A greedy algorithm for convex geometries*

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Abstract

Convex geometries are closure spaces which satisfy anti-exchange property, and they are known as dual of antimatroids. We consider functions defined on the sets of the extreme points of a convex geometry. Faigle–Kern (1996) presented a greedy algorithm to linear programming problems for shellings of posets, and Krüger (2000) introduced b-submodular functions and proved that Faigle–Kern's algorithm works for shellings of posets if and only if the given set function is b-submodular. We extend their results to all classes of convex geometries, that is, we prove that the same algorithm works for all convex geometries if and only if the given set function on the extreme sets is submodular in our sense.

Key words: Antimatroid; Convex geometry; Extreme set; Greedy algorithm; Submodularity

1 Introduction

The theory of submodular optimization has been developed since the greedy-algorithmic characterization of submodular functions by Edmonds [4]. See [9] for a comprehensive survey of this field. In particular, Faigle–Kern [5, 6] considered the dual greedy algorithm and submodular functions over the family of the antichains of a partially ordered set, or poset. Krüger [11] showed that submodular-type optimization problems on the family of the antichains of a poset are characterized by b-submodularity.

In this work, we extend the result of Krüger [11] to convex geometries. A convex geometry is a closure space which satisfies anti-exchange property, and known as dual of antimatroids [3, 10]. They are considered as a combinatorial abstraction of precedence relations or convexity. In researches of optimization problems for antimatroids, bottleneck-type optimization problems over them are characterized by a greedy algorithm [2, 13]. We consider

*The extended abstract version of this paper appeared in Proceedings of the 5th Japan-Korea Joint Workshop on Algorithms and Computation, 2000, 118–121.

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another optimization problem for them, which is linear optimization over their extreme sets.

Let E be a non-empty finite set. The family $\mathcal{L}$ of subsets of E is a *closure space* on E if it satisfies $\emptyset \in \mathcal{L}$, $E \in \mathcal{L}$ and $A \cap B \in \mathcal{L}$ for all $A, B \in \mathcal{L}$. For a closure space $\mathcal{L}$ on E, E is called the *ground set* of $\mathcal{L}$ and a member of $\mathcal{L}$ is called a *closed set*. We define an operator $\tau : 2^E \to 2^E$ associated with a closure space $\mathcal{L}$ as $\tau(A) = \bigcap\{B \in \mathcal{L} : A \subseteq B\}$. That is, $\tau(A)$ is the minimum closed set containing A. Then, τ satisfies the conditions that: $\tau(\emptyset) = \emptyset$; $A \subseteq \tau(A)$; $A \subseteq B$ implies $\tau(A) \subseteq \tau(B)$; and $\tau(\tau(A)) = \tau(A)$. Conversely, if we give an operator τ satisfying these four conditions, then $\mathcal{L} = \{A \subseteq E : A = \tau(A)\}$ forms a closure space. An operator satisfying these four conditions is called a *closure operator*. The property that $A \subseteq B$ implies $\tau(A) \subseteq \tau(B)$ is called *monotonicity*.

A closure space $\mathcal{L}$ on E is a *convex geometry* if it satisfies anti-exchange property. *Anti-exchange property* states that if, for all $A \subseteq E$, distinct $x, y \in E$ satisfy $x, y \notin \tau(A)$ and $y \in \tau(A \cup \{x\})$, then $x \notin \tau(A \cup \{y\})$ holds. A member of a convex geometry is called a *convex set*. A convex geometry can be also defined without the closure operator, as is shown in the next proposition.

Proposition 1.1 ([10]). *Let $\mathcal{L}$ be a closure space on E. Then $\mathcal{L}$ is a convex geometry if and only if $A \in \mathcal{L} \setminus \{E\}$ implies $A \cup \{e\} \in \mathcal{L}$ for some $e \in E \setminus A$.*

Various combinatorial objects yield convex geometries. We see two examples, convex shellings and poset shellings. For further examples, see [3, 10]. Suppose that E is a finite set of points on a Euclidean space. Then $\{A \subseteq E : A = \mathrm{conv}(A) \cap E\}$ forms a convex geometry called a *convex shelling*. In general, if a given convex geometry is isomorphic to some convex shelling, then it is also called a convex shelling. The second example is obtained from a partially ordered set. Let $(E, \leq)$ be a partially ordered set. An *order ideal* of E is a subset $I \subseteq E$ such that $y \in I$ and $x \leq y$ imply $x \in I$. Then the family of the order ideals of E satisfies the axioms of convex geometries. We call a convex geometry isomorphic to such a convex geometry a *poset shelling*. The next proposition is an important characterization of poset shellings.

Proposition 1.2 ([10]). *A convex geometry is a poset shelling if and only if it is closed under union.*

We call an element a of a subset $A \subseteq E$ an *extreme point* of A if $a \notin \tau(A \setminus \{a\})$. The set of the extreme points of A is denoted by $\mathrm{ex}(A)$, which is called the *extreme set* of A. Note that $\mathrm{ex}(A) \subseteq A$. For a convex geometry $\mathcal{L}$ we denote $\mathrm{ex}(\mathcal{L}) = \{\mathrm{ex}(A) : A \in \mathcal{L}\}$. We can easily check that $\emptyset \in \mathrm{ex}(\mathcal{L})$. If $\mathcal{L}$ is a convex shelling on E, then $\mathrm{ex}(A) = \{x \in A : x \text{ is a vertex of } \mathrm{conv}(A)\}$ for $A \subseteq E$. If $\mathcal{L}$ is a poset shelling on E, $\mathrm{ex}(A)$ is the set of the maximal elements of A for $A \subseteq E$. Convex geometries have an important characterization using extreme sets. This characterization is known as the *finite Minkowski-Krein-Milman property*: a closure space $\mathcal{L}$ is a convex geometry if and only if $A = \tau(\mathrm{ex}(A))$ for all closed set $A \in \mathcal{L}$. See [3, 10] for the proof. Also, it is known that a convex geometry $\mathcal{L}$ forms a lattice with respect to set-inclusion [3, 10]. Then the join $\vee_{\mathcal{L}}$ and the meet $\wedge_{\mathcal{L}}$ on $\mathcal{L}$ are defined as follows: for all $A, B \in \mathcal{L}$, $A \vee_{\mathcal{L}} B = \tau(A \cup B)$ and $A \wedge_{\mathcal{L}} B = A \cap B$. By the finite Minkowski-Krein-Milman property, we can show that the map $\mathrm{ex} : \mathcal{L} \to \mathrm{ex}(\mathcal{L})$ is bijective for a convex geometry $\mathcal{L}$. We call the map $\mathrm{ex} : \mathcal{L} \to \mathrm{ex}(\mathcal{L})$ the *extreme operator*. Therefore, by setting a partial order $\preceq$ on $\mathrm{ex}(\mathcal{L})$ so that $X \preceq Y$ if and only if $\tau(X) \subseteq \tau(Y)$ for all $X, Y \in \mathrm{ex}(\mathcal{L})$, $\mathrm{ex}(\mathcal{L})$ is a lattice isomorphic to $\mathcal{L}$. Then, we have the join $\vee_{\mathrm{ex}(\mathcal{L})}$ and the meet $\wedge_{\mathrm{ex}(\mathcal{L})}$ on $\mathrm{ex}(\mathcal{L})$: $X \vee_{\mathrm{ex}(\mathcal{L})} Y = \mathrm{ex}(\tau(X) \vee_{\mathcal{L}} \tau(Y))$ and $X \wedge_{\mathrm{ex}(\mathcal{L})} Y = \mathrm{ex}(\tau(X) \wedge_{\mathcal{L}} \tau(Y))$. Note that $X \prec Y$ means $X \preceq Y$ and $X \neq Y$. Also remark that for a convex geometry $\mathcal{L}$ with the closure operator τ, the restriction of τ to $\mathrm{ex}(\mathcal{L})$ is a lattice-isomorphism from $\mathrm{ex}(\mathcal{L})$ to $\mathcal{L}$, whose inverse operation is ex. When there is no risk of confusion, we use $\vee$ instead of $\vee_{\mathrm{ex}(\mathcal{L})}$, and $\wedge$ instead of $\wedge_{\mathrm{ex}(\mathcal{L})}$.

We introduce two new operators on the family $\mathrm{ex}(\mathcal{L})$ of the extreme sets of a convex geometry $\mathcal{L}$. For $X, Y \in \mathrm{ex}(\mathcal{L})$, the *reduced meet* $X \sqcap Y$ is defined as $X \sqcap Y = (X \wedge Y) \cap (X \cup Y)$, and the *residue* $X \diamond Y$ is defined as $X \diamond Y = (X \cap Y) \setminus (X \vee Y)$. Remark that $X \sqcap Y$ and $X \diamond Y$ are also in $\mathrm{ex}(\mathcal{L})$. This remark is led from Lemma 2.2 proved later.

Let $\mathcal{L}$ be a convex geometry on E, $c \in \mathbb{R}_+^E$ be a non-negative vector and $f : \mathrm{ex}(\mathcal{L}) \to \mathbb{R}$ be a set function on $\mathrm{ex}(\mathcal{L})$ with $f(\emptyset) = 0$. Then we consider the following linear programming problem (P):

$$\text{maximize} \qquad \sum_{e \in E} c(e)x(e) \tag{1.1}$$

$$\text{subject to} \qquad \sum_{e \in X} x(e) \leq f(X) \quad (X \in \mathrm{ex}(\mathcal{L})). \tag{1.2}$$

Problem (P) leads to the dual problem (D):

$$\text{minimize} \qquad \sum_{X \in \mathrm{ex}(\mathcal{L})} f(X)y(X) \tag{1.3}$$

$$\text{subject to} \qquad \sum_{\substack{X \in \mathrm{ex}(\mathcal{L}) \\ \text{s.t. } e \in X}} y(X) = c(e) \quad (e \in E), \tag{1.4}$$

$$y(X) \geq 0 \quad (X \in \mathrm{ex}(\mathcal{L})). \tag{1.5}$$

It is easily checked that the problems (P) and (D) always have optimums. In order to solve dual problem (D) Faigle–Kern [5] introduced a greedy algorithm (G) shown in Fig. 1, which is known as *Faigle–Kern's dual greedy algorithm*. For the algorithm (G) and submodular functions, the next theorem is important.

Theorem 1.3 (Krüger [11], also see Ando [1]). *Let $\mathcal{L}$ be a poset shelling on E. Then for $f : \mathrm{ex}(\mathcal{L}) \to \mathbb{R}$ with $f(\emptyset) = 0$, the greedy algorithm (G) gives an optimum of the problem (D) for all $c \in \mathbb{R}_+^E$ if and only if f is b-submodular, that is, $f(X) + f(Y) \geq f(X \vee Y) + f(X \sqcap Y)$ for all $X, Y \in \mathrm{ex}(\mathcal{L})$.*

Note that b-submodularity was refereed as "K-submodularity" in Ando [1].

In order to consider optimization problems for all convex geometries, we extend Krüger's b-submodular function to a slightly different submodular function in our sense, which we call a c-submodular function. Let $\mathcal{L}$ be a convex geometry on E. A function $f : \mathrm{ex}(\mathcal{L}) \to \mathbb{R}$ with $f(\emptyset) = 0$ is *c-submodular* if, for all $X, Y \in \mathrm{ex}(\mathcal{L})$ satisfying the condition that $\chi^X + \chi^Y = \chi^{X \vee Y} + \chi^{X \sqcap Y} + \chi^{X \diamond Y}$, it holds that $f(X) + f(Y) \geq f(X \vee Y) + f(X \sqcap Y) + f(X \diamond Y)$, where $\chi^X \in \{0, 1\}^E$ is the characteristic vector of X, that is, $\chi^X(e) = 1$ when $e \in X$ and $\chi^X(e) = 0$ when $e \notin X$. The next theorem shows the relationship between b-submodularity and c-submodularity, which is proved directly from Theorem 4.6 proved later.

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Algorithm  Faigle–Kern's dual greedy algorithm (G)
Input  c ∈ ℝ^E_+
Output  y ∈ ℝ^{ex(ℒ)} (an optimum of (D)) and π (a permutation of E)
Initialization:
   y(X) ← 0   (∀X ∈ ex(ℒ));
   T ← E;
   w(e) ← c(e)   (∀e ∈ E);
   π ← ∅;
Iteration:
   while T ≠ ∅ do
      determine e ∈ ex(T) s.t. w(e) = min{w(e′) : e′ ∈ ex(T)};
      y(ex(T)) ← w(e);
      π ← eπ;
      w(x) ← w(x) − w(e)   (∀x ∈ ex(T));
      T ← T \ {e}.
   end of while
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Figure 1: Faigle–Kern's dual greedy algorithm.

Theorem 1.4. *Let $\mathcal{L}$ be a convex geometry. Then the class of c-submodular functions on $\mathrm{ex}(\mathcal{L})$ coincides with the class of b-submodular functions if and only if $\mathcal{L}$ is a poset shelling.*

We extend Theorem 1.3 to the next theorem, which is the main theorem in this paper.

Theorem 1.5. *Let $\mathcal{L}$ be a convex geometry on E. Then for $f : \mathrm{ex}(\mathcal{L}) \to \mathbb{R}$ with $f(\emptyset) = 0$, the greedy algorithm (G) gives an optimum of the problem (D) for all $c \in \mathbb{R}^E_+$ if and only if f is c-submodular.*

In Section 2, we review some properties of convex geometries. The main theorem is proved in Section 3, and in Section 4 we discuss some consequences from the main theorem. The final section is devoted to the summary of this paper and some remarks.

2 Some properties of the extreme sets

In this section, we review some properties of convex geometries which we need for the proof of the main theorem (Theorem 1.5). Particularly, we investigate the lattice of the extreme sets of a convex geometry.

The next lemma is a characterization of extreme points of a closure space, which is frequently used in this paper, explicitly or implicitly.

Lemma 2.1. *Let $\mathcal{L}$ be a closure space on E, and $A \subseteq E$ be a closed set. An element $a \in A$ is an extreme point of A if and only if $A \setminus \{a\}$ is a closed set.*

Proof. The definition of extreme points directly leads to the if-part. We now show the only-if-part. That is, we must show that: if a is an extreme point of

A, then $\tau(A \setminus \{a\}) = A \setminus \{a\}$. By the definition of the closure operator, $\tau(A \setminus \{a\}) \supseteq A \setminus \{a\}$. Suppose that $e \in \tau(A \setminus \{a\})$. Then, $e \in \tau(A)$ by the monotonicity of τ. Since a is an extreme point of A, we have $e \neq a$. Since A is a closed set, we have $e \in A \setminus \{a\}$. Therefore, $\tau(A \setminus \{a\}) \subseteq A \setminus \{a\}$. □

From the next lemma, we can see that $X \sqcap Y, X \diamond Y \in \mathrm{ex}(\mathcal{L})$ for any $X, Y \in \mathrm{ex}(\mathcal{L})$ where $\mathcal{L}$ is a convex geometry.

Lemma 2.2. *For a convex geometry $\mathcal{L}$, $X \in \mathrm{ex}(\mathcal{L})$ and $Y \subseteq X$ imply $Y \in \mathrm{ex}(\mathcal{L})$.*

Proof. If we suppose that $X \in \mathrm{ex}(\mathcal{L})$, then $a \notin \tau(\tau(X) \setminus \{a\})$ for any $a \in X$ by the definition of extreme points. Take $b \in Y$ for $Y \subseteq X$. Now we show that $b \notin \tau(\tau(Y) \setminus \{b\})$. Since $b \in Y$, we have $b \in X$ and $b \notin \tau(\tau(X) \setminus \{b\})$. Moreover, by the fact that $X \subseteq Y$ and the monotonicity of the closure operator, $\tau(\tau(Y) \setminus \{b\}) \subseteq \tau(\tau(X) \setminus \{b\})$. Therefore $b \notin \tau(\tau(Y) \setminus \{b\})$, which concludes $Y \subseteq \mathrm{ex}(Y)$. Moreover, we have $Y \supseteq \mathrm{ex}(Y)$ by the definition of extreme points. So, we have $Y \in \mathrm{ex}(\mathcal{L})$. □

Now, we show some lemmas about $\mathrm{ex}(\mathcal{L})$ as a lattice. Let "$\preceq$" be the partial order on $\mathrm{ex}(\mathcal{L})$ defined above. We can show the next lemma using the monotonicity of the closure operator and the definition of "$\preceq$."

Lemma 2.3. *Let $\mathcal{L}$ be a convex geometry. Then, $X \subseteq Y$ implies $X \preceq Y$ for all $X, Y \in \mathrm{ex}(\mathcal{L})$.*

Lemma 2.4. *Let $\mathcal{L}$ be a convex geometry. For $X, Y \in \mathrm{ex}(\mathcal{L})$ with $X \preceq Y$ and for $e \in Y$ with $e \in \tau(X)$, we have $e \in X$.*

Proof. By Lemma 2.1, $\tau(Y) \setminus \{e\}$ is a convex set. So, $(\tau(Y) \setminus \{e\}) \cap \tau(X)$ is also a convex set by the definition of a closure space. Since $X \preceq Y$, we have $\tau(X) \subseteq \tau(Y)$. Hence, $(\tau(Y) \setminus \{e\}) \cap \tau(X) = \tau(X) \setminus \{e\}$. Using Lemma 2.1, we have $e \in X$. □

If $X, Y \in \mathrm{ex}(\mathcal{L})$, then by the definition of the residue we have $X \diamond Y \subseteq X \cap Y$. Also we have $X \cap Y \subseteq X \sqcap Y$. Why? Suppose that $e \in X \cap Y$. Then, since $e \in X$ and $e \in Y$, we have $e \in \tau(X)$ and $e \in \tau(Y)$. Therefore, $e \in \tau(X) \cap \tau(Y)$, and by Lemma 2.4 we have $e \in \mathrm{ex}(\tau(X) \cap \tau(Y)) = X \wedge Y$. Since $e \in X \cup Y$, we conclude that $X \cap Y \subseteq X \sqcap Y$. Combining the argument above and Lemma 2.3, we obtain the next important lemma.

Lemma 2.5. *Let $\mathcal{L}$ be a convex geometry. Then, $X \diamond Y \preceq X \sqcap Y \preceq X \vee Y$ for all $X, Y \in \mathrm{ex}(\mathcal{L})$.*

When we see convex geometries as lattices, they belong to the class of lower semimodular lattices [3]. This means that all the maximal chains of $\mathrm{ex}(\mathcal{L})$ have the same length $|E|$. See [15] for more discussion on semimodular lattices. The next lemma is shown directly from the isomorphism between $\mathcal{L}$ and $\mathrm{ex}(\mathcal{L})$, Proposition 1.1, Lemmas 2.3 and 2.4.

Lemma 2.6. *Let $\mathcal{L}$ be a convex geometry on E. If we take any maximal chain $\mathcal{C}: \emptyset = X_0 \prec X_1 \prec X_2 \prec X_3 \prec \cdots \prec X_{n-1} \prec X_n = \mathrm{ex}(E)$ of $\mathrm{ex}(\mathcal{L})$ where $n = |E|$, then for any $a \in E$ there exist natural numbers l, m such that $1 \leq l \leq m \leq n$ which satisfy: $0 \leq i < l$ if and only if $a \notin \tau(X_i)$; $l \leq i \leq m$ if and only if $a \in X_i$; $m < i \leq n$ if and only if $a \notin X_i$,$a \in \tau(X_i)$.*

Table 1: All the possible cases.

	$\mathfrak{A}$	$\mathfrak{B}$	$\mathfrak{C}$	$\mathfrak{D}$	$\mathfrak{E}$	$\mathfrak{F}$	$\mathfrak{G}$	$\mathfrak{H}$	$\mathfrak{I}$	$\mathfrak{J}$
X	(in)	(in)	(in)	(in)	(ex)	(ex)	(ex)	(ex)	(out)	(out)
Y	(in)	(in)	(ex)	(out)	(ex)	(ex)	(out)	(out)	(out)	(out)
$X \vee Y$	(in)	(in)	(in)	(in)	(in)	(ex)	(in)	(ex)	(in)	(out)
$X \wedge Y$	(in)	(ex)	(ex)	(out)	(ex)	(ex)	(out)	(out)	(out)	(out)

By Lemma 2.6, every extreme set $Z \in \mathrm{ex}(\mathcal{L})$ satisfies either of the followings: for each $a \in E$

$$a \notin Z, a \in \tau(Z) \quad \text{(This case is indicated by (in));} \tag{2.1}$$

$$a \in Z \quad \text{(This case is indicated by (ex));} \tag{2.2}$$

$$a \notin \tau(Z) \quad \text{(This case is indicated by (out)).} \tag{2.3}$$

This leads to the following remark.

Remark 2.7. *Let $X, Y \in \mathrm{ex}(\mathcal{L})$. If either X or Y is in the case (in) for $a \in E$, then $X \vee Y$ is also in the case (in) for the same $a \in E$. All the possible cases where either X or Y is in the case (in) for $a \in E$ are indicated in Table 1 as $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ and $\mathfrak{D}$.*

Proof. Without loss of generality, assume that X is in the case (in) for $a \in E$. We take a maximal chain $\mathcal{C} : \emptyset = X_0 \prec X_1 \prec X_2 \prec X_3 \prec \cdots \prec X_{n-1} \prec X_n = \mathrm{ex}(E)$ of $\mathrm{ex}(\mathcal{L})$ containing X and $X \vee Y$. Suppose that $X_i = X$ and $X_j = X \vee Y$, with $i \leq j$. From the assumption, X satisfies that $a \notin X, a \in \tau(X)$. Then, by Lemma 2.6, $X \vee Y$ also satisfies that $a \notin X \vee Y$ and $a \in \tau(X \vee Y)$ for the same $a \in E$. This means that $X \vee Y$ is in the case (in). $\square$

Similarly, we can obtain some consequences. We arrange them in Table 1, which shows all the possible cases. Now we show some non-trivial cases in Table 1.

Remark 2.8. *Let $X, Y \in \mathrm{ex}(\mathcal{L})$. If X and Y are in the case (ex) for $a \in E$, then $X \wedge Y$ is in the case (ex) for the same $a \in E$. This fact is indicated in Table 1 as $\mathfrak{E}$ and $\mathfrak{F}$.*

Proof. Suppose that $X \wedge Y$ is not in the case (ex) for a, that is, $a \notin X \wedge Y$. Since X and Y are in the case (ex) for a, we have $a \in \tau(X) \cap \tau(Y)$. Therefore, $a \in (\tau(X) \cap \tau(Y)) \setminus (X \wedge Y)$. This implies that $X \wedge Y \subseteq \tau(X) \setminus \{a\}$. By the monotonicity of τ, we have $\tau(X \wedge Y) \subseteq \tau(\tau(X) \setminus \{a\})$. However, $a \in \tau(X) \cap \tau(Y) = \tau(X \wedge Y)$ and $a \notin \tau(\tau(X) \setminus \{a\})$, which leads to a contradiction. $\square$

Remark 2.9. *Let $X, Y \in \mathrm{ex}(\mathcal{L})$. If X and Y are in the case (out) for $a \in E$, then $X \vee Y$ is in the case (in) or (out) for the same $a \in E$. This fact is indicated in Table 1 as $\mathfrak{I}$ and $\mathfrak{J}$.*

Proof. Suppose that $X \vee Y$ is in the case (ex) for $a \in E$, that is, $a \in X \vee Y$. This means that a is an extreme point of $\tau(\tau(X) \cup \tau(Y))$. So we have $\tau(\tau(X) \cup \tau(Y)) \setminus \{a\} \in \mathcal{L}$, which is equivalent to $a \notin \tau(\tau(\tau(X) \cup \tau(Y)) \setminus \{a\})$.

Since X and Y are in the case $(\mathfrak{ex})$ for a, we have $a \notin \tau(X), \tau(Y)$. So we have $a \notin \tau(X) \cup \tau(Y)$. Since $a \in \tau(\tau(X) \cup \tau(Y))$, we have $a \in \tau(\tau(X) \cup \tau(Y)) \setminus (\tau(X) \cup \tau(Y))$. Therefore, $\tau(X) \cup \tau(Y) \subseteq \tau(\tau(X) \cup \tau(Y)) \setminus \{a\}$. By the monotonicity of τ, we have $\tau(\tau(X) \cup \tau(Y)) \subseteq \tau(\tau(\tau(X) \cup \tau(Y)) \setminus \{a\})$. However, $a \in \tau(\tau(X) \cup \tau(Y))$ and $a \notin \tau(\tau(\tau(X) \cup \tau(Y)) \setminus \{a\})$, which leads to a contradiction. □

We have some additional remarks, which can be easily checked.

Remark 2.10. *1.* $X, Y \in \mathrm{ex}(\mathcal{L})$ *are in the case* $\mathfrak{B}$ *for* $a \in E$ *if and only if* $a \in (X \wedge Y) \setminus (X \sqcap Y)$.

2. $X, Y \in \mathrm{ex}(\mathcal{L})$ *are in the case* $\mathfrak{E}$ *for* $a \in E$ *if and only if* $a \in X \diamond Y$.

3. $X, Y \in \mathrm{ex}(\mathcal{L})$ *are in the case* $\mathfrak{G}$ *for* $a \in E$ *if and only if* $\chi^X(a) + \chi^Y(a) > \chi^{X \vee Y}(a) + \chi^{X \sqcap Y}(a) + \chi^{X \diamond Y}(a)$. *Otherwise,* $\chi^X(a) + \chi^Y(a) = \chi^{X \vee Y}(a) + \chi^{X \sqcap Y}(a) + \chi^{X \diamond Y}(a)$.

From the discussion above we can easily see the following lemma.

Lemma 2.11. *Let* $\mathcal{L}$ *be a convex geometry. Then* $X \vee Y \subseteq X \cup Y$ *for all* $X, Y \in \mathrm{ex}(\mathcal{L})$.

3 Proof of the main theorem

In this section, we give the proof of Theorem 1.5. First, we make some remarks on the greedy algorithm (G) without proofs, which can be easily checked.

Remark 3.1. *1. During the execution of* (G), $w(e) \geq 0$ *holds for all* $e \in E$. *Therefore,* $y(\mathrm{ex}(T))$ *in line 9 is always non-negative.*

2. During the execution of `Iteration` *in* (G), $\mathrm{ex}(T) \neq \emptyset$ *holds.*

3. Suppose that there exist more than one elements $e \in \mathrm{ex}(T)$ *such that* $w(e)$ *is minimum in line 8 of* (G) *at an* `Iteration` *step. Then, whichever one of the candidates we choose at the step, the output* y *is the same at the termination of* (G).

Now, we prove some lemmas for the main theorem. Let $n = |E|$. First we can easily show that the sequence $\pi = e_1 e_2 \cdots e_n$ of the elements in E as the output of (G) satisfies the next condition: $e_i \in \tau(\{e_1, \ldots, e_j\})$ implies $i \leq j$. This means $\{e_1, \ldots, e_j\}$ is a convex set. Therefore, we have $\mathrm{ex}(\{e_1, e_2, \ldots, e_i\}) \in \mathrm{ex}(\mathcal{L})$. We set $X_i^\pi = \mathrm{ex}(\{e_1, e_2, \ldots, e_i\})$. Let x^π be the unique solution of the system of the following linear equations:

$$\sum_{e \in X_i^\pi} x^\pi(e) = f(X_i^\pi) \quad (i = 1, \ldots, n). \tag{3.1}$$

We call x^π the *greedy vector* with respect to π. The uniqueness of the solution of (3.1) is ensured by the next lemma, which is directly shown by Lemma 2.6.

Lemma 3.2. *There exists* $l \in \{i, \ldots, n\}$ *for all* $i \in \{1, \ldots, n\}$ *such that* $e_i \in X_j^\pi$ *if and only if* $i \leq j \leq l$.

Notice that $X_i^\pi \prec X_j^\pi$ for all $i < j$ in $\mathrm{ex}(\mathcal{L})$. If $\mathcal{C}^\pi : \emptyset \prec X_1^\pi \prec X_2^\pi \prec \cdots \prec X_n^\pi$ is a maximal chain of $\mathrm{ex}(\mathcal{L})$, we call $\mathcal{C}^\pi$ the *greedy chain* with respect to π.

Here we have another lemma.

Lemma 3.3. *The vector $y \in \mathbb{R}_+^{\mathrm{ex}(\mathcal{L})}$ as the output of* (G) *coincides with the unique solution of the following system of equations:*

$$\sum_{\substack{X \in \{X_1^\pi, \dots, X_n^\pi\} \\ s.t.\ e_i \in X}} y^\pi(X) = c(e_i) \quad (i \in \{1, \dots, n\}), \tag{3.2}$$

$$y^\pi(X) = 0 \quad (X \in \mathrm{ex}(\mathcal{L}) \setminus \{X_1^\pi, \dots, X_n^\pi\}). \tag{3.3}$$

We call y^π the dual greedy vector with respect to π.

Proof. From the algorithm (G), we have $y^\pi(X) = 0$ for all $X \in \mathrm{ex}(\mathcal{L}) \setminus \{X_1^\pi, \dots, X_n^\pi\}$. For $i \in \{1, \dots, n\}$, by Lemma 3.2

$$\sum_{\substack{X \in \{X_1^\pi, \dots, X_n^\pi\} \\ \text{s.t. } e_i \in X}} y^\pi(X) = \sum_{i \le j \le l} y^\pi(X_j^\pi) = \sum_{i \le j \le l} w(e_j) = c(e_i). \tag{3.4}$$

The uniqueness is obtained from the form of the equations, determined by Lemma 3.2. ☐

From the definitions of the greedy vector and the dual greedy vector, we can easily see complementary slackness.

Lemma 3.4. *The greedy vector x^π and the dual greedy vector y^π satisfy complementary slackness, that is, $y^\pi(X) > 0$ implies $\sum\{x^\pi(e) : e \in X\} = f(X)$ for all $X \in \mathrm{ex}(\mathcal{L})$.*

Next, we show the feasibility of the greedy vector x^π and the dual greedy vector y^π. From the definition of the dual greedy vector, we have $y^\pi \ge 0$. By Lemma 3.3, we can easily check that $\sum\{y^\pi(X) : X \in \mathrm{ex}(\mathcal{L}) \text{ s.t. } e \in X\} = c(e)$. Thus, we have the next lemma, which says the feasibility of the dual greedy vector y^π.

Lemma 3.5. *The dual greedy vector y^π is a feasible solution of the problem* (D).

For a convex geometry $\mathcal{L}$ and $A \in \mathcal{L}$, define the *contraction* of $\mathcal{L}$ by A as $\mathcal{L}/A = \{B \setminus A : B \in \mathcal{L}, A \subseteq B\}$. Similarly, for a convex geometry $\mathcal{L}$ and $E \setminus A \in \mathcal{L}$, define the *deletion* of $\mathcal{L}$ by A as $\mathcal{L} \setminus A = \{B \in \mathcal{L} : B \subseteq E \setminus A\}$. Notice that every contraction of a convex geometry is also a convex geometry and so is every deletion.

Let $\mathcal{L}$ be a convex geometry on E satisfying $|E| > 1$. Define $\mathrm{atom}(\mathcal{L}) = \{e \in E : \{e\} \in \mathcal{L}\}$. For $e \in \mathrm{atom}(\mathcal{L})$, we consider the contraction $\mathcal{L}/\{e\}$. We use τ' and ex' for the closure operator and the extreme operator on $\mathcal{L}/\{e\}$ respectively, because of the distinction from the operators on $\mathcal{L}$. So we write $\mathrm{ex}'(\mathcal{L}/\{e\}) = \{\mathrm{ex}'(X) : X \in \mathcal{L}/\{e\}\}$. Furthermore, the join, the meet, the reduced meet and the residue on $\mathrm{ex}'(\mathcal{L}/\{e\})$ are denoted by $\vee'$, $\wedge'$, $\sqcap'$ and $\diamond'$, respectively.

Now we consider the relationship between $\mathrm{ex}'(\mathcal{L}/\{e\})$ and $\mathrm{ex}(\mathcal{L})$. First, we can easily check that for $A \subseteq E \setminus \{e\}$, $\tau'(A) = \tau(A \cup \{e\}) \setminus \{e\}$, and $\mathrm{ex}'(A) = \mathrm{ex}(A \cup \{e\}) \setminus \{e\}$. Here we define the map $\iota : \mathrm{ex}'(\mathcal{L}/\{e\}) \to \{X \in \mathrm{ex}(\mathcal{L}) : e \in \tau(X)\} \subseteq \mathrm{ex}(\mathcal{L})$ as $\iota(X) = \mathrm{ex}(\tau'(X) \cup \{e\})$ for all $X \in \mathrm{ex}'(\mathcal{L}/\{e\})$. Then we have for $X, Y \in \mathrm{ex}'(\mathcal{L}/\{e\})$

$$\iota(X \vee' Y) = \iota(X) \vee \iota(Y), \quad \iota(X \wedge' Y) = \iota(X) \wedge \iota(Y). \tag{3.5}$$

Therefore, ι is an isomorphism.

We notice that if $X \in \mathrm{ex}'(\mathcal{L}/\{e\})$, then $\iota(X)$ is in the case (in) or (ex) for e since $e \in \tau(\iota(X))$. In fact, we have the following.

Lemma 3.6. *Let $\iota : \mathrm{ex}'(\mathcal{L}/\{e\}) \to \{X \in \mathrm{ex}(\mathcal{L}) : e \in \tau(X)\}$ be defined above. Then $\iota(X) = X \cup \{e\}$ when $\iota(X)$ is in the case* (ex) *for e, and $\iota(X) = X$ when $\iota(X)$ is in the case* (in) *for e.*

Proof. First suppose that $\iota(X)$ is in the case (ex) for e. We have $e \in \iota(X)$. Moreover we have $\iota(X) \setminus \{e\} = \mathrm{ex}(\tau'(X) \cup \{e\}) \setminus \{e\} = \mathrm{ex}'(\tau'(X)) = X$. So it holds that $\iota(X) = X \cup \{e\}$.

Secondly suppose that $\iota(X)$ is in the case (in) for e. For this case, we have $e \notin \iota(X)$ and we can see that $\iota(X) \setminus \{e\} = X$ in the same way as discussed in the first case. So we have $\iota(X) = X$. $\square$

Remark that we can rewrite ι as $\iota(X) = X \cup \{e\}$ when $X \cup \{e\} \in \mathrm{ex}(\mathcal{L})$ and $\iota(X) = X$ when $X \cup \{e\} \notin \mathrm{ex}(\mathcal{L})$, which is ensured by the next lemma.

Lemma 3.7. *Let $X \in \mathrm{ex}'(\mathcal{L}/\{e\})$. Then $\iota(X)$ is in the case* (ex) *for e if and only if $X \cup \{e\} \in \mathrm{ex}(\mathcal{L})$. In other words, $\iota(X)$ is in the case* (in) *for e if and only if $X \cup \{e\} \notin \mathrm{ex}(\mathcal{L})$.*

Proof. Lemma 3.6 immediately implies that if $\iota(X)$ is in the case (ex) for e, then $X \cup \{e\} \in \mathrm{ex}(\mathcal{L})$. Let $\iota(X)$ be in the case (in) for e. By Lemma 3.6, we have $\iota(X) = X$. Suppose $X \cup \{e\} \in \mathrm{ex}(\mathcal{L})$. By Lemma 2.3, $X \preceq X \cup \{e\}$. So $X \cup \{e\}$ should be in the case (in) for e by Lemma 2.6. This contradicts to $e \in X \cup \{e\}$. $\square$

Lemma 3.6 implies that $X \subseteq \iota(X)$ for $X \in \mathrm{ex}'(\mathcal{L}/\{e\})$. Therefore, by Lemma 2.2, if $X \in \mathrm{ex}'(\mathcal{L}/\{e\})$ then also $X \in \mathrm{ex}(\mathcal{L})$. Moreover, we can show the next lemma.

Lemma 3.8. *Let $X \in \mathrm{ex}'(\mathcal{L}/\{e\})$. Then $\iota(X)$ is in the case* (ex) *for e if and only if $X \vee \{e\} = X \cup \{e\}$.*

Proof. By Lemma 2.11, we have $X \vee \{e\} \subseteq X \cup \{e\}$. Suppose that $\iota(X)$ is in the case (ex) for e. Then we have $X \cup \{e\} \in \mathrm{ex}(\mathcal{L})$. By Lemma 2.3, we have $X \vee \{e\} \preceq X \cup \{e\}$. Now we show that $X \cup \{e\} \preceq X \vee \{e\}$. Since $X \subseteq \tau(X)$, we have $X \cup \{e\} \subseteq \tau(X) \cup \{e\}$. From the monotonicity of τ, we have $\tau(X \cup \{e\}) \subseteq \tau(\tau(X) \cup \{e\}) = \tau(X \vee \{e\})$. Hence, $X \cup \{e\} \preceq X \vee \{e\}$. It concludes that $X \vee \{e\} = X \cup \{e\}$.

Conversely, suppose that $\iota(X)$ is in the case (in) for e. By Lemma 3.7, we have $X \cup \{e\} \notin \mathrm{ex}(\mathcal{L})$. Since $X \vee \{e\} \in \mathrm{ex}(\mathcal{L})$, it concludes that $X \vee \{e\} \neq X \cup \{e\}$. $\square$

Now we define $f' : \mathrm{ex}'(\mathcal{L}/\{e\}) \to \mathbb{R}$ as

$$f'(X) = \begin{cases} f(X \cup \{e\}) - f(\{e\}) & (\iota(X) \text{ is in the case (ex) for } e), \\ f(X) & (\iota(X) \text{ is in the case (in) for } e). \end{cases} \tag{3.6}$$

Lemma 3.9. *Let* $e \in \mathrm{atom}(\mathcal{L})$ *and* $f : \mathrm{ex}(\mathcal{L}) \to \mathbb{R}$ *be c-submodular. Then,* $f'(X) \leq f(X)$ *for all* $X \in \mathrm{ex}'(\mathcal{L}/\{e\})$.

Proof. Suppose that $\iota(X)$ is in the case (in) for e. Then $f'(X) = f(X)$ by the definition of f'.

Suppose that $\iota(X)$ is in the case (ex) for e. Since f is c-submodular, $e \notin X$ and $X \sqcap \{e\} = X \diamond \{e\} = \emptyset$, we have $f(X) + f(\{e\}) \geq f(X \vee \{e\}) = f(X \cup \{e\})$. Here we use Lemma 3.8. Therefore, $f'(X) = f(X \cup \{e\}) - f(\{e\}) \leq f(X)$. □

The next is a key lemma for the main theorem.

Lemma 3.10. *Let* $|E| > 1$ *and* $e \in \mathrm{atom}(\mathcal{L})$. *If* $f : \mathrm{ex}(\mathcal{L}) \to \mathbb{R}$ *is c-submodular, then* $f' : \mathrm{ex}'(\mathcal{L}/\{e\}) \to \mathbb{R}$ *defined by (3.6) is also c-submodular.*

Proof. Since $e \in \tau(\iota(X)), \tau(\iota(Y))$ for all $X, Y \in \mathrm{ex}'(\mathcal{L}/\{e\})$, we only have to consider the cases $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{E}$, $\mathfrak{F}$ in Table 1 with respect to $\iota(X)$, $\iota(Y)$, $\iota(X \vee' Y)$ and $\iota(X \wedge' Y)$.

Case: $\mathfrak{A}$. *That is,* $\iota(X) = X$, $\iota(Y) = Y$, $\iota(X \vee' Y) = X \vee' Y$ *and* $\iota(X \wedge' Y) = X \wedge' Y$. By (3.5), $\iota(X \vee' Y) = \iota(X) \vee \iota(Y) = X \vee Y$. Similarly, by (3.5), $\iota(X \wedge' Y) = \iota(X) \wedge \iota(Y) = X \wedge Y$. Therefore we have $X \sqcap' Y = X \sqcap Y$ and $X \diamond' Y = X \diamond Y$. Hence, if $\chi^X + \chi^Y = \chi^{X \vee' Y} + \chi^{X \sqcap' Y} + \chi^{X \diamond' Y}$ for $\mathrm{ex}'(\mathcal{L}/\{e\})$, then we also have $\chi^X + \chi^Y = \chi^{X \vee Y} + \chi^{X \sqcap Y} + \chi^{X \diamond Y}$ for $\mathrm{ex}(\mathcal{L})$. By Lemma 3.9,

$$\begin{aligned} f'(X) + f'(Y) &= f(X) + f(Y) \\ &\geq f(X \vee Y) + f(X \sqcap Y) + f(X \diamond Y) \\ &\geq f'(X \vee Y) + f'(X \sqcap Y) + f'(X \diamond Y) \\ &= f'(X \vee' Y) + f'(X \sqcap' Y) + f'(X \diamond' Y). \end{aligned}$$

Case: $\mathfrak{B}$. *That is,* $\iota(X) = X$, $\iota(Y) = Y$, $\iota(X \vee' Y) = X \vee' Y$ *and* $\iota(X \wedge' Y) = (X \wedge' Y) \cup \{e\}$. By (3.5), $\iota(X \vee' Y) = \iota(X) \vee \iota(Y) = X \vee Y$. Similarly, by (3.5), $\iota(X \wedge' Y) = \iota(X) \wedge \iota(Y) = X \wedge Y$. Therefore, we have $X \diamond' Y = X \diamond Y$. Moreover, since $e \notin X \cup Y$,

$$\begin{aligned} X \sqcap Y &= (X \wedge Y) \cap (X \cup Y) = \iota(X \wedge' Y) \cap (X \cup Y) \\ &= ((X \wedge' Y) \cup \{e\}) \cap (X \cup Y) = (X \wedge' Y) \cap (X \cup Y) = X \sqcap' Y. \end{aligned}$$

Hence, if $\chi^X + \chi^Y = \chi^{X \vee' Y} + \chi^{X \sqcap' Y} + \chi^{X \diamond' Y}$ for $\mathrm{ex}'(\mathcal{L}/\{e\})$, then we have $\chi^X + \chi^Y = \chi^{X \vee Y} + \chi^{X \sqcap Y} + \chi^{X \diamond Y}$ for $\mathrm{ex}(\mathcal{L})$. By Lemma 3.9,

$$\begin{aligned} f'(X) + f'(Y) &= f(X) + f(Y) \\ &\geq f(X \vee Y) + f(X \sqcap Y) + f(X \diamond Y) \\ &\geq f'(X \vee Y) + f'(X \sqcap Y) + f'(X \diamond Y) \\ &= f'(X \vee' Y) + f'(X \sqcap' Y) + f'(X \diamond' Y). \end{aligned}$$

Case: $\mathfrak{C}$. *That is,* $\iota(X) = X$, $\iota(Y) = Y \cup \{e\}$, $\iota(X \vee' Y) = X \vee' Y$, $\iota(X \wedge' Y) = (X \wedge' Y) \cup \{e\}$. By (3.5), $\iota(X \vee' Y) = \iota(X) \vee \iota(Y) = X \vee (Y \cup \{e\})$. Similarly, by (3.5), $\iota(X \wedge' Y) = \iota(X) \wedge \iota(Y) = X \wedge (Y \cup \{e\})$. Therefore, $X \vee' Y = X \vee (Y \cup \{e\})$ and $(X \wedge' Y) \cup \{e\} = X \wedge (Y \cup \{e\})$. Since $Y \subseteq Y \cup \{e\}$ and Lemma 2.3, we have $Y \preceq Y \cup \{e\}$. Also, since $Y \cup \{e\} \subseteq \tau(Y) \cup \{e\} \subseteq \tau(X) \cup \tau(Y) \subseteq \tau(\tau(X) \cup \tau(Y)) = \tau(X \vee Y)$, we have $\tau(Y \cup \{e\}) \subseteq \tau(X \vee Y)$. Therefore, $Y \cup \{e\} \preceq X \vee Y$. Hence, we also have $X \vee Y = X \vee (Y \cup \{e\}) = X \vee' Y$, and $X \diamond (Y \cup \{e\}) = X \diamond Y = X \diamond' Y$. Moreover, we have

$$\begin{aligned}(X \sqcap' Y) \cup \{e\} &= ((X \wedge' Y) \cap (X \cup Y)) \cup \{e\} \\ &= ((X \wedge' Y) \cup \{e\}) \cap ((X \cup Y) \cup \{e\}) \\ &= (X \wedge (Y \cup \{e\})) \cap (X \cup (Y \cup \{e\})) \\ &= X \sqcap (Y \cup \{e\}).\end{aligned}$$

Now, if $\chi^X + \chi^Y = \chi^{X \vee' Y} + \chi^{X \sqcap' Y} + \chi^{X \diamond' Y}$ for $\mathrm{ex}'(\mathcal{L}/\{e\})$, then we also have $\chi^X + \chi^{Y \cup \{e\}} = \chi^{X \vee (Y \cup \{e\})} + \chi^{X \sqcap (Y \cup \{e\})} + \chi^{X \diamond (Y \cup \{e\})}$ for $\mathrm{ex}(\mathcal{L})$. By Lemma 3.9,

$$\begin{aligned}f'(X) + f'(Y) &= f(X) + f(Y \cup \{e\}) - f(\{e\}) \\ &\geq f(X \vee (Y \cup \{e\}) + f(X \sqcap (Y \cup \{e\})) \\ &\quad + f(X \diamond (Y \cup \{e\})) - f(\{e\}) \\ &= f(X \vee' Y) + f((X \sqcap' Y) \cup \{e\}) + f(X \diamond' Y) - f(\{e\}) \\ &\geq f'(X \vee' Y) + f'(X \sqcap' Y) + f'(X \diamond' Y).\end{aligned}$$

Case: $\mathfrak{E}$. *That is,* $\iota(X) = X \cup \{e\}$, $\iota(Y) = Y \cup \{e\}$, $\iota(X \vee' Y) = X \vee' Y$ *and* $\iota(X \wedge' Y) = (X \wedge' Y) \cup \{e\}$. By (3.5), $\iota(X \vee' Y) = \iota(X) \vee \iota(Y) = (X \cup \{e\}) \vee (Y \cup \{e\})$. Similarly, by (3.5), $\iota(X \wedge' Y) = \iota(X) \wedge \iota(Y) = (X \cup \{e\}) \wedge (Y \cup \{e\})$. Since $X \subseteq X \cup \{e\}$ and Lemma 2.3, we have $X \preceq X \cup \{e\}$. Also, since $X \cup \{e\} \subseteq \tau(X) \cup \{e\} \subseteq \tau(X) \cup \tau(Y) \subseteq \tau(\tau(X) \cup \tau(Y)) = \tau(X \vee Y)$, we have $\tau(X) \cup \{e\} \subseteq \tau(X \vee Y)$. Therefore, $X \preceq X \cup \{e\} \preceq X \vee Y$. Similarly, we have $Y \preceq Y \cup \{e\} \preceq X \vee Y$. Hence, we also have $X \vee Y = (X \cup \{e\}) \vee (Y \cup \{e\}) = X \vee' Y$. Moreover, we have

$$\begin{aligned}(X \sqcap' Y) \cup \{e\} &= ((X \wedge' Y) \cap (X \cup Y)) \cup \{e\} \\ &= ((X \wedge' Y) \cup \{e\}) \cap ((X \cup Y) \cup \{e\}) \\ &= ((X \cup \{e\}) \wedge (Y \cup \{e\})) \cap ((X \cup \{e\}) \cup (Y \cup \{e\})) \\ &= (X \cup \{e\}) \sqcap (Y \cup \{e\}), \\ (X \diamond' Y) \cup \{e\} &= ((X \cap Y) \setminus (X \vee' Y)) \cup \{e\} \\ &= ((X \cap Y) \setminus (X \vee Y)) \cup \{e\} \\ &= ((X \cap Y) \cup \{e\}) \setminus (X \vee Y) \\ &= ((X \cup \{e\}) \cap (Y \cup \{e\})) \setminus ((X \cup \{e\}) \vee (Y \cup \{e\})) \\ &= (X \cup \{e\}) \diamond (Y \cup \{e\}).\end{aligned}$$

Hence, if $\chi^X + \chi^Y = \chi^{X \vee' Y} + \chi^{X \sqcap' Y} + \chi^{X \diamond' Y}$ for $\mathrm{ex}'(\mathcal{L}/\{e\})$, then we also have $\chi^{X \cup \{e\}} + \chi^{Y \cup \{e\}} = \chi^{(X \cup \{e\}) \vee (Y \cup \{e\})} + \chi^{(X \cup \{e\}) \sqcap (Y \cup \{e\})} + \chi^{(X \cup \{e\}) \diamond (Y \cup \{e\})}$ for $\mathrm{ex}(\mathcal{L})$. By Lemma 3.9,

$$\begin{aligned}f'(X) + f'(Y) &= f(X \cup \{e\}) + f(Y \cup \{e\}) - 2f(\{e\}) \\ &\geq f((X \cup \{e\}) \vee (Y \cup \{e\})) + f((X \cup \{e\}) \sqcap (Y \cup \{e\})) \\ &\quad + f((X \cup \{e\}) \diamond (Y \cup \{e\})) - 2f(\{e\}) \\ &= f(X \vee' Y) + f((X \sqcap' Y) \cup \{e\}) + f((X \diamond' Y) \cup \{e\}) \\ &\quad - 2f(\{e\}) \\ &\geq f'(X \vee' Y) + f'(X \sqcap' Y) + f'(X \diamond' Y).\end{aligned}$$

Case: $\mathfrak{F}$. *That is,* $\iota(X) = X \cup \{e\}$, $\iota(Y) = Y \cup \{e\}$, $\iota(X \vee' Y) = (X \vee' Y) \cup \{e\}$ *and* $\iota(X \wedge' Y) = (X \wedge' Y) \cup \{e\}$. By (3.5), $\iota(X \vee' Y) = \iota(X) \vee \iota(Y) = (X \cup \{e\}) \vee (Y \cup \{e\})$. Similarly, by (3.5), $\iota(X \wedge' Y) = \iota(X) \wedge \iota(Y) = (X \cup \{e\}) \wedge (Y \cup \{e\})$. Therefore, $(X \vee' Y) \cup \{e\} = (X \cup \{e\}) \vee (Y \cup \{e\})$ and $(X \wedge' Y) \cup \{e\} = (X \cup \{e\}) \wedge (Y \cup \{e\})$. Similarly as in the case $\mathfrak{C}$, we have $(X \cup \{e\}) \vee (Y \cup \{e\}) = (X \vee' Y) \cup \{e\}$, $(X \cup \{e\}) \sqcap (Y \cup \{e\}) = (X \sqcap' Y) \cup \{e\}$ and $X \diamond' Y = (X \cup Y) \setminus (X \vee' Y) = ((X \cup Y) \cup \{e\}) \setminus ((X \vee' Y) \cup \{e\}) = ((X \cup \{e\}) \cap (Y \cup \{e\})) \setminus ((X \cup \{e\}) \vee (Y \cup \{e\})) = (X \cup \{e\}) \diamond (Y \cup \{e\})$. Hence, if $\chi^X + \chi^Y = \chi^{X \vee' Y} + \chi^{X \sqcap' Y} + \chi^{X \diamond' Y}$ for $\mathrm{ex}'(\mathcal{L}/\{e\})$, then we also have $\chi^{X \cup \{e\}} + \chi^{Y \cup \{e\}} = \chi^{(X \cup \{e\}) \vee (Y \cup \{e\})} + \chi^{(X \cup \{e\}) \sqcap (Y \cup \{e\})} + \chi^{(X \cup \{e\}) \diamond (Y \cup \{e\})}$ for $\mathrm{ex}(\mathcal{L})$. By Lemma 3.9,

$$\begin{aligned}
f'(X) + f'(Y) &= f(X \cup \{e\}) + f(Y \cup \{e\}) - 2f(\{e\}) \\
&\geq f((X \cup \{e\}) \vee (Y \cup \{e\})) + f((X \cup \{e\}) \sqcap (Y \cup \{e\})) \\
&\quad + f((X \cup \{e\}) \diamond (Y \cup \{e\})) - 2f(\{e\}) \\
&= f((X \vee' Y) \cup \{e\}) + f((X \sqcap' Y) \cup \{e\}) + f(X \diamond' Y) \\
&\quad - 2f(\{e\}) \\
&\geq f'(X \vee' Y) + f'(X \sqcap' Y) + f'(X \diamond' Y).
\end{aligned}$$

□

In order to prove Lemma 3.12 we need the next lemma. Let $e \in \mathrm{atom}(\mathcal{L})$ and $X \in \mathrm{ex}(\mathcal{L})$ be in the case (out) for e, i.e., $e \notin \tau(X)$. Set $Y = (X \cup \{e\}) \setminus (X \vee \{e\})$. Remark that $e \in X \vee \{e\}$ and we have $Y \subseteq X$. So, by Lemma 2.3 $Y \in \mathrm{ex}(\mathcal{L})$.

Lemma 3.11. *Let $e \in \mathrm{atom}(\mathcal{L})$ and $X \in \mathrm{ex}(\mathcal{L})$ be in the case (out) for e, i.e., $e \notin \tau(X)$. Also let $f : \mathrm{ex}(\mathcal{L}) \to \mathbb{R}$ be c-submodular. Then it holds that $f(X) + f(\{e\}) \geq f(X \vee \{e\}) + f(Y)$ where $Y = (X \cup \{e\}) \setminus (X \vee \{e\})$.*

Proof. Since $X \vee \{e\} \subseteq X \cup \{e\}$, we have the following two cases.

Case 1: $X \vee \{e\} = X \cup \{e\}$. In this case, we have $f(Y) = 0$ since $Y = \emptyset$. So by the c-submodularity of f, we have $f(X) + f(\{e\}) \geq f(X \vee \{e\})$.

Case 2: $X \vee \{e\} \subsetneq X \cup \{e\}$. First remark that Y is also in the case (out) for e by Lemmas 2.3 and 2.6. So we use the induction in terms of the rank of X, i.e., the size of $\tau(X)$. The base case is trivial. Let $W = (Y \cup \{e\}) \setminus (Y \vee \{e\}) \in \mathrm{ex}(\mathcal{L})$. Then W is also in the case (out) for e. Also we have $W \prec Y \prec X$ by Lemma 2.3. From the assumption of induction, we have

$$f(Y) + f(\{e\}) \geq f(Y \vee \{e\}) + f(W). \tag{3.7}$$

Moreover, we have $X \sqcap (Y \vee \{e\}) = (X \wedge (Y \vee \{e\})) \cap (X \cup (Y \vee \{e\})) = Y \cap (X \cup ((Y \cup \{e\}) \setminus W)) = Y$, $X \diamond (Y \vee \{e\}) = (X \cap (Y \vee \{e\})) \setminus (X \vee (Y \vee \{e\})) = ((Y \vee \{e\}) \setminus \{e\}) \setminus (X \vee \{e\}) = (Y \setminus W) \setminus ((X \cup \{e\}) \setminus Y) = Y \setminus W$, and $(Y \setminus X) \sqcap W = (Y \setminus W) \diamond W = \emptyset$. So the c-submodularity of f induces

$$f(Y \vee \{e\}) + f(X) \geq f(X \vee \{e\}) + f(Y) + f(Y \setminus W), \tag{3.8}$$

$$f(Y \setminus W) + f(W) \geq f(Y). \tag{3.9}$$

Then, summing up the inequalities (3.7)–(3.9) results in the conclusion. □

Now, we prove the feasibility of the greedy vector x^π when f is c-submodular.

Lemma 3.12. *If f is c-submodular, then the greedy vector x^π is a feasible solution of the problem* (P).

Proof. We will prove $\sum\{x^\pi(e) : e \in X\} \leq f(X)$ for all $X \in \mathrm{ex}(\mathcal{L})$ by induction on $|E|$ and the rank of X. It trivially holds for the case of $|E| = 1$ and the cases when the rank of X is 0.

For the output $\pi = e_1 e_2 \cdots e_n$ of (G), set $\pi' = e_2 \cdots e_n$. Define $f' : \mathrm{ex}'(\mathcal{L}/\{e_1\}) \to \mathbb{R}$ as (3.6). Note that $e_1 \in \mathrm{atom}(\mathcal{L})$ and that f' is c-submodular by Lemma 3.10. Let $x^{\pi'}$ be the unique solution of the system of the following equations:

$$\sum_{e \in X_i^\pi \setminus \{e_1\}} x^{\pi'}(e) = f'(X_i^\pi \setminus \{e_1\}) \quad (i = 2, \ldots, n),$$

where $X_i^\pi = \mathrm{ex}(\{e_1, \ldots, e_i\})$. Notice that $x^{\pi'}(e) = x^\pi(e)$ for all $e \in \{e_2, \ldots, e_n\}$. Now assume that $\sum\{x^{\pi'}(e) : e \in Z\} \leq f'(Z)$ for all $Z \in \mathrm{ex}'(\mathcal{L}/\{e_1\})$ by the induction hypothesis.

Consider the case when $|E| = n > 1$. Fix $X \in \mathrm{ex}(\mathcal{L})$ such that the rank of X is greater than zero. By the induction hypothesis, we assume that $\sum\{x(e) : e \in \tilde{X}\} \leq f(\tilde{X})$ for all $\tilde{X} \in \mathrm{ex}(\mathcal{L})$ such that the rank of $\tilde{X}$ is less than that of X.

Now we consider the following three cases for each $X \in \mathrm{ex}(\mathcal{L})$.

Case 1: X is in the case (in) *for* e_1. In this case, we have $\iota(X) = X$. Therefore,

$$\sum_{e \in X} x^\pi(e) = \sum_{e \in X} x^{\pi'}(e) \leq f'(X) = f(X).$$

Case 2: X is in the case (ex) *for* e_1. We have $X \setminus \{e_1\} \in \mathrm{ex}'(\mathcal{L}/\{e_1\})$ and $\iota(X \setminus \{e_1\}) = X$. Therefore,

$$\begin{aligned}
\sum_{e \in X} x^\pi(e) &= x^\pi(e_1) + \sum_{e \in X \setminus \{e_1\}} x^\pi(e) \\
&= x^\pi(e_1) + \sum_{e \in X \setminus \{e_1\}} x^{\pi'}(e) \\
&\leq x^\pi(e_1) + f'(X \setminus \{e_1\}) \\
&= x^\pi(e_1) + f(X) - f(\{e_1\}) \\
&= x^\pi(e_1) + f(X) - x^\pi(e_1) \\
&= f(X).
\end{aligned}$$

Case 3: X is in the case (out) *for* e_1. Let $Y = (X \cup \{e_1\}) \setminus (X \vee \{e_1\})$. By Lemma 3.11 we have $f(\{e_1\}) + f(X) \geq f(X \vee \{e_1\}) + f(Y)$. Then, we have

$$\begin{aligned}
\sum_{e \in X} x^\pi(e) &= \sum_{e \in X} x^{\pi'}(e) \\
&= \sum_{e \in (X \vee \{e_1\}) \setminus \{e_1\}} x^{\pi'}(e) + \sum_{e \in Y} x^{\pi'}(e) \\
&\leq f'((X \vee \{e_1\}) \setminus \{e_1\}) + f(Y) \\
&= f(X \vee \{e_1\}) - f(\{e_1\}) + f(Y) \\
&\leq f(X).
\end{aligned}$$

□

From the discussion above, we have shown the if-part of Theorem 1.5. Now we show the only-if-part, and it completes the proof of Theorem 1.5.

Proof of Theorem 1.5. We obtain the if-part from Lemmas 3.3–3.5 and 3.12. Now we prove the only-if-part. By Lemma 2.5, $X \diamond Y \preceq X \sqcap Y \preceq X \vee Y$ for all $X, Y \in \mathrm{ex}(\mathcal{L})$ satisfying $\chi^X + \chi^Y = \chi^{X \vee Y} + \chi^{X \sqcap Y} + \chi^{X \diamond Y}$. Hence, there exists a maximal chain $\mathcal{C}$ of $\mathrm{ex}(\mathcal{L})$ containing these three sets. Let π be the permutation such that the greedy chain with respect to π is $\mathcal{C}$. Consider the greedy vector x^π. From the assumption that the algorithm (G) is valid, for all $Z \in \mathcal{C}$, we have $\sum\{x^\pi(e) : e \in Z\} = f(Z)$. Hence, $f(X \vee Y) + f(X \sqcap Y) + f(X \diamond Y) = \sum\{x^\pi(e) : e \in X \vee Y\} + \sum\{x^\pi(e) : e \in X \sqcap Y\} + \sum\{x^\pi(e) : e \in X \diamond Y\} = \sum\{x^\pi(e) : e \in X\} + \sum\{x^\pi(e) : e \in Y\} \leq f(X) + f(Y)$. We have the c-submodularity of f and this completes the proof. □

4 Consequences from the main theorem

In this section, we discuss some consequences obtained from the main theorem.

4.1 The Lovász extensions of c-submodular functions

Ando [1] gave the characterization of b-submodular functions by the Lovász extension. Now, we give the characterization of c-submodular functions by the Lovász extension. To do that, we need the next lemma.

Lemma 4.1. *For any* $c \in \mathbb{R}^E_+$ *there uniquely exist a chain* $\tilde{\mathcal{C}} : \emptyset \prec X_1 \prec X_2 \prec \cdots \prec X_m$ *of* $\mathrm{ex}(\mathcal{L})$ *and positive numbers* $\lambda_i > 0$ $(i = 1, \ldots, m)$ *such that*

$$c = \sum_{i=1}^{m} \lambda_i \chi^{X_i}. \tag{4.1}$$

Proof. For the greedy chain $\mathcal{C}^\pi$ and the dual greedy vector y^π, $\tilde{\mathcal{C}} = \{X \in \mathcal{C}^\pi : y^\pi(X) > 0\}$ is the chain above.

The uniqueness is shown by induction on the size of the ground set E as follows. Suppose that the statement holds for any proper subset of E. Let $Z = \{e \in E : c(e) > 0\}$. Assume that c is represented as (4.1). We consider the following two cases.

Case 1: $\tau(Z) = E$. We now show that $X_m = \mathrm{ex}(Z)$. Trivially we have $X_m \preceq \mathrm{ex}(Z)$. Suppose that $X_m \neq \mathrm{ex}(Z)$. That is, there exists $e \in \mathrm{ex}(Z) \setminus X_m$. Therefore, we have $e \notin X_i$ for all $i \in \{1, \ldots, m\}$ from Lemma 2.6. This contradicts the fact $c(e) > 0$.

Next, we show that $\lambda_m = \min\{c(e) : e \in \mathrm{ex}(Z)\}$. Since $e \in X_m \setminus X_{m-1}$ for some $e \in \mathrm{ex}(Z)$ and $e \notin X_i$ for all $i \in \{1, \ldots, m-1\}$, we have $c(e) = \lambda_m$. Suppose that e is not a minimizer of $\{c(e) : e \in \mathrm{ex}(Z)\}$. That is, there exists $e' \in \mathrm{ex}(Z)$ such that $c(e') < c(e)$. Then, since $X_m = \mathrm{ex}(Z)$, we have $c(e') = \sum\{\lambda_i : i \in \{1, \ldots, m\} \text{ s.t. } e' \in X_i\} = \lambda_m + \sum\{\lambda_i : i \in \{1, \ldots, m-1\} \text{ s.t. } e' \in X_i\} > \lambda_m = c(e)$. This leads to a contradiction.

Now, we have $X_m = \mathrm{ex}(Z)$ and $\lambda_m = \min\{c(e) : e \in \mathrm{ex}(Z)\}$. Let $\bar{e}$ be a minimizer of $\{c(e) : e \in \mathrm{ex}(Z)\}$. Consider the deletion $\mathcal{L} \setminus \{\bar{e}\}$ and $\bar{c} = c - \lambda_m \chi^{X_m}$.

From the induction hypothesis, we have the unique representation of $\bar{c}$ as $\bar{c} = \sum\{\lambda_i \chi^{X_i} : i = 1, \ldots m-1\}$. Hence, we have the uniqueness.

Case 2: $\tau(Z) \subsetneq E$. Delete $E \setminus \tau(Z)$ from E and consider $\mathcal{L} \setminus (E \setminus \tau(Z))$. Then it concludes the proof by the induction hypothesis. $\square$

When $c \in \mathbb{R}^E_+$ is decomposed as (4.1), then for a function $f : \mathrm{ex}(\mathcal{L}) \to \mathbb{R}$ we define $\hat{f} : \mathbb{R}^E_+ \to \mathbb{R}$ as

$$\hat{f}(c) = \sum_{i=1}^{m} \lambda_i f(X_i), \tag{4.2}$$

and $\hat{f}$ is called the *Lovász extension* of f. The name of the Lovász extension is derived from Lovász [12]. Notice that $\hat{f}$ is positively homogeneous, that is, $\hat{f}(\mu c) = \mu \hat{f}(c)$ for any positive number $\mu > 0$.

Theorem 4.2. *Let $\mathcal{L}$ be a convex geometry on E. Then, $f : \mathrm{ex}(\mathcal{L}) \to \mathbb{R}$ is c-submodular if and only if the Lovász extension $\hat{f} : \mathbb{R}^E_+ \to \mathbb{R}$ defined by (4.2) is convex.*

Proof. Suppose that f is c-submodular. By Theorem 1.5, we have $\hat{f}(c) = \max\{\sum\{c(e)x(e) : e \in E\} : \sum\{x(e) : e \in X\} \leq f(X) \text{ for all } X \in \mathrm{ex}(\mathcal{L})\}$. Therefore $\hat{f}$ is convex.

We now show the converse. Suppose that $\hat{f}$ is convex. Then we have $\hat{f}(\chi^X) + \hat{f}(\chi^Y) \geq 2\hat{f}\left((\chi^X + \chi^Y)/2\right) = \hat{f}(\chi^X + \chi^Y)$ for all $X, Y \in \mathrm{ex}(\mathcal{L})$. Therefore, if X and Y satisfy $\chi^X + \chi^Y = \chi^{X \vee Y} + \chi^{X \sqcap Y} + \chi^{X \diamond Y}$, then $f(X) + f(Y) = \hat{f}(\chi^X) + \hat{f}(\chi^Y) \geq \hat{f}(\chi^X + \chi^Y) = \hat{f}(\chi^{X \vee Y} + \chi^{X \sqcap Y} + \chi^{X \diamond Y})$. Since $X \diamond Y \preceq X \sqcap Y \preceq X \vee Y$ by Lemma 2.5, $\hat{f}(\chi^{X \vee Y} + \chi^{X \sqcap Y} + \chi^{X \diamond Y}) = f(X \vee Y) + f(X \sqcap Y) + f(X \diamond Y)$. Hence f is c-submodular. $\square$

4.2 Total dual integrality of the problem (P)

We consider the total dual integrality of the system of the constraint inequalities (1.2) of the problem (P). For the total dual integrality, see [14] etc. If c is integral for the problem (P), then the greedy vector y^π must be integral from the description of the greedy algorithm (G). Moreover, by Lemma 4.1, we have $\lambda_i = y^\pi(X_i)$ for any $i \in \{1, \ldots, m\}$. This fact and Theorem 1.5 lead to the following theorem.

Theorem 4.3. *If $f : \mathrm{ex}(\mathcal{L}) \to \mathbb{R}$ is c-submodular, the system of the constraint inequalities of the problem* (P) *is totally dual integral.*

4.3 Characterization of poset shellings by b-submodular functions

Krüger [11] asserted that for a poset shelling $\mathcal{L}$ the greedy algorithm (G) works if and only if a function $f : \mathrm{ex}(\mathcal{L}) \to \mathbb{R}$ is b-submodular. Now, we prove that the greedy algorithm (G) works for all b-submodular functions if and only if $\mathcal{L}$ is a poset shelling as Theorem 4.6. Theorem 1.4 noted in Section 1 is a direct consequence of Theorems 1.5 and 4.6.

First, we have a characterization of poset shellings by means of the residue operator "$\diamond$."

Lemma 4.4. *A convex geometry $\mathcal{L}$ on E is a poset shelling if and only if $X \diamond Y = \emptyset$ for all $X, Y \in \mathrm{ex}(\mathcal{L})$.*

Proof. Suppose that $\mathcal{L}$ is not a poset shelling. Then by Proposition 1.2, there exist $X, Y \in \mathrm{ex}(\mathcal{L})$ such that $\tau(X) \cup \tau(Y) \subsetneq \tau(\tau(X) \cup \tau(Y)) = \tau(X \vee Y)$. That is, there exists some $a \in E$ such that $a \in \tau(X \vee Y) \setminus (\tau(X) \cup \tau(Y))$. Set $A = \mathrm{ex}(\tau(X) \cup \{a\})$ and $B = \mathrm{ex}(\tau(Y) \cup \{a\})$. Now we show that $A \diamond B \neq \emptyset$.

Since $a \in \tau(X \vee Y) \setminus (\tau(X) \cup \tau(Y))$, we have $a \notin \tau(X) = \tau(\tau(X)) = \tau((\tau(X) \cup \{a\}) \setminus \{a\})$. Therefore, a is an extreme point of $\tau(X) \cup \{a\}$, and we obtain that $a \in \mathrm{ex}(\tau(X) \cup \{a\}) = A$. Similarly, we obtain that $a \in B$. Hence, $a \in A \cap B$. By Lemma 2.11, we have $X \vee Y \subseteq X \cup Y$. Since $a \notin X \cup Y$, we have $a \notin X \vee Y$.

We have $A \preceq X \vee Y$ since $\tau(A) = \tau(\mathrm{ex}(\tau(X) \cup \{a\})) \subseteq \tau(\tau(X) \cup \{a\}) \subseteq \tau(\tau(X) \cup \tau(Y) \cup \{a\}) = \tau(\tau(X) \cup \tau(Y)) = \tau(X \vee Y)$. Since $\tau(X) \subseteq \tau(\tau(X) \cup \{a\}) = \tau(A)$, $X \preceq A$. Similarly, $Y \preceq B \preceq X \vee Y$. Therefore, $X \vee Y = A \vee B$, and we obtain that $a \notin A \vee B$. Hence we have $a \in (A \cap B) \setminus (A \vee B) = A \diamond B$.

Conversely, suppose that $X \diamond Y \neq \emptyset$ for some $X, Y \in \mathrm{ex}(\mathcal{L})$. That is, there exists $a \in E$ such that $a \in X$, $a \in Y$ and $a \notin X \vee Y$. By Lemma 2.1, $\tau(X) \setminus \{a\} \in \mathcal{L}$ and $\tau(Y) \setminus \{a\} \in \mathcal{L}$. Now suppose that $\mathcal{L}$ is a poset shelling. Then, by Proposition 1.2 we obtain that $(\tau(X) \setminus \{a\}) \cup (\tau(Y) \setminus \{a\}) \in \mathcal{L}$. Therefore, $(\tau(X) \cup \tau(Y)) \setminus \{a\} \in \mathcal{L}$. By Proposition 1.2, Lemma 2.1 and $\tau(X \vee Y) = \tau(X) \cup \tau(Y) \in \mathcal{L}$, we have $a \in X \vee Y$. This is a contradiction. $\square$

Next, we prove the following lemma for the theorem of this subsection.

Lemma 4.5. *Let $\mathcal{L}$ be a convex geometry on E. Suppose that $\mathcal{L}$ is not a poset shelling. Then, there exist $X, Y \in \mathrm{ex}(\mathcal{L})$ such that $X \diamond Y \neq \emptyset$ and $\chi^X + \chi^Y = \chi^{X \vee Y} + \chi^{X \sqcap Y} + \chi^{X \diamond Y}$.*

Proof. Since $\mathcal{L}$ is not a poset shelling, there exist $X, Y \in \mathrm{ex}(\mathcal{L})$ such that $X \diamond Y \neq \emptyset$ by Lemma 4.4. Let $(X^*, Y^*) \in \arg\min\{|\tau(X \vee Y) \setminus \tau(X)| + |\tau(X \vee Y) \setminus \tau(Y)| : X \diamond Y \neq \emptyset\}$. By Remark 2.10.3, $\chi^{X^*} + \chi^{Y^*} \geq \chi^{X^* \vee Y^*} + \chi^{X^* \sqcap Y^*} + \chi^{X^* \diamond Y^*}$. Now suppose that $\chi^{X^*} + \chi^{Y^*} > \chi^{X^* \vee Y^*} + \chi^{X^* \sqcap Y^*} + \chi^{X^* \diamond Y^*}$. Then, by Remark 2.10.3, without loss of generality, we have $b \in X^*$, $b \notin \tau(Y^*)$, $b \notin X^* \vee Y^*$ and $b \in \tau(X^* \vee Y^*)$ for some $b \in E$. By Lemma 2.6, there exists $Y' \in \mathrm{ex}(\mathcal{L})$ such that $b \in Y'$ and $Y^* \prec Y' \prec X^* \vee Y^*$. Therefore $X^* \vee Y^* = X^* \vee Y'$. Hence, we have $|\tau(X^* \vee Y^*) \setminus \tau(X^*)| + |\tau(X^* \vee Y^*) \setminus \tau(Y^*)| > |\tau(X^* \vee Y') \setminus \tau(X^*)| + |\tau(X^* \vee Y') \setminus \tau(Y')|$, since $b \notin \tau(X^* \vee Y^*) \setminus \tau(Y')$. This contradicts the minimality of X^* and Y^*. $\square$

Theorem 4.6. *Let $\mathcal{L}$ be a convex geometry. Algorithm (G) gives an optimum of the problem (D) for all b-submodular functions if and only if $\mathcal{L}$ is a poset shelling.*

Proof. The if-part is Theorem 1.3. We show the only-if-part. Suppose that $\mathcal{L}$ is not a poset shelling. Then, we set $\tilde{f}$ as $\tilde{f}(X) = 0$ when $X = \emptyset$ and $\tilde{f}(X) = 1$ when $X \neq \emptyset$. We can easily check that $\tilde{f}$ is b-submodular but not c-submodular using Lemma 4.5. From Theorem 1.5, Algorithm (G) does not always give an optimum for $\tilde{f}$. $\square$

5 Summary and remarks

In this paper we have considered the lattice of the extreme sets of a convex geometry and have investigated the relationship between a dual greedy algorithm and c-submodularity. Moreover, we have characterized c-submodular functions in terms of their Lovász extensions and have given a characterization of a poset shelling.

Our work generalizes results of Krüger [11] and Ando [1]. Furthermore, other directions of generalization are discussed by some authors. First, Faigle–Kern [7] recently gave a general framework for the greedy algorithm, considering the so-called algebraic posets. Second, Frank [8] gave another greedy algorithm for a class of lattice polyhedra and applied it to connectivity augmentation problems. However, we have not yet seen the exact relationship among these results.

Acknowledgments

The authors are very grateful to Kazutoshi Ando, Masataka Nakamura, Takashi Takabatake and Masahiro Hachimori for their valuable comments. The authors also thank the anonymous referees for careful reading and useful comments for this paper.

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