The Free Complex of a Two-Dimensional Generalized Convex Shelling

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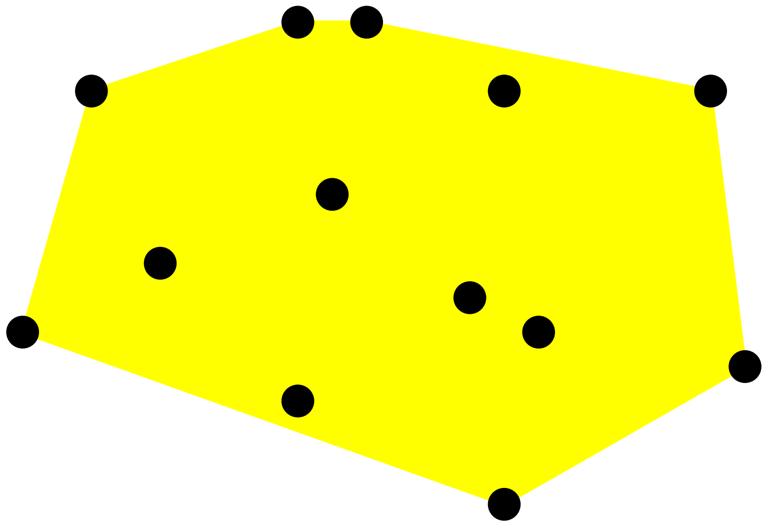
September 8–12, 2003
Eurocomb'03
— European Conference on Combinatorics, Graph Theory and Applications —
Czech University of Agriculture, Prague, Czech Republic

Supported by the Berlin-Zürich Joint Graduate Program

CGC
Combinatorics
Geometry
Computation

How many interior points are there in a finite point configuration $\mathcal{P}$?



An Euler-type formula:

$$\#\text{ of int. pts in } \mathcal{P} = (-1)^{d-1} \sum_{\text{free } A \subseteq \mathcal{P}} (-1)^{|A|} |A|.$$

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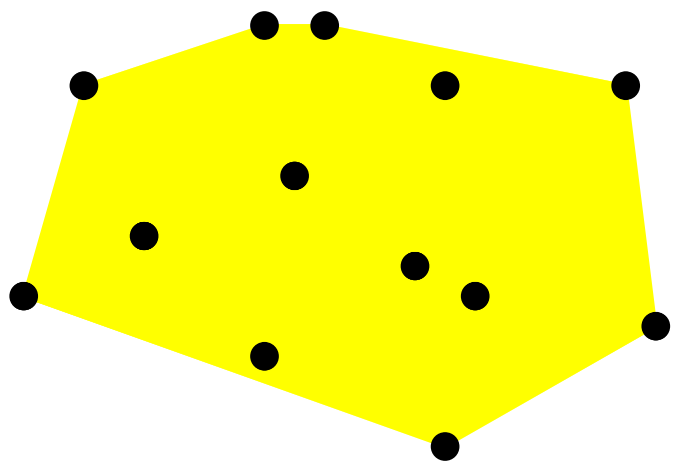
Proved by:

- Ahrens, Gordon & McMahon (DCG '99)
  for $d = 2$, geometric proof
- Klain (Adv Math '99)
  for general d, using a valuation
- **Edelman & Reiner** (DCG '00)
  for general d, topological proof
  $\longrightarrow\longrightarrow\longrightarrow$ making use of **free complexes**

$\mathcal{P}$ a finite point configuration in $\mathbb{R}^d$.

Def. $X \subseteq \mathcal{P}$ is **free** if

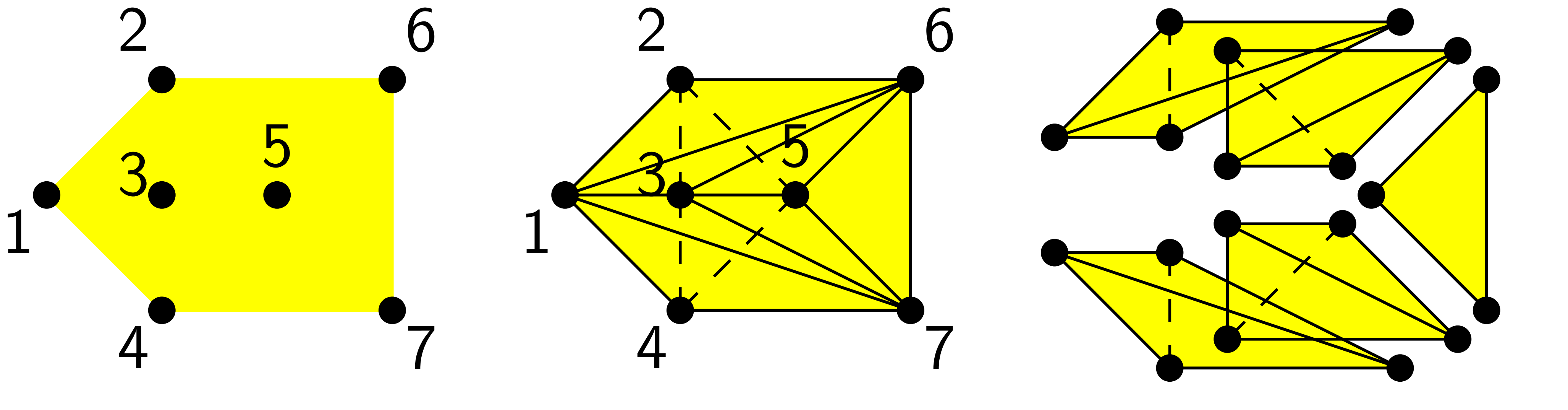
- $\text{conv}(X) \cap \mathcal{P} = X$ (convexity)
- the extreme points of $\text{conv}(X) = X$ (the points of X lie in convex position) (independence).



Free complex of a point configuration

Def. The **free complex** of $\mathcal{P}$ is the family of all free sets in $\mathcal{P}$, denoted by $\mathrm{Free}(\mathcal{P})$.

Remark $\mathrm{Free}(\mathcal{P})$ is a simplicial complex.



$\mathcal{P}$ $\qquad$ $\mathrm{Free}(\mathcal{P})$ $\qquad$ the facets of $\mathrm{Free}(\mathcal{P})$

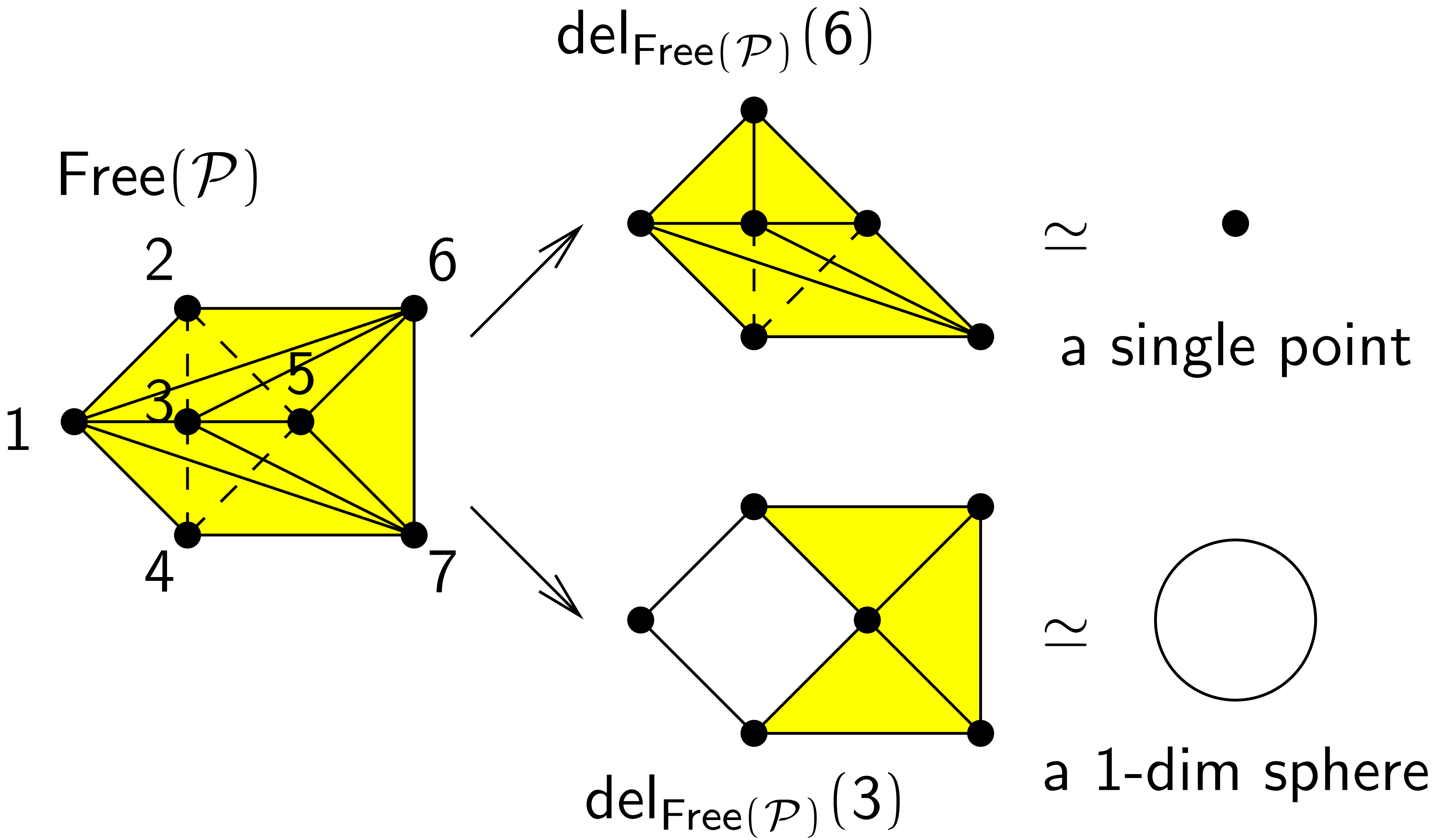
$\mathcal{P}$ a finite point configuration in $\mathbb{R}^d$

Consider the free complex $\mathrm{Free}(\mathcal{P})$ of $\mathcal{P}$.

In Edelman & Reiner's proof, it was a key that

- **$\mathrm{del}_{\mathrm{Free}(\mathcal{P})}(x)$ is contractible if $x \in \mathcal{P}$ lies on the bd of $\mathrm{conv}(\mathcal{P})$**
  (implying $\tilde{\chi}(\mathrm{del}_{\mathrm{Free}(\mathcal{P})}(x)) = 0$),
- **$\mathrm{del}_{\mathrm{Free}(\mathcal{P})}(x)$ has the integral homology of a $(d-1)$-dim sphere if $x \in \mathcal{P}$ lies in the interior of $\mathrm{conv}(\mathcal{P})$**
  (implying $\tilde{\chi}(\mathrm{del}_{\mathrm{Free}(\mathcal{P})}(x)) = (-1)^{d-1}$).

Proof by Edelman & Reiner: Example



Q.

How about a generalization to abstract convex geometries??

This work

- Study on their problems for a special case (2-dim generalized convex shellings).
- Result for this special case.

(1) (Abstract) convex geometries and Free complexes

(2) Questions by Edelman & Reiner for (abstract) convex geometries

(3) 2-dim generalized convex shellings

(4) Results

Convex geometries (Edelman & Jamison '85)

E a nonempty finite set,
$\mathcal{L} \subseteq 2^{E}$ a family of subsets of E.

$\boxed{\text{Def.}}$ $\mathcal{L}$ is called a **convex geometry** on E if $\mathcal{L}$ satisfies the following conditions.

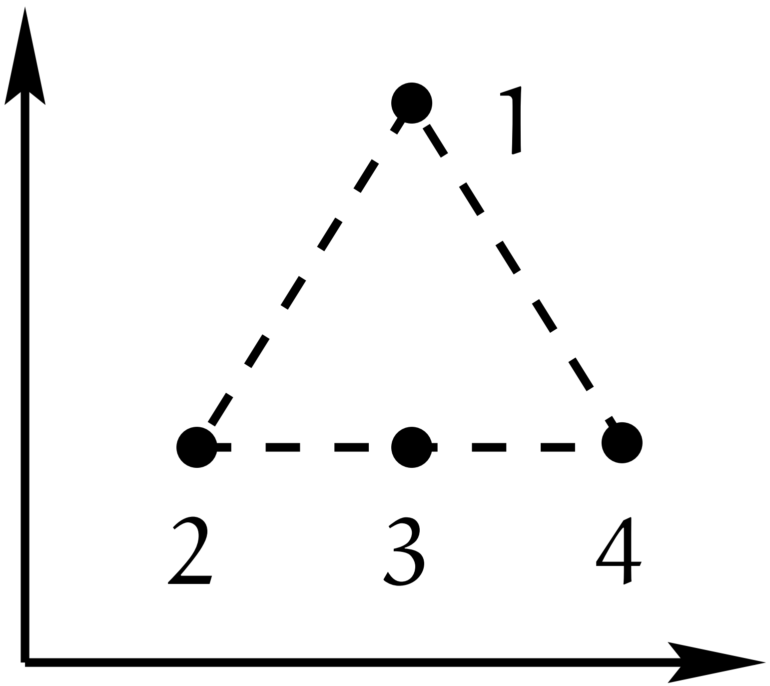
(1) $\emptyset \in \mathcal{L}$, $E \in \mathcal{L}$.
(2) $X, Y \in \mathcal{L} \Rightarrow X \cap Y \in \mathcal{L}$.
(3) $X \in \mathcal{L} \setminus \{E\} \Rightarrow \exists\, e \in E \setminus X$: $X \cup \{e\} \in \mathcal{L}$.

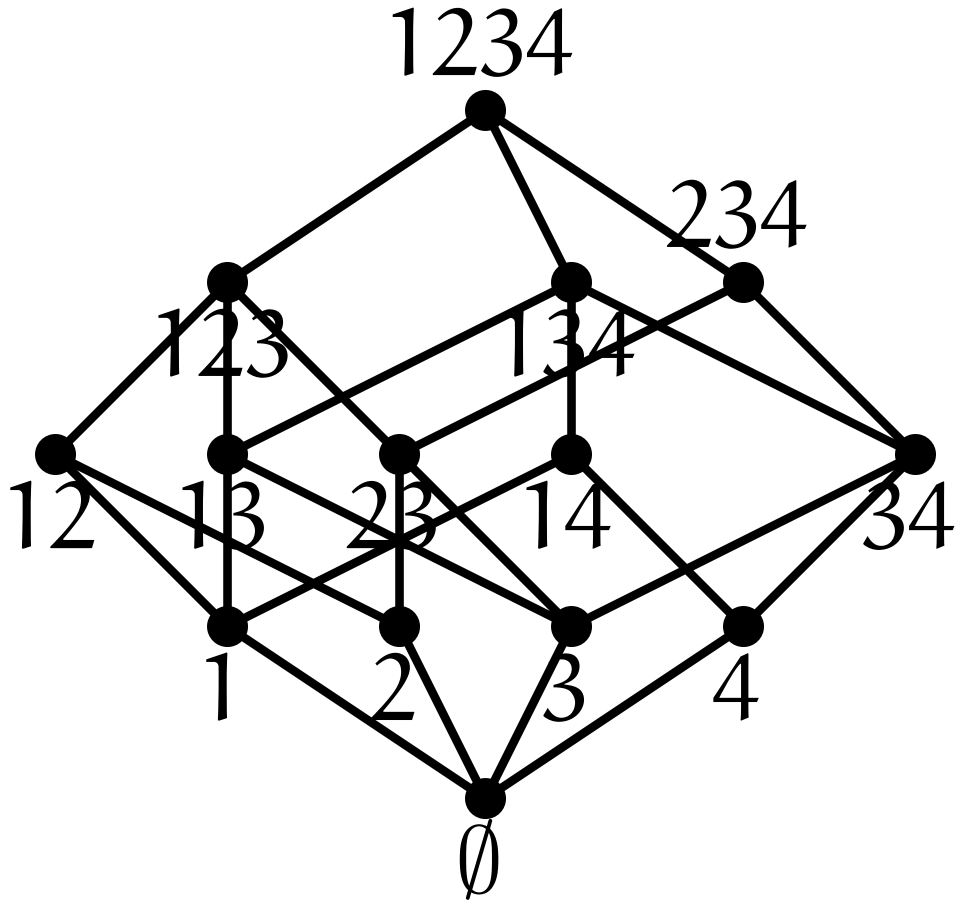
$X \subseteq E$ is called **convex** if $X \in \mathcal{L}$.

Example 1: convex shelling

$\mathcal{P}$ a finite point set in $\mathbb{R}^d$.

Define: $\mathcal{L} = \{X \subseteq \mathcal{P} : \mathbf{conv}(X) \cap \mathcal{P} = X\}$.



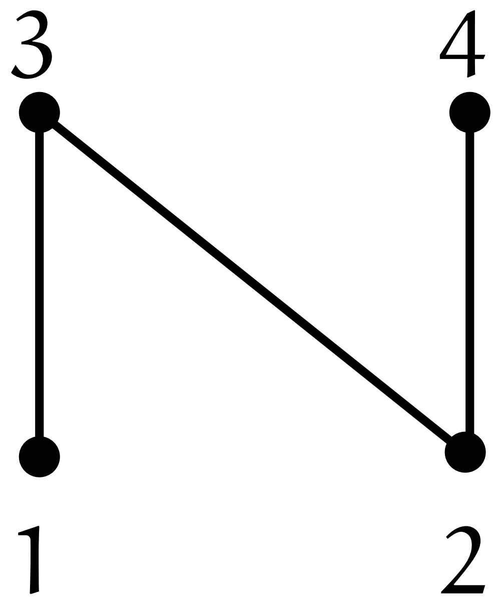


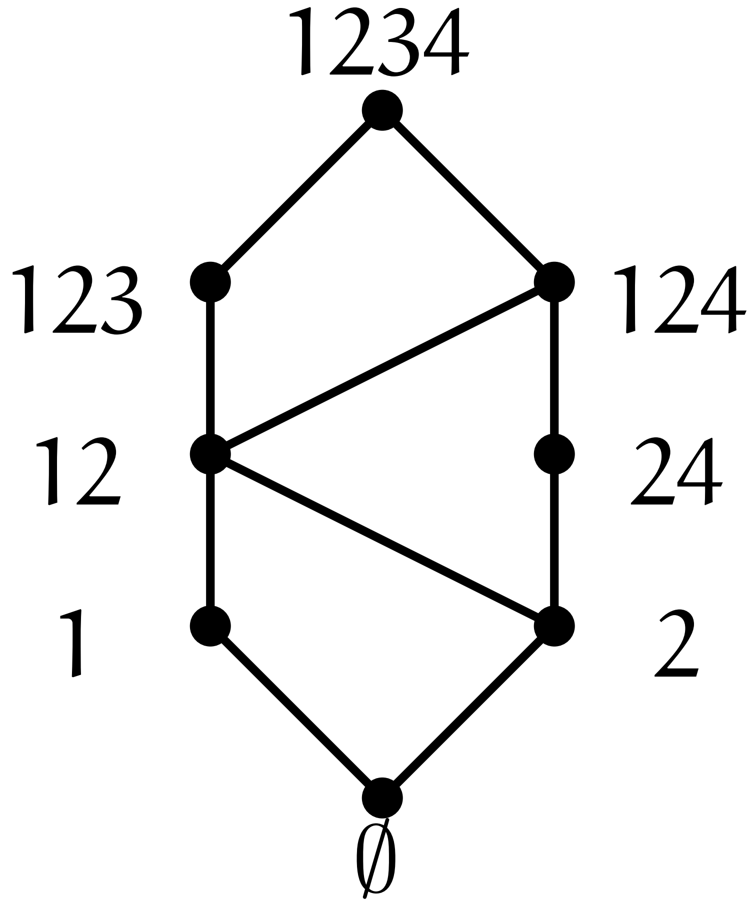
$\mathcal{L}$ is called the **convex shelling** on $\mathcal{P}$.

Example 2: poset shelling

$P = (E, \leq)$ a partially ordered set.

Define: $\mathcal{L} = \{X \subseteq E : e \in X, f \leq e \Rightarrow f \in X\}$.





$\mathcal{L}$ is called the **poset shelling** of P.

Convex geometries arise from various objects.

- From graphs
  - Tree shellings
  - Graph searches
  - Simplicial elimination of chordal graphs
- From partially ordered sets
  - Poset double shellings
  - k-chains
- From finite point sets in $\mathbb{R}^d$
  - Lower convex shellings
- From oriented matroids
  - Convex shellings of acyclic OMs
- ...

$\mathcal{L}$ a convex geometry on E.

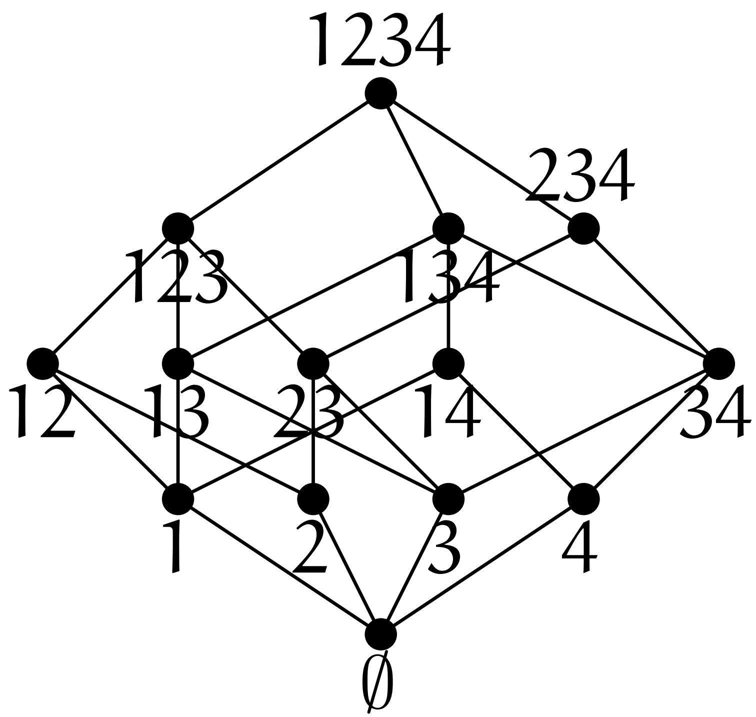
Def. $X \subseteq E$ is **free** in $\mathcal{L}$ if

- $X \in \mathcal{L}$ (convexity)
- the set of "extreme points" of $X = X$ (independence).

Extreme points

$\mathcal{L}$ a convex geometry on E, $X \in \mathcal{L}$ a convex set.

$\boxed{\text{Def.}}$ $e \in X$ is an **extreme point** of X if $X \setminus \{e\} \in \mathcal{L}$.



$X = \{2, 3, 4\} \in \mathcal{L}$

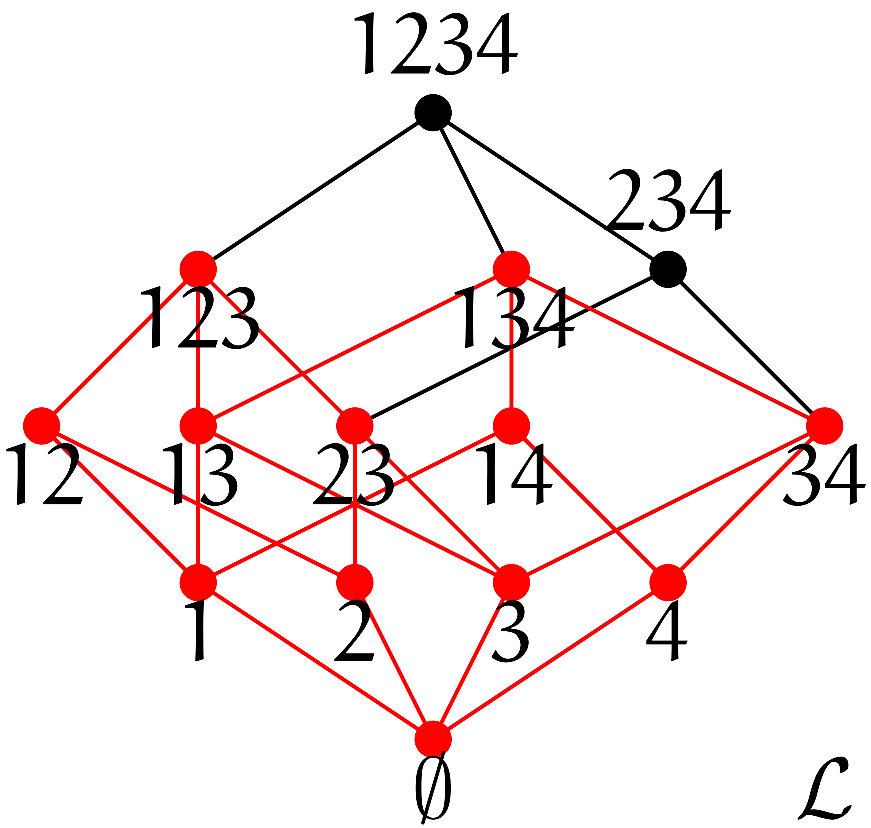
2	extreme
3	not extreme
4	extreme

X is **independent** if every $e \in X$ is extreme in X.

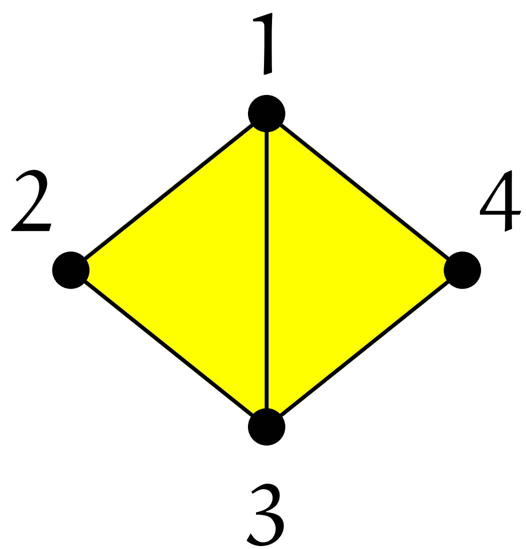
Free complex of a convex geometry

Def. The **free complex** $\mathrm{Free}(\mathcal{L})$ of $\mathcal{L}$ is the family of all free sets in $\mathcal{L}$,

Remark $\mathrm{Free}(\mathcal{L})$ is a simplicial complex.



$\mathcal{L}$



$\mathrm{Free}(\mathcal{L})$

Remark

$\mathcal{P}$ a point configuration in $\mathbb{R}^d$,
$\mathcal{L}$ the convex shelling of $\mathcal{P}$.

Then

$$\mathrm{Free}(\mathcal{P}) = \mathrm{Free}(\mathcal{L}).$$

To generalize Edelman & Reiner's result,

We also need to generalize
"the boundary" and "the interior."

$\Longrightarrow$ a concept of "dependency sets"
(we omit the definition).

Dependency sets and convex geometries

$\mathcal{P}$ a point configuration in $\mathbb{R}^{\mathrm{d}}$,
$\mathcal{L}$ the convex shelling of $\mathcal{P}$.

$\mathrm{Dep}_{\mathcal{L}}(e)$ the dependency set of $e \in \mathcal{P}$ in $\mathcal{L}$.

Then

- e lies on the boundary of $\mathrm{conv}(\mathcal{P})$ $\iff \mathrm{Dep}_{\mathcal{L}}(e) \neq \mathcal{P}$.
- e lies in the interior of $\mathrm{conv}(\mathcal{P})$ $\iff \mathrm{Dep}_{\mathcal{L}}(e) = \mathcal{P}$.

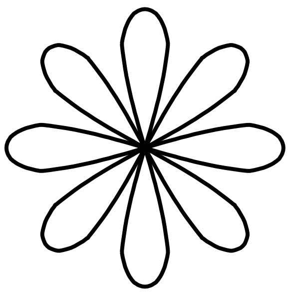
This leads to the following open problems.

$\boxed{\text{Open Problems}}$ (Edelman & Reiner '00)

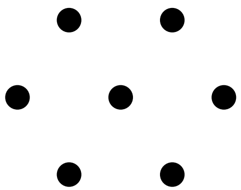
E a finite set; $\mathcal{L}$ a convex geometry on E.

(1) **Is $\mathrm{del}_{\mathrm{Free}(\mathcal{L})}(x)$ contractible if $\mathrm{Dep}_{\mathcal{L}}(x) \neq E$?**

(2) **Is $\mathrm{del}_{\mathrm{Free}(\mathcal{L})}(x)$ homotopy equivalent to a bouquet of equidimensional spheres if $\mathrm{Dep}_{\mathcal{L}}(x) = E$?**



a bouquet of eight 1-dim spheres



a bouquet of six 0-dim spheres

a bouquet of zero sphere

Partial results in the literature

Both problems have been solved affirmatively for the following classes of convex geometries.

- Convex shellings of point configurations (Edelman & Reiner '00, Dong '02)
- Poset double shellings (Edelman & Reiner '00)
- Simplicial eliminations of chordal graphs (Edelman & Reiner '00)
- Conv shellings of acyclic oriented matroids (Edelman, Reiner & Welker '02)
- Poset shellings. (Easy)

Consider another class of convex geometries, the 2-dim separable generalized convex shellings.

(1) **If $\mathrm{Dep}_{\mathcal{L}}(x) \neq E$,**
$\mathrm{del}_{\mathrm{Free}(\mathcal{L})}(x)$ is contractible.

(2) **If $\mathrm{Dep}_{\mathcal{L}}(x) = E$,**
$\mathrm{del}_{\mathrm{Free}(\mathcal{L})}(x)$ is either contractible or homotopy equiv to a 0-dim sphere.

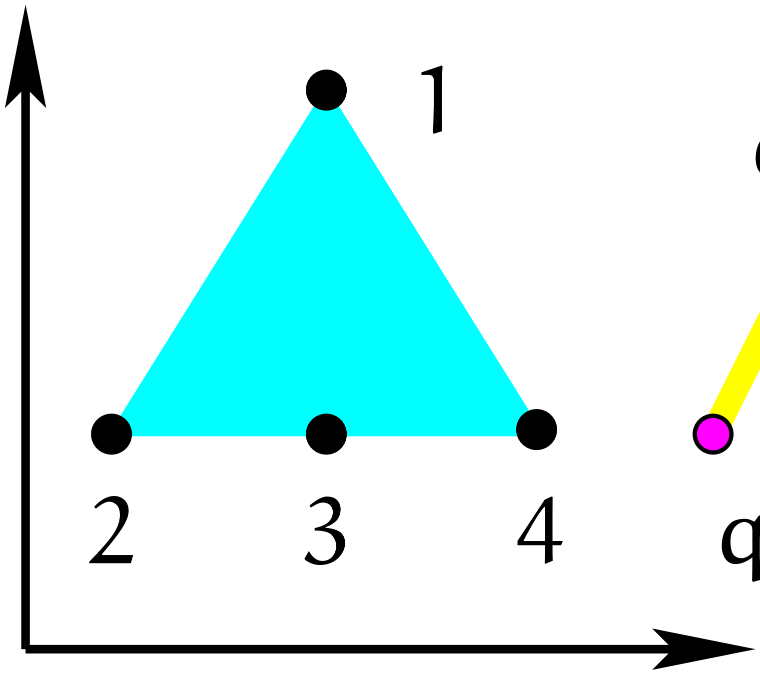
★ Verifies Open Problems for this special case!

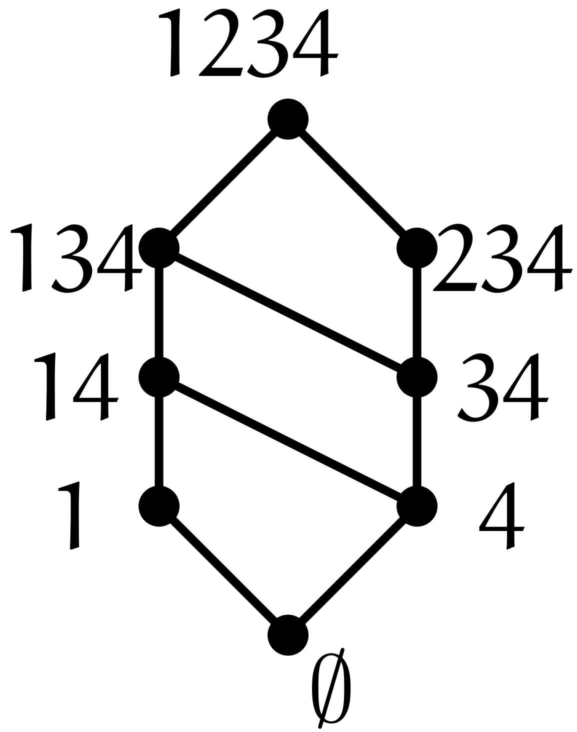
★ Gives the first example of $\mathcal{L}$ and x s.t. $\mathrm{del}_{\mathrm{Free}(\mathcal{L})}(x)$ is contractible & $\mathrm{Dep}_{\mathcal{L}}(x) = E$.

$\mathcal{P}, \mathcal{Q}$ point sets in $\mathbb{R}^d$ with $\mathcal{P} \cap \mathrm{conv}(\mathcal{Q}) = \emptyset$.

Define:

$$\mathcal{L} = \{X \subseteq \mathcal{P} : \mathbf{conv}(X \cup \mathcal{Q}) \cap \mathcal{P} = X\}.$$





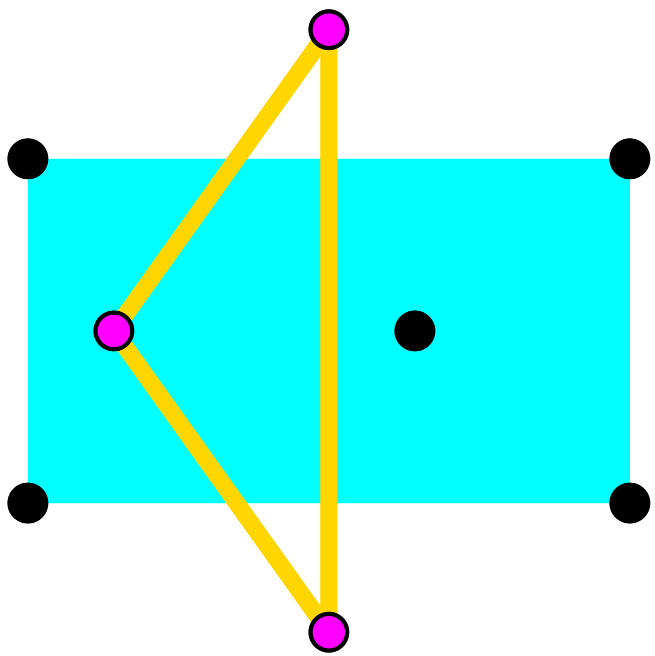
$\mathcal{L}$ is a convex geometry on $\mathcal{P}$ and called
the generalized conv shelling on $\mathcal{P}$ w.r.t. $\mathcal{Q}$.

Technical terminology

$\mathcal{P}$,$\mathcal{Q}$ point sets in $\mathbb{R}^d$ with $\mathcal{P} \cap \text{conv}(\mathcal{Q}) = \emptyset$,

$\mathcal{L}$ the generalized convex shelling on $\mathcal{P}$ w.r.t. $\mathcal{Q}$.

- $\mathcal{L}$ is **2-dimensional** if $d = 2$.
- $\mathcal{L}$ is **separable** if $\text{conv}(\mathcal{P}) \cap \text{conv}(\mathcal{Q}) = \emptyset$.



a non-separable case

Why separable generalized convex shellings?

$\boxed{\text{Thm}}$ (Kashiwabara, Nakamura & Okamoto, '03)

For every convex geometry $\mathcal{L}$,
there exist point sets $\mathcal{P}$, $\mathcal{Q}$ with
conv$(\mathcal{P}) \cap$ conv$(\mathcal{Q}) = \emptyset$ s.t.

$\mathcal{L} \cong$ the gen conv shelling on $\mathcal{P}$ w.r.t. $\mathcal{Q}$.

(Separable generalized convex shellings
represent all convex geometries.)

$\longrightarrow\longrightarrow\longrightarrow$ The 2-dim case is a first step
for resolution of Open Problems.

Our results (again)

$\mathcal{L}$ the 2-dim sep gen conv shelling on $\mathcal{P}$ w.r.t. $\mathcal{Q}$, $x \in \mathcal{P}$.

(1) **If $\mathbf{Dep}_{\mathcal{L}}(x) \neq \mathcal{P}$, $\mathbf{del}_{\mathbf{Free}(\mathcal{L})}(x)$ is contractible.**

(2) **If $\mathbf{Dep}_{\mathcal{L}}(x) = \mathcal{P}$, $\mathbf{del}_{\mathbf{Free}(\mathcal{L})}(x)$ is either contractible or homotopy equiv to a 0-dim sphere.**

Problems are still open

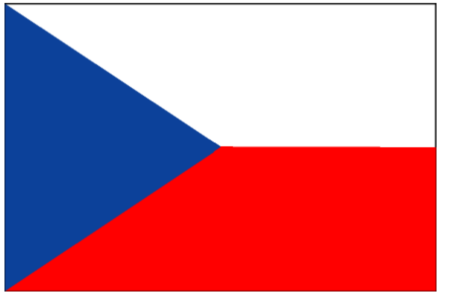
- **We don't know yet**
  the problems are affirmative or negative in the general case!
- How about a 3-dim case??
- How about a non-separable 2-dim case??

Acknowledgements

The speaker wants to thank

the financial support for Eurocomb'03
by DIMATIA and
the European project COMBSTRU.

Děkuji vám mnohokrát.



Here are extra slides for possible questions from the audience.

Closure operators, extreme point operators

$\mathcal{L}$ a convex geometry on E.

$\boxed{\text{Def.}}$ The **closure operator**

$\tau_{\mathcal{L}} : 2^E \to 2^E$ is defined as

$$\tau_{\mathcal{L}}(A) = \bigcap \{X \in \mathcal{L} : A \subseteq X\}.$$

$\boxed{\text{Def.}}$ The **extreme point operator**

$\mathrm{ex}_{\mathcal{L}} : 2^E \to 2^E$ is defined as

$$\mathrm{ex}_{\mathcal{L}}(A) = \{e \in A : e \notin \tau_{\mathcal{L}}(A \setminus \{e\})\}.$$

$\mathcal{L}$ a convex geometry on E.

Def. $A \subseteq E$ is **independent** if $\mathrm{ex}_{\mathcal{L}}(A) = A$.

Def. The **dependency set** of $e \in E$ in $\mathcal{L}$ is

$$\mathrm{Dep}_{\mathcal{L}}(e) = \left\{ f \in E : \begin{array}{l} \exists \text{ independent } A \text{ s.t.} \\ f \in A, e \in \tau_{\mathcal{L}}(A), \\ e \notin \tau_{\mathcal{L}}(A \setminus \{f\}) \end{array} \right\}.$$

$\mathcal{L}$ the 2-dim sep gen conv shelling on $\mathcal{P}$ w.r.t. $\mathcal{Q}$, $\mathcal{Q} \neq \emptyset$.

(1) Free$(\mathcal{L})$ is the clique complex of a graph G.

I.e., the family of all cliques of G.

(2) G is chordal & connected.

Chordal $\Leftrightarrow$ every ind. cycle is C_3.

(3) **(2)** $\Rightarrow$ Free$(\mathcal{L})$ contractible.

(4) $G - x$ has at most 2 connected components.

(5) x a cut-vertex of G $\Rightarrow$ $\mathrm{Dep}_{\mathcal{L}}(x) = \mathcal{P}$.