure 5; $C_{m,n} = (V(C_{m,n}), E(C_{m,n}), r)$ is defined as

$$\begin{aligned} V(C_{m,n}) &= \{a, b, c = x_0, d = y_0, e, x_1, \ldots, x_{m-1}, y_1, \ldots, y_{n-1}\}, & (11)\\ E(C_{m,n}) &= \{(r, a), (r, b), (a, c), (b, d), (c, x_1), (d, y_1), (e, c), (e, d), & (12)\\ &\quad (x_1, x_2), \ldots, (x_{m-2}, x_{m-1}), (x_{m-1}, e), \\ &\quad (y_1, y_2), \ldots, (y_{n-2}, y_{n-1}), (y_{n-1}, e)\},\end{aligned}$$

where $m, n \geq 1$, which is shown in Figure 6; $D_{l,m,n} = (V(D_{l,m,n}), E(D_{l,m,n}), r)$ is defined as

$$\begin{aligned} V(D_{l,m,n}) &= \{a, b, c = x_0, d = y_0, e, f = z_0, & (13)\\ &\quad x_1, \ldots, x_{l-1}, y_1, \ldots, y_{m-1}, z_1, \ldots, z_{n-1}\}, \\ E(D_{l,m,n}) &= \{(r, a), (r, b), (a, c), (b, d), (c, x_1), (d, y_1), (e, c), (e, d), & (14)\\ &\quad (f, z_1), (x_1, x_2), \ldots, (x_{l-2}, x_{l-1}), (x_{l-1}, f), \\ &\quad (y_1, y_2), \ldots, (y_{m-2}, y_{m-1}), (y_{m-1}, f), \\ &\quad (z_1, z_2), \ldots, (z_{n-2}, z_{n-1}), (z_{n-1}, e)\},\end{aligned}$$

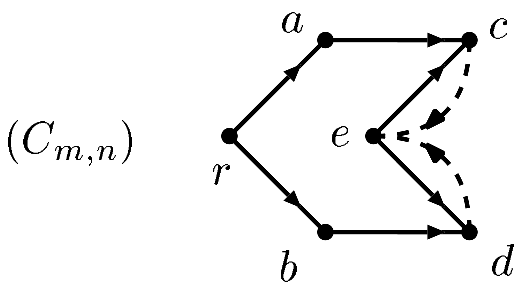

Figure 6: The rooted digraph $C_{m,n}$ $(m, n \geq 1)$ which violates the H-condition, where the broken arrows represent arbitrarily long paths.

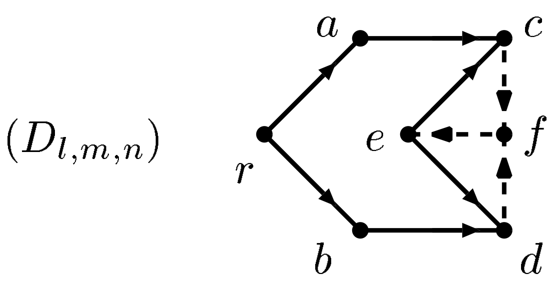

Figure 7: The rooted digraph $D_{l,m,n}$ $(l, m, n \geq 1)$ which violates the H-condition, where the broken arrows represent arbitrarily long paths.

where $l, m, n \geq 1$, which is shown in Figure 7.

Therefore, it is clear that if G is a rooted line graph then it cannot contain the above rooted digraphs as its rooted minors. Indeed, it turns out to be sufficient to exclude these minors to get a rooted line graph.

Theorem 3. *Let G be an irredundant rooted digraph. Then, G is a rooted line graph if and only if G has no rooted minor isomorphic to A, B, $C_{m,n}$ or $D_{l,m,n}$ $(l, m, n \geq 1)$.*

Proof. We only need to show the sufficiency. Let $G = (V, E, r)$ be an irredundant rooted digraph containing four vertices x, y, z, w which violate the H-condition and is minor-minimal with respect to this property. Let $\mathcal{W} = \{x, y, z, w\}$.

A vertex $a \in \mathcal{W}$ is the *joint* of a straight path P from r to a vertex of $\mathcal{W}$ if a is the first vertex of $\mathcal{W}$ along the path P from r. Let T be the set of joints for straight paths in G. From the assumption, we have $T \neq \emptyset$ and there must exist a path supporting each of the edges xy, zy, zw, which we denote by P, Q, R, respectively. We consider the following cases according to the size of T.

Case 1. $|T| = 1$. It is easily checked that this case leads to a contradiction.

Case 2. $|T| = 2$. This has the following six subcases.

Case 2-1. $T = \{x, y\}$. The path Q is not straight since Q must go through x or y. This is a contradiction.

Case 2-2. $T = \{x, z\}$. A path with the joint x supports the edge xy, and a path with the joint z supports the edges zy and zw. From the minimality of G, the vertices of G must be $\{r, x, y, z, w\}$. If we consider all the possible edges among them, then we obtain A and B.

Case 2-3. $T = \{x, w\}$. Suppose that the path Q goes through x, then the edge xy is a short-cut. This is a contradiction. Therefore, Q must go through w but not through x. Moreover, Q is $r \cdots w \cdots zy$ since Q does not go through y. If a path with the joint w has no vertex between r and w, then it is a short-cut of the path R. Therefore, it has an extra vertex p between r and w, namely the path is rpw, from the minimality of G. Moreover, the path with the joint x is rx from the minimality of G as a rooted minor. Since the path R does not go through w, it must go through x. We consider the subcases according to whether R goes through the edge xy or not.

Case 2-3-1. R goes through xy. R is $r \cdots xy \cdots z$. If there is a common vertex of the part $y \cdots z$ of R and the part $w \cdots z$ of Q except for z, then G must contain $D_{l,m,n}$ as a subgraph. Otherwise, G must contain $C_{m,n}$ as a subgraph.

Now we should check that if G has no rooted minor isomorphic to $C_{m,n}$ and $D_{l,m,n}$, then G must have A or B as its rooted minor, or it leads to a contradiction.

Case 2-3-1-1. $C_{m,n}$ has extra edges. Refer the definition (11, 12) of $C_{m,n}$.

Case 2-3-1-1-1. the edge cd exists. If we shrink a to r and we set $a = c$ and $c = x_1$, then we can reduce this case to A or B.

Case 2-3-1-1-2. the edge $x_i y_j$ exists $(0 < i < m, 0 < j < n)$. If we shrink a, b, c, x_1, $\ldots$, x_{i-1}, y_0, $\ldots$, y_{j-2} to r and we set $a = x_i$, $b = y_{j-1}$, $c = x_{i+1}$ and $d = y_j$, then we reduce this case to A or B.

Case 2-3-1-1-3. the edge $x_i e$ exists. A contradiction since the edge $x_{m-1}e$ is redundant.

Case 2-3-1-2. $D_{l,m,n}$ has extra edges. We can check similarly to Case 2-3-1-1.

Case 2-3-2. R does not go through xy. Then, we obtain the graphs shown in Figure 8, where I is a path from x to z and J is a path from w to z. In the left case, I and J have a unique common vertex z, and in the right case they have at least two common vertices.

Now we show that these graphs have A or B as a rooted minor. We consider the left case. The right case is shown similarly.

Case 2-3-2-1. the length of I is one, and the length of J is also one. If we shrink p to r, then it is reduced to B.

Case 2-3-2-2. the length of I is one, and the length of J is more than one. Let $J = wj_1 j_2 \ldots j_h z$ for $h \geq 1$. If we shrink p, w, j_1, $\ldots$, j_{h-1} to r, then it is reduced to B.

Case 2-3-2-3. the length of I is two, and the length of J is one. If we shrink p and w to r, then it is reduced to A.

Case 2-3-2-4. the length of I is more than two, and the length of J is one. Let $I = xi_1 i_2 \ldots i_k z$ for $k \geq 2$. If we delete $i_2, \ldots, i_k$ and shrink p and w to r, then it is reduced to A.

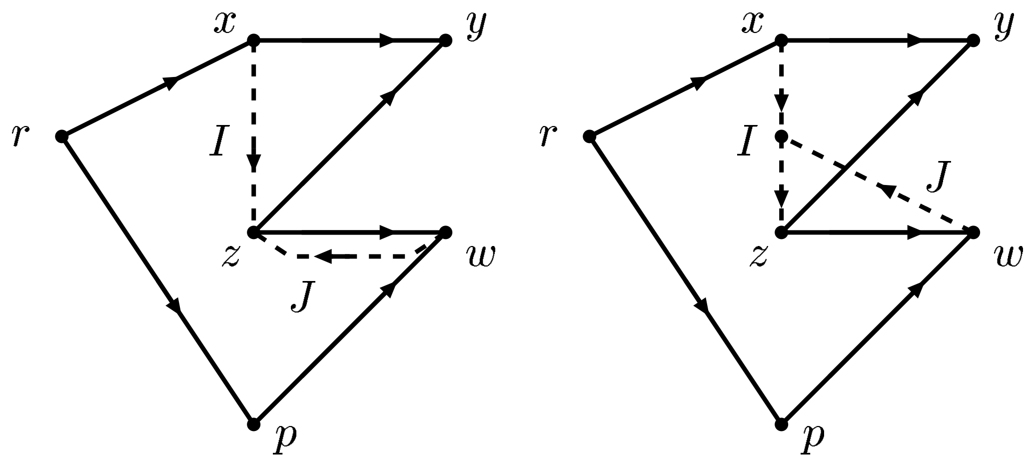

Figure 8: Case 2-3-2. Broken arrows represent arbitrarily long paths.

Case 2-3-2-5. the lengths of both I and J are more than one. Let $I = xi_1i_2\ldots i_kz$ for $k \geq 1$, and $J = wj_1\ldots j_hz$ for $h \geq 1$. If we delete $i_2, \ldots, i_k$ and shrink $p, w, j_1, \ldots, j_h$ to r, then it is reduced to A.

Case 2-4. $T = \{y, w\}$**.** From the minimality and the irredundancy of G, the length of a path with the joint y is two, and let it be rpy. Similarly, the length of a path with the joint w is two, and let it be rqw. If $p = q$, then the three edges xy,zy and zw are always redundant. Therefore, we have $p \neq q$.

The path Q goes through neither x nor y. Therefore, Q is $rqw\cdots zy$.

The path R does not go through w. Hence, it must go through y. If we delete x, then it is reduced to $C_{m,n}$ or $D_{l,m,n}$.

Case 2-5. $T = \{y, z\}$**.** The path P does not go through y. Therefore, it must go through z. Then, it is a contradiction since the edge zy is a short-cut.

Case 2-6. $T = \{z, w\}$**.** Since the path P does not go through z, it must go through w. From the minimality of G, the length of a path with the joint w is two, and the length of a path with the joint z is one. Now, we obtain the graph shown in Figure 9. Then, if we delete the vertices of the path $w\cdots x$ except for w, then it is reduced to A.

Case 3. $|T| = 3$**.** This has the following four subcases.

Case 3-1. $T = \{x, y, z\}$**.** The path P has the joint x. Moreover, the paths Q and R have the joint z. Suppose that the length of a path Y with the joint y is one. Then the edges xy and zy are redundant. Therefore, the length of Y is more than one, that is, $Y = ry_1\cdots y_kpy$ for $k \geq 0$. Note that p is contained neither in P nor in Q.

Let $P = ru_1\cdots u_lx$ and $Q = rv_1\cdots v_mz$ for $l, m \geq 0$. If we delete p and shrink $u_1, \ldots, u_l, v_1, \ldots, v_m, y_1, \ldots, y_k$ to r, then it is reduced to A or B.

Case 3-2. $T = \{x, y, w\}$**.** Suppose that the length of a path Y with the joint y is one. Then the edges xy and zy are redundant. Therefore, the length of Y is more than one, that is, $Y = ry_1\cdots y_kpy$ for $k \geq 0$. If we delete x, then $\{p, y, z, w\}$ is the set of vertices which violates the H-condition. Therefore, it is reduced to Case 2-3.

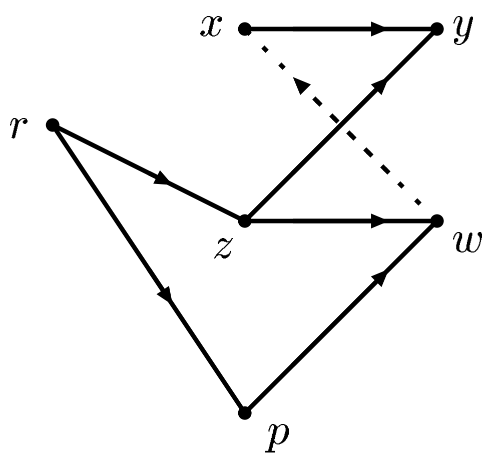

Figure 9: Case 2-6.

Case 3-3. $T = \{x, z, w\}$. The path P has the joint x. Moreover, the paths Q and R have the joint z. Suppose that the length of a path Y with the joint w is one. Then the edge zw is redundant. Therefore, the length of Y is more than one, that is, $Y = ry_1 \cdots y_k pw$ for $k \geq 0$. Note that p is contained neither in P nor in Q.

Let $P = ru_1 \cdots u_l x$ and $Q = rv_1 \cdots v_m z$ for $l, m \geq 0$. If we delete p, and shrink $u_1, \ldots,$ u_l, $v_1, \ldots, v_m$, $y_1, \ldots, y_k$ to r, then it is reduced to A or B.

Case 3-4. $T = \{y, z, w\}$. The paths Q and R have the joint z. Let Y be the path with the joint y. Note that the length of Y is more than one. Similarly, let W be the path with the joint w, then its length is more than one. The path P supporting the edge xy has the joint w. Let p be the vertex of Y which precedes y and q be the vertex of W which precedes w. Suppose that $p = q$, and consider the path P supporting the edge xy. The joint of P is not y. If the joint of P is z, then the edge zy is a short-cut of P. If the joint of P is w, then the edge py is a short-cut of P. Therefore, we have $p \neq q$.

Let $Y = ry_1 \cdots y_l py$, $W = rw_1 \cdots w_m qw$ and $Q = rq_1 \cdots q_n z$ for $l, m, n \geq 0$. If we delete p and x, and shrink $y_1, \ldots, y_l$, $w_1, \ldots, w_m$, $q_1, \ldots, q_n$ to r, then it is reduced to A or B.

Case 4. $|T| = 4$. It is easily checked that this case is reduced to Case 3-1 or Case 3-3. $\square$

Theorem 3 directly gives the forbidden minor characterization of line-search antimatroids of rooted digraphs as below.

Corollary 4. *Let $\mathcal{F}$ be an antimatroid. Then, $\mathcal{F}$ is a line-search antimatroid of a rooted digraph if and only if $\mathcal{F}$ has no minor isomorphic to D_5 or the point-search antimatroids of A, B, $C_{m,n}$ or $D_{l,m,n}$ $(l, m, n \geq 1)$.*

Robertson–Seymour [12] have shown the Graph Minor Theorem, that is, in every infinite set of graphs there are two graphs such that one is a minor of the other. From this theorem, we conclude that every minor-closed property of graphs can be characterized by finitely many forbidden minors. But for antimatroids, Theorem 3 implies that there exists an infinite set of antimatroids such that any of them is not a proper minor of the other one.

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