The free complex of a two-dimensional generalized convex shelling

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Abstract

In this article, we study two open problems posed by Edelman & Reiner about topology of certain simplicial complexes associated with abstract convex geometries. As a special case, we concentrate on generalized convex shellings, which were introduced by Kashiwabara, Nakamura & Okamoto for their representation theorem of abstract convex geometries. We answer one of the problems affirmatively for 2-dimensional separable generalized convex shellings, and the other problem negatively already for this case.

1 Introduction

An (abstract) convex geometry was introduced by Edelman & Jamison [2] as a combinatorial abstraction of "convexity" appearing in a lot of objects. Recently, a representation theorem for convex geometries has been established by Kashiwabara, Nakamura & Okamoto [6], which states that any convex geometry is isomorphic to some "separable generalized convex shelling." A generalized convex shelling is defined via two finite point sets in a certain dimension. Therefore, their representation theorem gives a stratification of the convex geometries by the minimum dimension in which a convex geometry can be realized as a generalized convex shelling. We will study the topology of the free complex of a 2-dimensional generalized convex shelling.

The motivation of this work stems from Edelman & Reiner [3]. An Euler-Poincaré type formula for the number of interior points in a d-dimensional point configuration was proved by Ahrens, Gordon & McMahon [1] when $d = 2$, and proved by Edelman & Reiner [3] and Klain [7] independently for any d. The approach by Klain [7] used a more general theorem on valuation, while that by

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Edelman & Reiner [3] was topological. (Their topological proof was later simplified with use of oriented matroids by Edelman, Reiner & Welker [4].) In the paper by Edelman & Reiner [3], they studied the topology of deletions in the free complex of a convex shelling (which arises from a point configuration), and also asked a possible generalization to any convex geometry, i.e., the following problems.

Open Problems 1 (Edelman & Reiner [3]). *Let $\mathcal{L}$ be a convex geometry on E and denote the free complex of $\mathcal{L}$ by* $\mathrm{Free}(\mathcal{L})$.

1. *Is* $\mathrm{del}_{\mathrm{Free}(\mathcal{L})}(\mathsf{x})$ *of* $\mathsf{x} \in \mathsf{E}$ *contractible if* $\mathrm{Dep}_{\mathcal{L}}(\mathsf{x}) \neq \mathsf{E}$*?*
2. *Is* $\mathrm{del}_{\mathrm{Free}(\mathcal{L})}(\mathsf{x})$ *homotopy equivalent to a bouquet of some equidimensional spheres*[1] *if* $\mathrm{Dep}_{\mathcal{L}}(\mathsf{x}) = \mathsf{E}$*? Does* $\mathrm{del}_{\mathrm{Free}(\mathcal{L})}(\mathsf{x})$ *have the same homology type as a bouquet of some equidimensional spheres for such a case?*

Edelman & Reiner showed, in the same paper [3], that Open Problems 1 are valid for some special cases and conjectured that the first part of the problems is valid for all convex geometries. Later, Edelman, Reiner & Welker [4] showed that Open Problems 1 are also valid for another special case.

Open Problems 1 ask for a generalization to all convex geometries, and thanks to Kashiwabara, Nakamura & Okamoto [6] every convex geometry is isomorphic to some separable generalized convex shelling. That is why it is worth looking at 2-dimensional separable generalized convex shellings as a simpler case. Our result states that Open Problem 1.1 is valid for 2-dimensional separable generalized convex shellings, but Open Problem 1.2 is not valid. Namely, to be more precise for the first statement, we will prove the following.

Theorem 2. *Let* P *and* Q *be nonempty finite point sets in* $\mathbb{R}^2$ *such that* $\mathrm{conv}(\mathsf{P}) \cap \mathrm{conv}(\mathsf{Q}) = \emptyset$*. In addition, let $\mathcal{L}$ be the generalized convex shelling on* P *with respect to* Q*. Consider the free complex* $\mathrm{Free}(\mathcal{L})$ *of $\mathcal{L}$. Then the deletion* $\mathrm{del}_{\mathrm{Free}(\mathcal{L})}(\mathsf{x})$ *of an element* $\mathsf{x} \in \mathsf{P}$ *is contractible if* $\mathrm{Dep}_{\mathcal{L}}(\mathsf{x}) \neq \mathsf{P}$.

2 Preliminaries

Due to the space limitation, we omit the necessary definitions of graph theory and topology.

A *convex geometry* on a nonempty finite set E is a family $\mathcal{L}$ of subsets of E satisfying the following three conditions: (1) $\emptyset, \mathsf{E} \in \mathcal{L}$, (2) if $\mathsf{X}, \mathsf{Y} \in \mathcal{L}$ then

[1] Note that a contractible complex is not considered as a bouquet of some equidimensional spheres.

$X \cap Y \in \mathcal{L}$, and (3) if $X \in \mathcal{L} \setminus \{E\}$ then there exists $e \in E \setminus X$ such that $X \cup \{e\} \in \mathcal{L}$. For a convex geometry $\mathcal{L}$ on E, we define an operator $\tau_{\mathcal{L}} : 2^E \to 2^E$ as $\tau_{\mathcal{L}}(A) = \bigcap\{X \in \mathcal{L} : A \subseteq X\}$. The operator $\tau_{\mathcal{L}}$ is called the *closure operator* of $\mathcal{L}$.

For a set $A \subseteq E$, an element $e \in A$ is called an *extreme point* if $e \notin \tau_{\mathcal{L}}(A \setminus \{e\})$. We denote the set of extreme points of A by $\mathrm{ex}_{\mathcal{L}}(A)$. Namely, define the operator $\mathrm{ex}_{\mathcal{L}} : 2^E \to 2^E$ as $\mathrm{ex}_{\mathcal{L}}(A) = \{e \in A : e \text{ is an extreme point of } A\}$. We call $\mathrm{ex}_{\mathcal{L}}$ the *extreme point operator*.

A set $A \subseteq E$ is called *independent* if $\mathrm{ex}_{\mathcal{L}}(A) = A$. We say that e *depends on* f if there exists an independent set A such that $f \in A$, $e \in \tau_{\mathcal{L}}(A)$ and $e \notin \tau_{\mathcal{L}}(A \setminus \{f\})$. We denote the set of all elements f on which e depends by $\mathrm{Dep}_{\mathcal{L}}(e)$ and call it the *dependency set* of e. A set $X \subseteq E$ is called *free* if $X \in \mathcal{L}$ and $\mathrm{ex}_{\mathcal{L}}(X) = X$. We denote the family of free sets of a convex geometry $\mathcal{L}$ by $\mathrm{Free}(\mathcal{L})$. Note that $\mathrm{Free}(\mathcal{L})$ forms a simplicial complex for any convex geometry $\mathcal{L}$. Thus, it is natural that we call $\mathrm{Free}(\mathcal{L})$ the *free complex* of a convex geometry $\mathcal{L}$.

Now we will define a generalized convex shelling. Let P and Q be finite point sets in $\mathbb{R}^d$ (where d is a positive integer) such that $P \cap \mathrm{conv}(Q) = \emptyset$. Then the *generalized convex shelling* on P with respect to Q is a convex geometry $\mathcal{L}$ defined as follows: $\mathcal{L} = \{X \subseteq P : P \cap \mathrm{conv}(X \cup Q) = X\}$.[2] We also call a convex geometry $\mathcal{L}$ a d-*dimensional generalized convex shelling* if there exist finite point sets P and Q in $\mathbb{R}^d$ such that $P \cap \mathrm{conv}(Q) = \emptyset$ and $\mathcal{L}$ is isomorphic to the generalized convex shelling on P with respect to Q. A generalized convex shelling on P with respect to Q is called *separable* if $\mathrm{conv}(P) \cap \mathrm{conv}(Q) = \emptyset$; otherwise it is *non-separable*. The next lemma tells us the closure operator and the extreme point operator of a generalized convex shelling.

Lemma 3. *Let $\mathcal{L}$ be a generalized convex shelling on P with respect to Q. Then, for a set $X \subseteq P$ we have $\tau_{\mathcal{L}}(X) = P \cap \mathrm{conv}(X \cup Q)$ and $\mathrm{ex}_{\mathcal{L}}(X) = \{x \in X : x$ is an extreme point of $\mathrm{conv}(X \cup Q)\}$.[3] In particular, $X \subseteq P$ is free if and only if $P \cap \mathrm{conv}(X \cup Q) = X$ and every element of X is an extreme point of $\mathrm{conv}(X \cup Q)$.*

In this paper, we study the free complex of a 2-dimensional separable generalized convex shelling. Since Open Problems 1 are valid if $Q = \emptyset$ [3], we want to concentrate on the case in which $Q \neq \emptyset$. Let us state that as an assumption.

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[2]Actually, Q does not have to be a finite point set. It can be any subset of $\mathbb{R}^d$ satisfying that $P \cap \mathrm{conv}(Q) = \emptyset$. But for the simplicity, we will require Q to be a finite set.

[3]Here, you would notice that we are using the words "extreme point" in two different ways. One for an extreme point of a convex geometry, one for an extreme point of the convex hull. But they should be clear from the context.

Assumption 4. *When we talk about the generalized convex shelling on* P *with respect to* Q *in the rest of this paper,* Q *is always nonempty unless stated otherwise.*

Here, we define the clique complex of a graph. Let G be a graph. A *clique* of G is a vertex subset of G which induces a complete subgraph. The *clique complex* of G is the family of cliques of G. We treat the empty set and the single vertices as cliques, so the clique complex is actually a simplicial complex. In the literature, a clique complex is also called a *flag complex*.

3 On Open Problem 1.1 (proof of Theorem 2)

Now we concentrate on 2-dimensional separable generalized convex shellings. Namely, P and Q are nonempty finite point sets in $\mathbb{R}^2$ satisfying that $\mathrm{conv}(P) \cap \mathrm{conv}(Q) = \emptyset$ and $\mathcal{L}$ is the generalized convex shelling on P with respect to Q.

To prove Theorem 2, we will use the following fact, which is a consequence of a proposition by Hachimori & Nakamura [5].

Lemma 5. *A minimal nonface of the free complex* $\mathrm{Free}(\mathcal{L})$ *of a* d*-dimensional generalized convex shelling is of size at most* d*.*

It is well known that a simplicial complex whose minimal nonfaces are of size 2 is a clique complex of some graph. Therefore, so is the free complex of a 2-dimensional generalized convex shelling $\mathcal{L}$ by Lemma 5 (and this graph is actually the 1-dimensional skeleton of $\mathrm{Free}(\mathcal{L})$). Denote by $G(\mathcal{L})$ a unique graph whose clique complex is $\mathrm{Free}(\mathcal{L})$. The next is a key lemma.

Lemma 6. $G(\mathcal{L})$ *is chordal and connected.*

Note that Lemma 6 fails for a non-separable case. Since the clique complex of a connected chordal graph is contractible (not difficult to show), we can immediately find that the free complex of a 2-dimensional separable generalized convex shelling is contractible. Note that this holds for all d-dimensional (possibly non-separable) generalized convex shellings even if $Q = \emptyset$. A proof of this fact has already been given by Edelman & Reiner [3] (based on a theorem in Edelman & Jamison [2]). Notice that our approach is discrete-geometric while they used tools from combinatorial topology. However, we are not aware of a discrete-geometric proof for the higher-dimensional case.

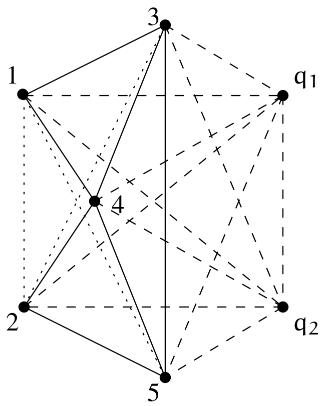
Since an induced subgraph of a chordal graph is also chordal, we can immediately see that if x be a vertex of $G(\mathcal{L})$ and c_x be the number of connected components of $G(\mathcal{L}) - x$ then $\mathrm{del}_{\mathrm{Free}(\mathcal{L})}(x)$ is homotopy equivalent to c_x distinct points. Therefore, to prove Theorem 2, we only have to show the following lemma.

Lemma 7. *A vertex* x *of* $G(\mathcal{L})$ *is not a cut-vertex of* $G(\mathcal{L})$ *if* $\mathrm{Dep}_{\mathcal{L}}(x) \neq P$.

Thus, we are able to finish the proof of Theorem 2. Note that the converse of Lemma 7 does not hold in general. That is exactly the reason why Open Problem 1.2 is invalid, as we will see in the next section.

4 On Open Problem 1.2 (a negative answer)

Here we will give an example which answers Open Problem 1.2 negatively. Look at the figure. In this example, $P = \{1, 2, 3, 4, 5\}$ and $Q = \{q_1, q_2\}$. Let $\mathcal{L}$ be the generalized convex shelling on P with respect to Q. The solid lines show the edges of $G(\mathcal{L})$. We can observe that $\mathrm{Dep}_{\mathcal{L}}(4) = P$. However, the deletion of 4 from $G(\mathcal{L})$ results in a connected graph, therefore $\mathrm{del}_{\mathrm{Free}(\mathcal{L})}(4)$ is contractible, which implies that this is not homotopy equivalent to (or does not have the same homology type as) a bouquet of equidimensional spheres.



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