Fair cost allocations under conflicts
— a game-theoretic point of view —

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3rd CGC Annual Workshop

Park Hotel Fasanerie, Neustrelitz, Germany
September 29 – October 1, 2003

Supported by the Berlin-Zürich Joint Graduate Program

CGC
Combinatorics
Geometry
Computation

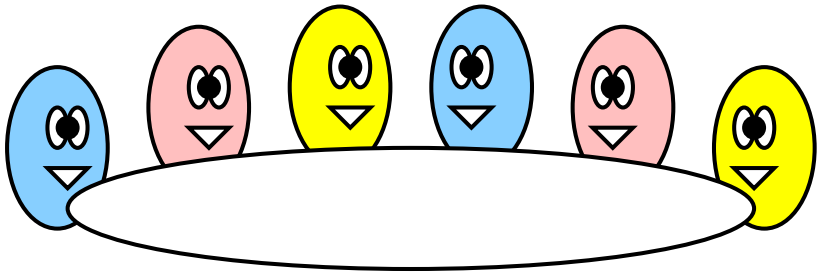
What is all about?

Framework: Several people are willing to work together...

- They want to have a largest possible benefit.
  ..... optimization problem
- They want to allocate the benefit in a fair way.
  ..... game-theoretic problem

Game Theory?

- Noncooperative Game Theory
- Cooperative Game Theory



Def.: A cooperative game (or a game) is a pair (N, γ) of

- a finite set N (set of players)
- a function $\gamma : 2^{\mathrm{N}} \to \mathbb{R}$ with $\gamma(\emptyset) = 0$ (characteristic function).

Interpretation: For $S \subseteq \mathrm{N}$,

$\gamma(S)$ represents $\left\{ \begin{array}{l} \text{the max. benefit gained by } S \\ \text{the min. cost owed by } S \end{array} \right\}$ when the players in S work in cooperation.

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Goal: To allocate $\gamma(N)$ to each player in a "fair" way.

This work: study on "minimum coloring games."

Definition: minimum coloring game

$G = (V, E)$ an undirected graph

- A proper k-coloring of G

  is a surjective map $c : V \to \{1, \ldots, k\}$ s.t.
  if $\{u, v\} \in E$, then $c(u) \neq c(v)$.

- The chromatic number $\chi(G)$ of G

  $= \min\{\ k\ :$ a proper k-coloring of G exists $\}$.

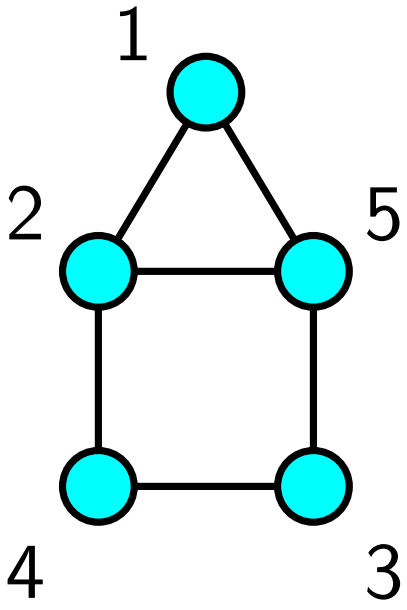
- The minimum coloring game on G

  is a cooperative game (V, χ_G).

  $\chi_G : 2^V \to \mathbb{N}$ is defined as $\chi_G(S) = \chi(G[S])$,
  where $G[S]$ is the subgraph induced by $S \subseteq V$.

Example: minimum coloring game

$\chi_G(S) = \chi(G[S])$ for $S \subseteq V$.



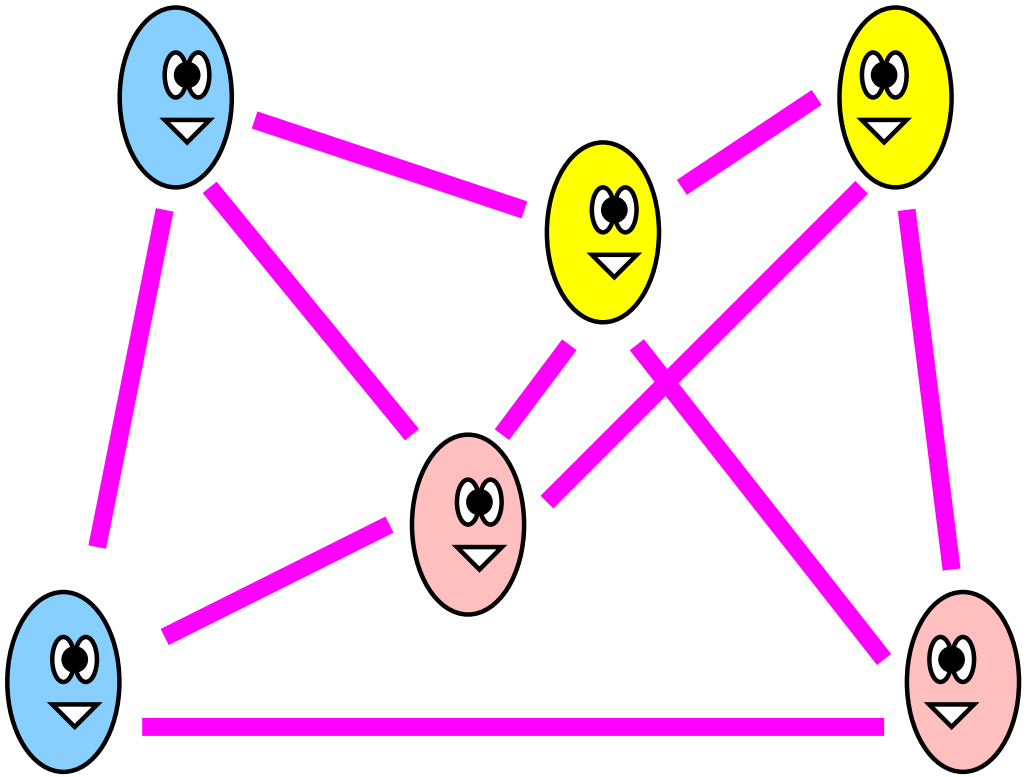
S	χ_G	S	χ_G	S	χ_G	S	χ_G
$\emptyset$	0	14	1	123	2	245	2
1	1	15	2	124	2	345	2
2	1	23	2	125	3	1234	2
3	1	24	1	134	2	1235	3
4	1	25	2	135	2	1245	3
5	1	34	2	145	2	1345	2
12	2	35	1	234	2	2345	2
13	1	45	2	235	2	12345	3

Goal: To allocate $\chi(G)$ to each vertex in a fair way.

Background: model of conflicts

Conflict graph: a model of conflict

- the vertices = the agents, the principals...
- the edges = between two in conflict.



min. coloring game:

a simplest model of the fair cost allocation problem in conflict situations

Objective of our work

We study minimum coloring games, and investigate the following kinds of fairness concepts:

- Core (Gillies '53)
- Nucleolus (Schmeidler '69)
- τ-value (Tijs '81)
- Shapley value (Shapley '53).

Past works on minimum coloring games:

- Deng, Ibaraki & Nagamochi '99
- Deng, Ibaraki, Nagamochi & Zang '00
- Okamoto '03.

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Why fair cost allocation problems??

Background: Operations Research

Fair cost allocation problems are studied in OR community from the game-theoretic viewpoint.

- Megiddo '87

  First noticed the computational issue in fair cost allocation problems.

- So far, a lot of results have appeared in

  Mathematics of Operations Research,
  Mathematical Programming,
  Mathematical Methods of Operations Research,
  Discrete Applied Mathematics,
  International Journal of Game Theory,
  Games and Economic Behavior,
  etc.

- They assume practical applications.

Background: Algorithmic game theory

There are many kinds of "fairness" concepts (called "solutions") in cooperative game theory.

Thesis: Bounded Rationality (Simon '70s)

Decisions by realistic economic agents cannot involve unbounded resources for reasoning.

Thesis: (Deng & Papadimitriou '94)

[Economic concept]		[Algorithmic concept]
A solution follows Bounded Rationality	$\simeq$	Computation can be done in poly. time

$\Longrightarrow$ Algorithmic study of cooperative game theory

$\mathbb{E}$ a finite set

Def.: A set function on $\mathbb{E}$ is a function $f : 2^{\mathbb{E}} \to \mathbb{R}$.

Appearance:

- Cooperative game theory
- Combinatorial optimization
- Pseudo-boolean functions
- Nonadditive measure theory (fuzzy measure theory)
- ...

They study different aspects of set functions.

Scope of today's talk

Focus on cores and nucleoli of minimum coloring games

- Def.: cost allocation
- Def.: nucleolus
- Characterization: the nucleolus for a chordal graph
- Open problems

Cost allocation in a min coloring game

Def.: A cost allocation for a game (N, γ) is a vector $z \in \mathbb{R}^N$ such that

$$\sum \{z[i] : i \in N\} = \gamma(N).$$

(Often in cooperative game theory, this is called a pre-imputation.)

. . . . . .

Q. What kinds of cost allocations are considered fair??

..... Core, Nucleolus, τ-value, Shapley value, etc.

Let (N, γ) a game
$z \in \mathbb{R}^N$ a cost allocation
$S \subseteq N$ (often called a coalition)

Def.: An excess $e(S, z)$ is defined as

$$e(S, z) := \sum_{i \in S} z[i] - \gamma(S).$$

Interpretation: The smaller $e(S, z)$, the happier S with z.

$\sum_{i \in S} z[i]$: cost owed by S when people in N work together

$\gamma(S)$: cost owed by S when people in S work together.

Enumerating the excesses...

Let (N, γ) be a game, $z \in \mathbb{R}^N$ a cost allocation

Consider the following procedure.

- Enumerate $e(S, z)$ for all $S \in 2^N \setminus \{\emptyset, N\}$.

Example:

$$z = \left(1, \frac{1}{2}, \frac{1}{2}\right)^\top$$

S	$\gamma(S)$	$e(S,z)$
$\emptyset$	0	(0)
$\{1\}$	1	0
$\{2\}$	1	$-1/2$
$\{3\}$	1	$-1/2$
$\{1,2\}$	1	$1/2$
$\{1,3\}$	2	$-1/2$
$\{2,3\}$	2	-1
$\{1,2,3\}$	2	(0)

Enumerating the excesses...

Let (N, γ) be a game, $z \in \mathbb{R}^N$ a cost allocation

Consider the following procedure.

- Enumerate $e(S, z)$ for all $S \in 2^N \setminus \{\emptyset, N\}$.
- Arrange these excesses in non-increasing order to obtain $\theta_z \in \mathbb{R}^{2^{|N|}-2}$. ($\theta_z[i] \geq \theta_z[j]$ if $i \leq j$.)

Example:

$$z = \left(1, \frac{1}{2}, \frac{1}{2}\right)^\top$$

$$\theta_z = \left(\frac{1}{2}, 0, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -1\right)^\top$$

S	$\gamma(S)$	$e(S, z)$
$\emptyset$	0	(0)
$\{1\}$	1	0
$\{2\}$	1	$-1/2$
$\{3\}$	1	$-1/2$
$\{1,2\}$	1	$1/2$
$\{1,3\}$	2	$-1/2$
$\{2,3\}$	2	-1
$\{1,2,3\}$	2	(0)

Def.: The nucleolus of (N, γ) is defined as

$$\boldsymbol{\nu}(N,\gamma) = \left\{ \boldsymbol{z} \in \mathbb{R}^N : \begin{array}{l} \boldsymbol{z} \text{ lex-mins } \boldsymbol{\theta}_z \text{ over all cost alloc's } \boldsymbol{y} \\ \text{s.t. } \boldsymbol{y}[i] \leq \gamma(\{i\}) \quad \forall i \in N \end{array} \right\}.$$

Interpretation: The smaller $e(S, z)$, the happier S with z.

$\Rightarrow$ Want an allocation which minimizes max excess.

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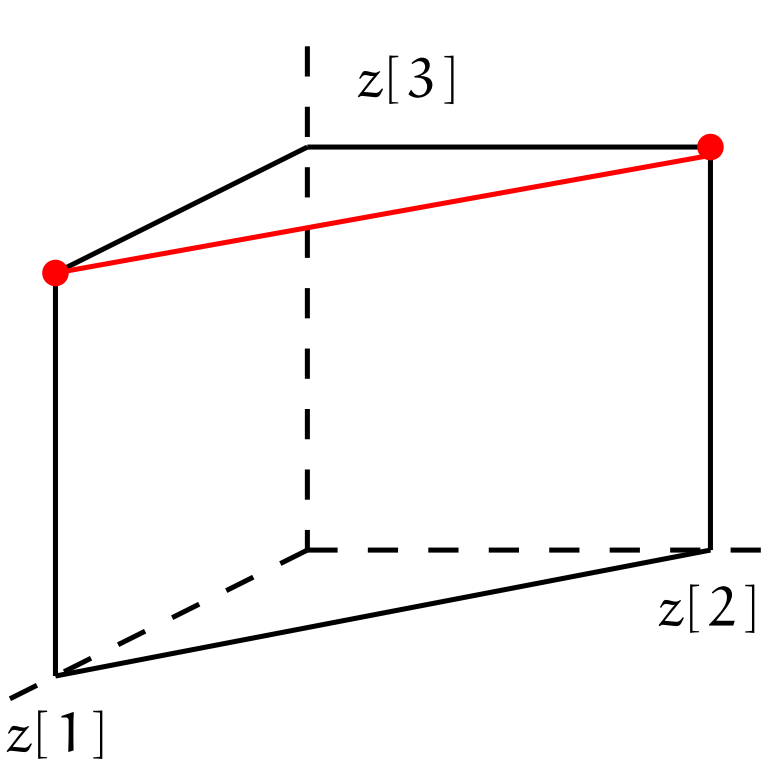
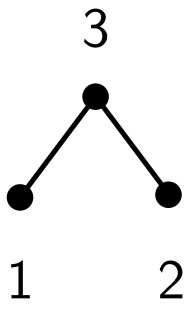
Thm. (Schmeidler '69)

The nucleolus consists of a single vector.

So we usually say $\boldsymbol{\nu}(N,\gamma) = \boldsymbol{z}$ instead of $\boldsymbol{\nu}(N,\gamma) = \{\boldsymbol{z}\}$.

Example: nucleolus





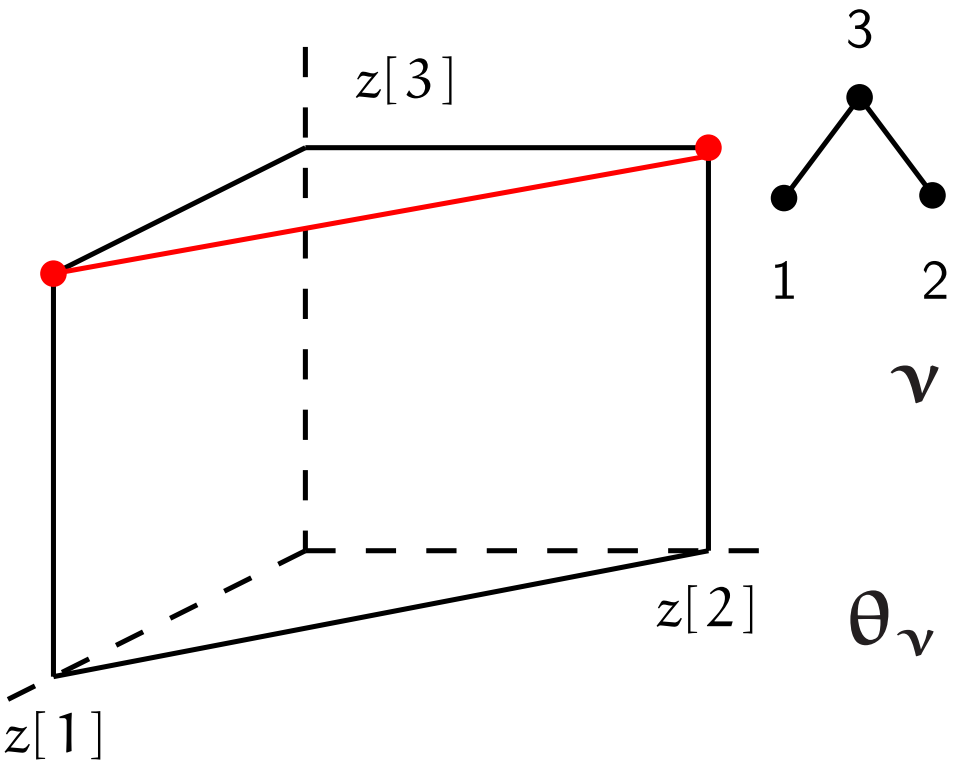
$\nu = \left(\frac{1}{2}, \frac{1}{2}, 1\right)^\top$ is the nucleolus.

$$\theta_\nu = \left(0, 0, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)^\top.$$

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$
$\gamma(S)$	1	1	1	1	2	2
$e(S,\nu)$	$-1/2$	$-1/2$	0	0	$-1/2$	$-1/2$

$$e(S,\nu) := \sum_{i \in S} \nu[i] - \gamma(S).$$

Example: nucleolus



$$\boldsymbol{\nu} = \left(\frac{1}{2}, \frac{1}{2}, 1\right)^{\top} \text{ is the nucleolus.}$$

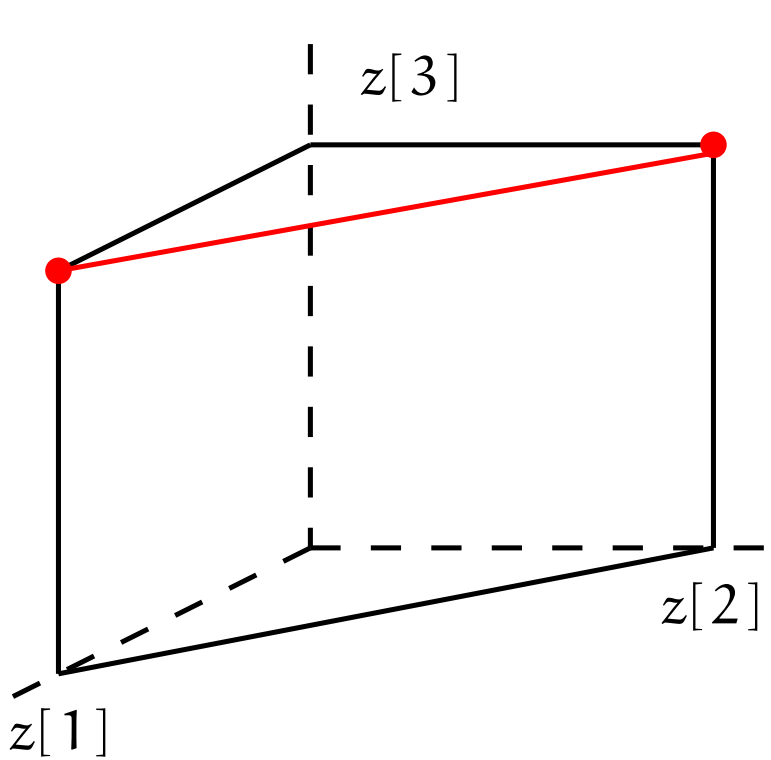
$$\theta_{\boldsymbol{\nu}} = \left(0, 0, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)^{\top}.$$

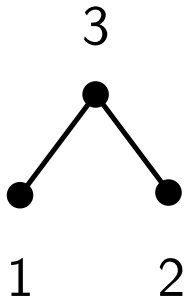
Fact: the core is nonempty

$\Rightarrow$ the nucleolus $\in$ the core.

Def.: A cost allocation $z \in$ the core of (N, γ) if $e(S, z) \leq 0$ $(\forall\ S \subseteq N)$.

Example: nucleolus





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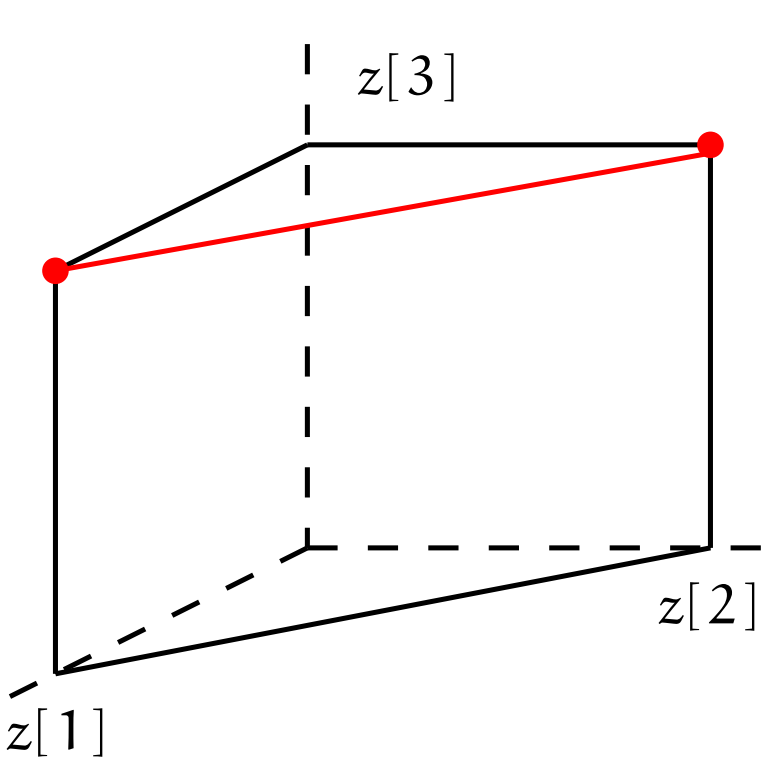
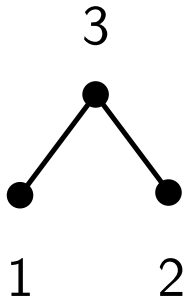
$$\theta_{\boldsymbol{\nu}} = \left(0, 0, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)^\top .$$

Let $\boldsymbol{z} = \lambda(1,0,1)^\top + (1-\lambda)(0,1,1)^\top \qquad (0 \le \lambda \le 1)$
$= (\lambda, 1-\lambda, 1)^\top .$

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$
$\gamma(S)$	1	1	1	1	2	2
$e(S,\boldsymbol{z})$	$\lambda-1$	$-\lambda$	0	0	$\lambda-1$	$-\lambda$

Example: nucleolus





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$$\theta_{\boldsymbol{\nu}} = \left(0, 0, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)^\top .$$

$$\theta_z = \begin{cases} (0, 0, -\lambda, -\lambda, \lambda - 1, \lambda - 1)^\top & \text{if } 0 \le \lambda \le 1/2 \\ (0, 0, \lambda - 1, \lambda - 1, -\lambda, -\lambda)^\top & \text{if } 1/2 \le \lambda \le 1 \end{cases}$$

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$
$\gamma(S)$	1	1	1	1	2	2
$e(S,z)$	$\lambda - 1$	$-\lambda$	0	0	$\lambda - 1$	$-\lambda$

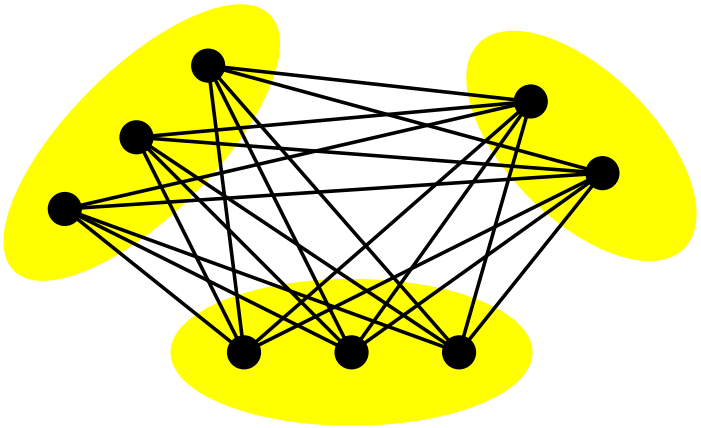
Computation of the nucleolus

Thm. (Kuipers '96, Faigle, Kern & Kuipers '01)

The nucleolus can be computed in polynomial time for submodular games.

Thm. (Okamoto '03)

χ_G is submodular $\Leftrightarrow$ G is complete multipartite.



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Cor.

G complete multipartite
$\Rightarrow$ the nucleolus of χ_G computed in poly. time.

On the computation of the nucleolus of a min coloring game

Graph	$\leftrightarrow$	Min col. game
general		NP-hard
$\cup\mid$		
zero duality gap		???
$\cup\mid$		
perfect		???
$\cup\mid$		
complete multipartite		Poly

Nucleolus of a min coloring game

Obs. The computation of the nucleolus of a min coloring game is NP-hard.

Proof Suppose we get the nucleolus $\boldsymbol{\nu}$ in poly time.

$\Rightarrow$ Compute $\sum_{i \in V} \boldsymbol{\nu}[i] = \chi(G)$.

$\Rightarrow$ We have obtained $\chi(G)$ in poly time.

$\Rightarrow$ P $=$ NP. [qed]

On the computation of the nucleolus of a min coloring game

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On the computation of the nucleolus of a min coloring game

Graph	$\leftrightarrow$	Min col. game
general		NP-hard
$\cup\mid$		
zero duality gap		???
$\cup\mid$		
perfect		???
$\cup\mid$		
O-good		characterization
$\cup\mid$		
complete multipartite		Poly

Thm.

The nucleolus for an O-good perfect graph G is the barycenter of the characteristic vectors of the maximum cliques of G.

Namely,

$$\boldsymbol{\nu}[i] = \frac{\#\text{ of maximum cliques containing } i}{\#\text{ of maximum cliques}}.$$

O-good perfect graphs

Thm.

The nucleolus for an O-good perfect graph G
is the barycenter of the characteristic vectors of
the maximum cliques of G.

Remark:

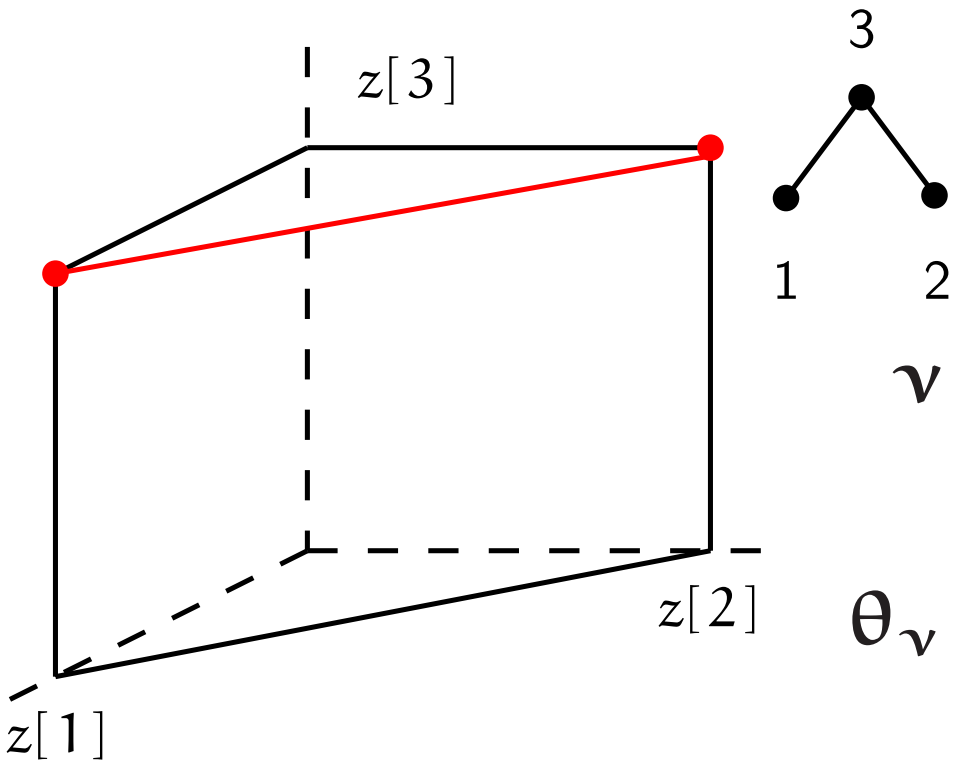
(1) We omit the def. of O-good perfect graphs.

(2) The class of O-good perfect graphs contains

- the graphs with unique maximum cliques
- the complete multipartite graphs
- the chordal graphs (especially the forests).

(3) A graph is chordal if every induced cycle is of length 3.

Example: nucleolus



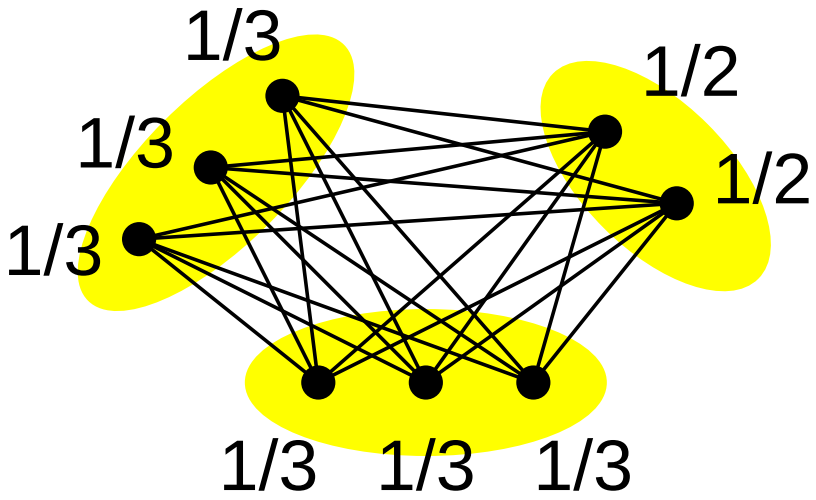
$$\boldsymbol{\nu} = \left(\frac{1}{2}, \frac{1}{2}, 1\right)^\top \text{ is the nucleolus.}$$

$$\theta_{\boldsymbol{\nu}} = \left(0, 0, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)^\top.$$

Indeed, this graph is $\begin{cases} \text{a forest} \\ \text{complete multipartite.} \end{cases}$

Example: nucleolus

Consider a complete multipartite graph.



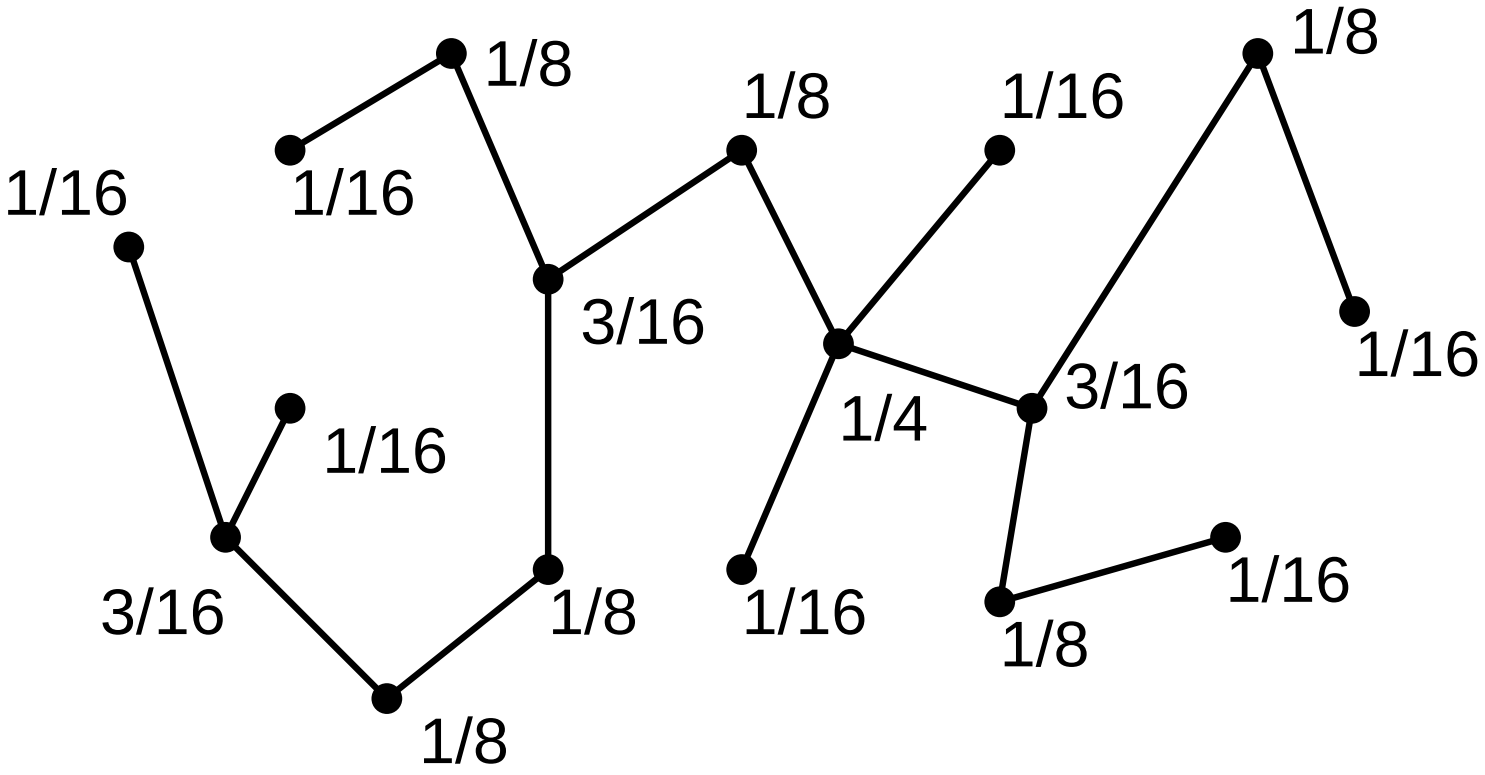
We have

$$\boldsymbol{\nu}[\,\mathrm{i}\,] = \frac{1}{\mathrm{n_i}},$$

where $\mathrm{n_i}$ is $\#$ of vertices of the class to which i belongs.

Example: nucleolus

Consider a forest.



We have

$$\boldsymbol{\nu}[\,i\,] = \frac{\deg(i)}{|E|},$$

where $\deg(i)$ is # of edges incident to i.

Example: nucleolus

For chordal graphs, we can use the following theorem to compute the nucleoli.

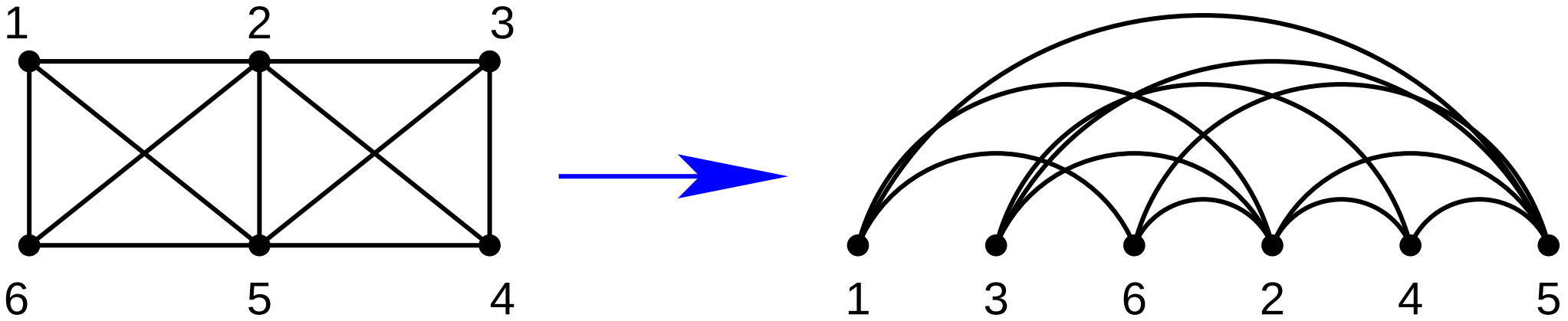
Thm. (Fulkerson & Gross '65)

In a chordal graph,

$$\#\text{ of maximal cliques} \leq \#\text{ of vertices.}$$

We can enumerate them in poly time.

Proof Sketch



Outline of the proof

(1) Use the LP formulation for the nucleolus computation. (Peleg)

By solving a sequence of LP problems, we can obtain the nucleolus (not in poly time).

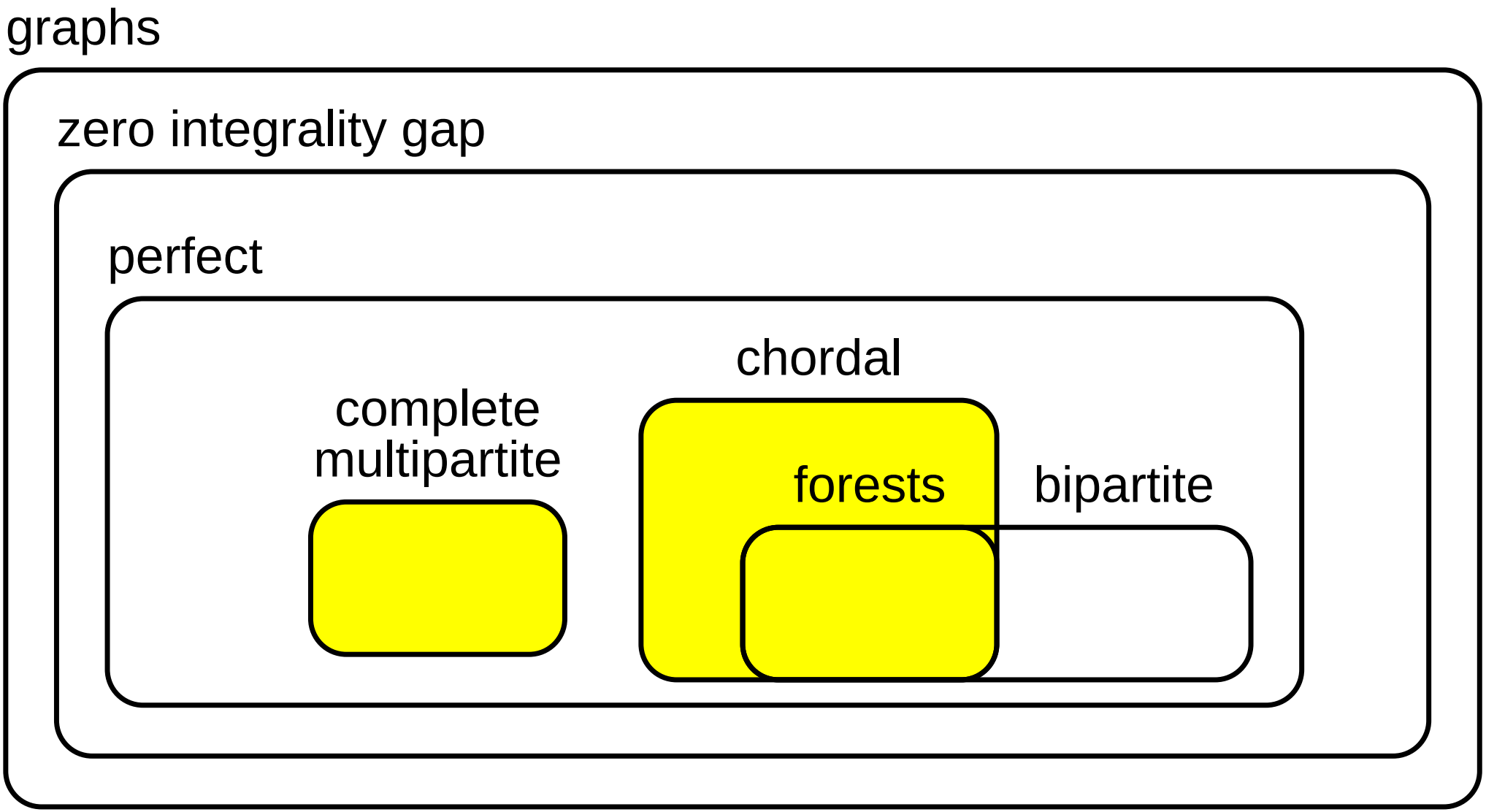
(2) Identify the essential coalitions. (Huberman '80)

The essential coalitions reduce the work load.
$S \subseteq V$ is essential $\Leftrightarrow$ S is an independent set.

(3) Analyze the LP.

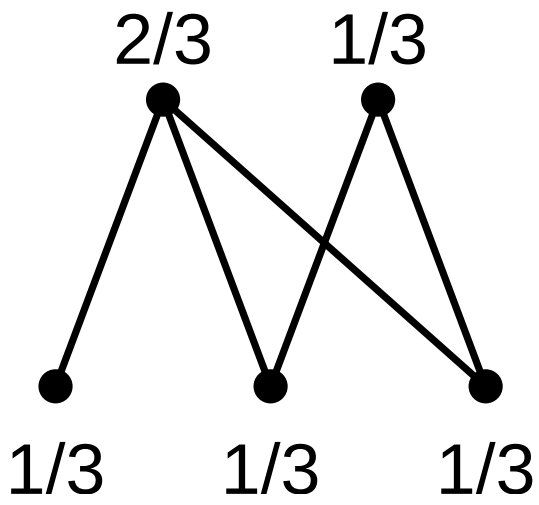
For O-good perfect graphs, we can precisely tell what are the optimal solutions in the LP problems with help of the characterization of the extreme points of the core.

Summary: the nucleolus



Example: bipartite graph

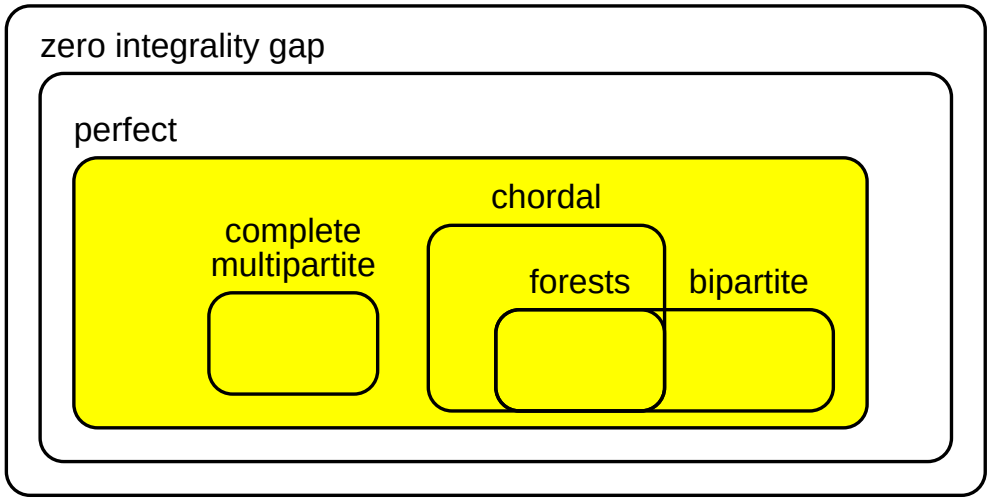
There is a bipartite graph for which the nucleolus is not the barycenter of the char. vectors of the maximum cliques.



The state of the art

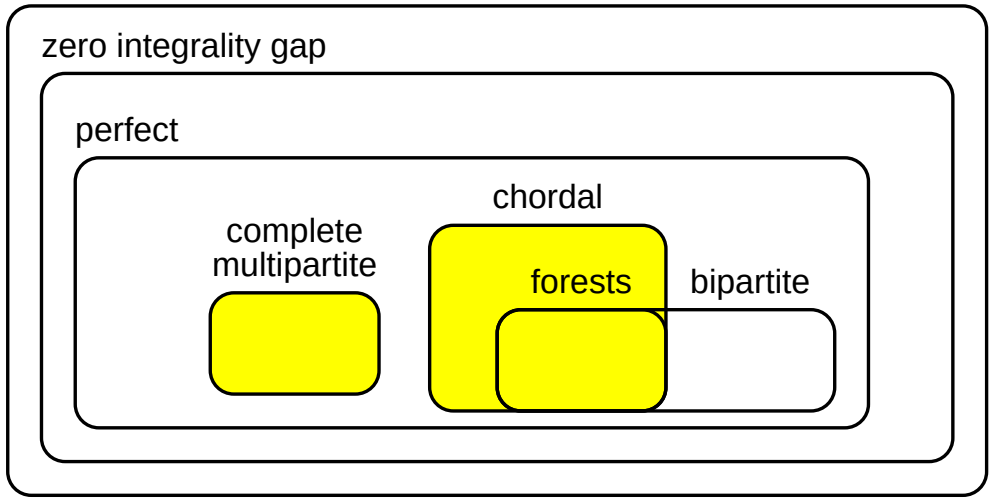
Core

graphs



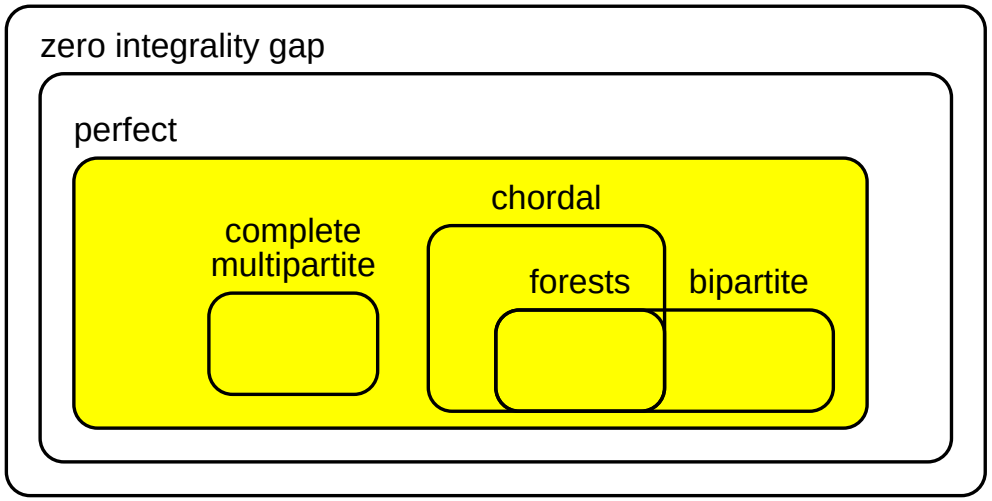
Nucleolus

graphs



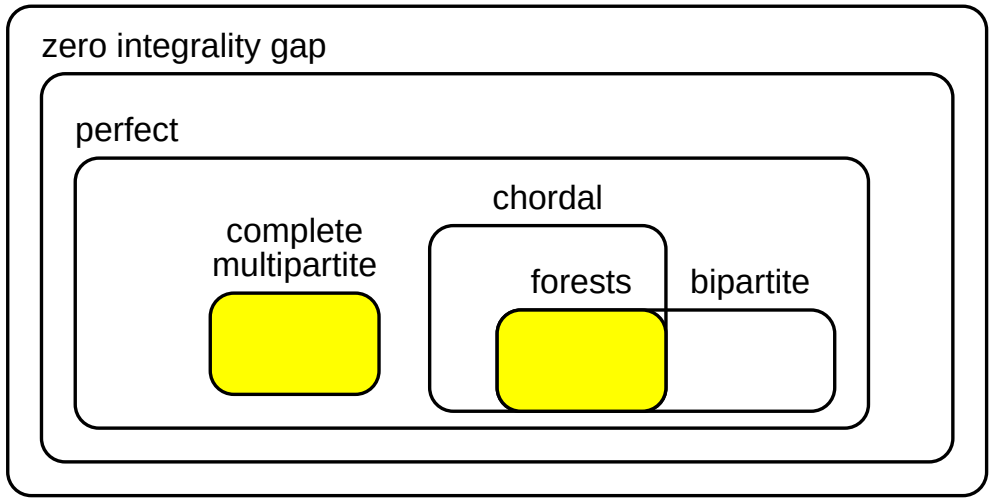
τ-value

graphs



Shapley value

graphs



Open problems: minimum coloring games

Computation of the nucleoli for

- perfect graphs ??
- bipartite graphs ??
- outerplanar graphs ??
- cographs ??

The list of what we discussed

- Def.: minimum coloring game
- Def.: nucleolus
- Characterization: the nucleolus for a chordal graph
- Open problems

Framework: Several people are willing to work together...

- They want to have a largest possible benefit.
  (optimization theory)
- They want to allocate the benefit in a fair way.
  (cooperative game theory)

Status of algorithmic problems on cooperative games

- As many cooperative games as optimization problems!!
- Many algorithmic problems remain unsolved!!

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$\Longrightarrow$ Why not work on them??

[End of the talk]