Matroid Representation of Clique Complexes

Kenji Kashiwabara (Univ Tokyo)
Yoshio Okamoto* (ETH Zurich)
Takeaki Uno (NII Japan)

July 25–28, 2003
COCOON 2003 @ Big Sky Resort, MT

* Supported by the Berlin-Zürich Joint Graduate Program

Combinatorics
Geometry
Computation

Authors

K. Kashiwabara
Univ Tokyo
Japan

Y. Okamoto
ETH
Switzerland

T. Uno
NII
Japan

Background

- An independence system can be represented as the intersection of finitely many matroids.
- The clique complex of a graph is an independence system.

Question

- How many matroids do we need for clique complexes?

Motivation

Given later.

Def.: independence system

V a nonempty finite set

Def.: an independence system on V is a nonempty family $\mathcal{I}$ of subsets of V s.t.

$$X \in \mathcal{I} \implies Y \in \mathcal{I} \quad \text{for all } Y \subseteq X.$$

Example: $V = \{1, 2, 3, 4\}$;

$$\{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 4\}, \{1, 2, 3\}\}$$

Def.: independence system

V a nonempty finite set

Def.: an independence system on V is a nonempty family $\mathcal{I}$ of subsets of V s.t.

$$X \in \mathcal{I} \implies Y \in \mathcal{I} \quad \text{for all } Y \subseteq X.$$

Example: $V = \{1,2,3,4\}$;

$$\{\emptyset, \{1\}, \{2\}, \{3\}, \quad \{1,2\}, \{1,3\}, \{2,3\}, \{3,4\}, \{1,2,3\}\}.$$

Def.: independence system

V a nonempty finite set

Def.: an independence system on V is a nonempty family $\mathcal{I}$ of subsets of V s.t.

$$X \in \mathcal{I} \implies Y \in \mathcal{I} \quad \text{for all } Y \subseteq X.$$

Remark:

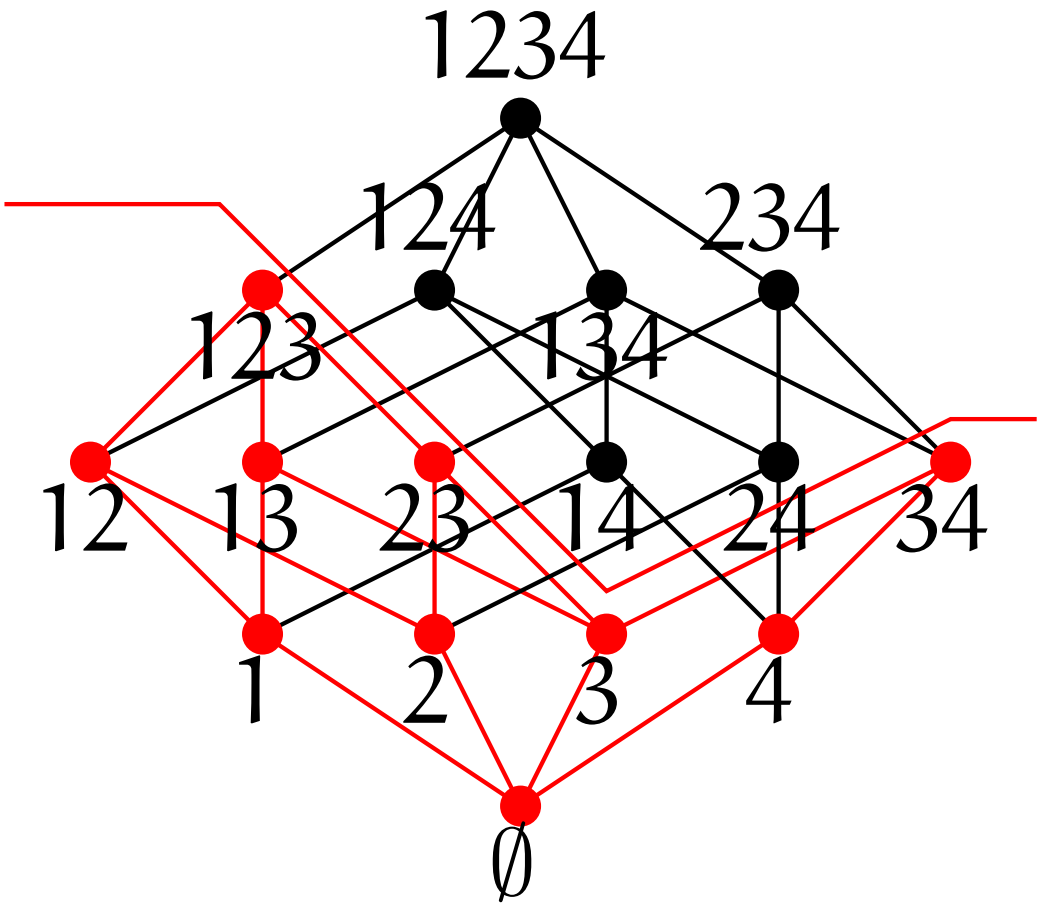
also called an abstract simplicial complex.

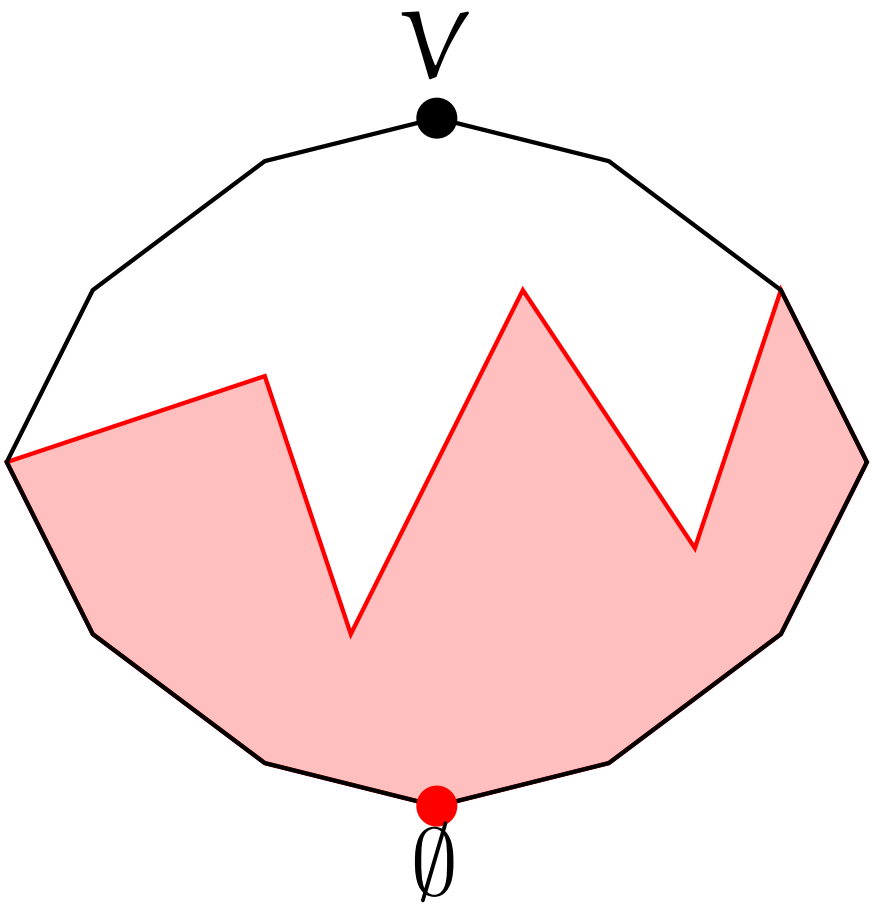
Typical independence systems from ...

- From graphs
  - the family of the cliques
  - the family of the forests
  - the family of the matchings
- From partially ordered sets
  - the family of all chains
- From polytopes
  - the family of all faces of a simplicial polytope
- From game theory
  - the family of losing coalitions of a weighted majority game

Understanding independence systems

$\mathcal{I} = \{\emptyset, 1, 2, 3, 4, 12, 13, 23, 34, 123\}$.



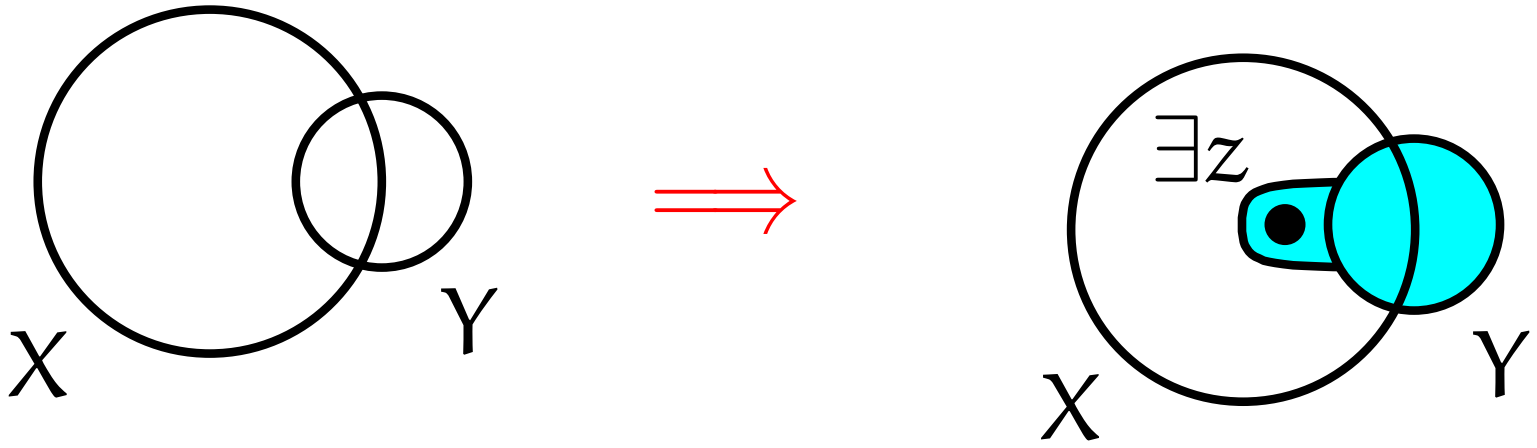


(known as the face poset of $\mathcal{I}$,
when we see $\mathcal{I}$ as an abstract simplicial complex)

Def.: matroids

Def.: An indep. system $\mathcal{I}$ is called a matroid if $\mathcal{I}$ satisfies the augmentation axiom:

$$X, Y \in \mathcal{I}, |X| > |Y| \Longrightarrow \exists z \in X \setminus Y \text{ s.t. } Y \cup \{z\} \in \mathcal{I}.$$



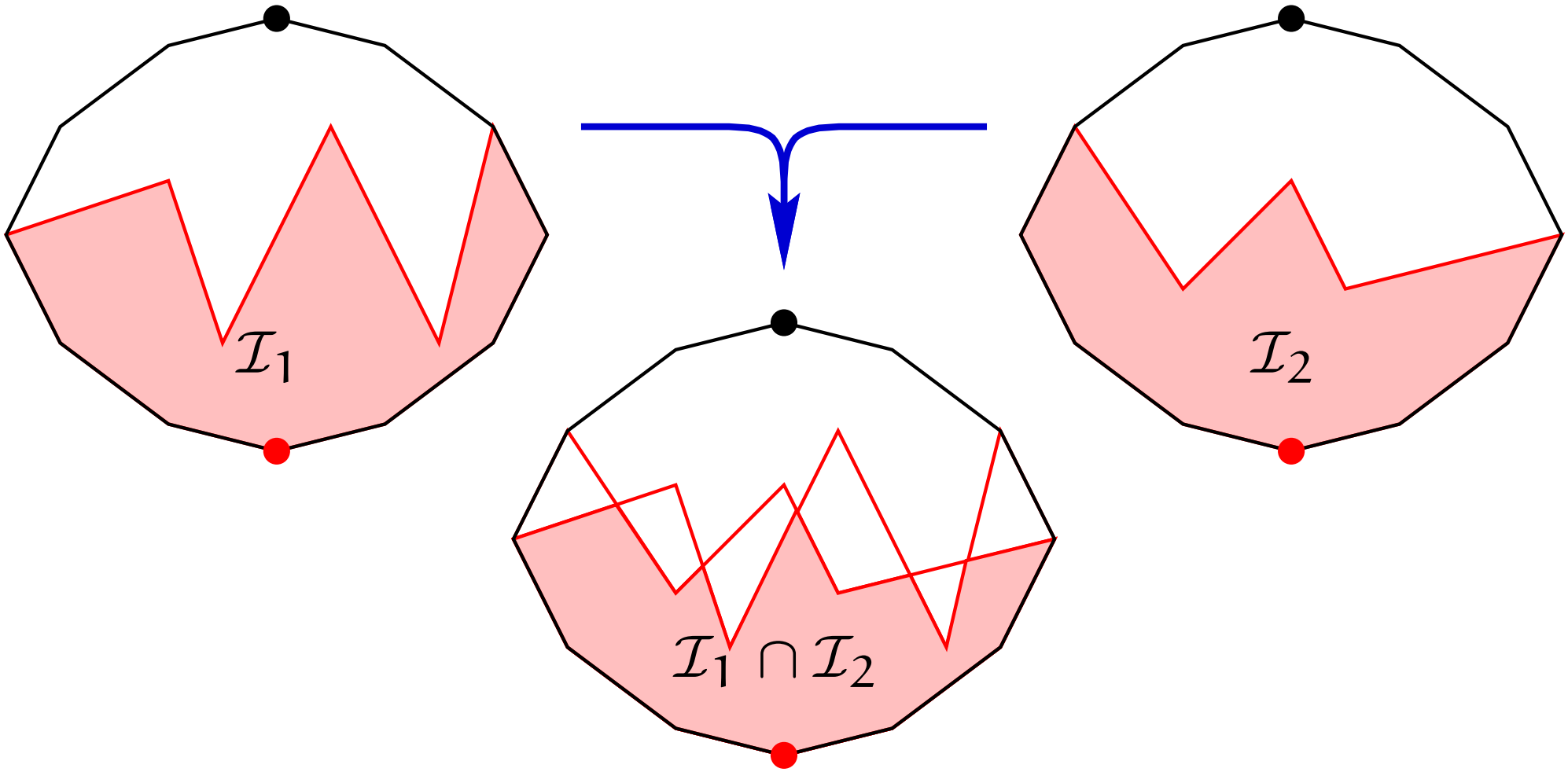
Appearance of matroids:

Combinatorial optimization, Combinatorial geometry, Combinatorial topology, Coding theory, Design theory, etc.

Intersection of independence systems

The intersection of indep. systems $\mathcal{I}_1, \mathcal{I}_2$ on V is

$$\mathcal{I}_1 \cap \mathcal{I}_2 = \{X \subseteq V : X \in \mathcal{I}_1 \text{ and } X \in \mathcal{I}_2\}.$$



Intersection of independence systems

The intersection of indep. systems $\mathcal{I}_1, \mathcal{I}_2$ on V is

$$\mathcal{I}_1 \cap \mathcal{I}_2 = \{X \subseteq V : X \in \mathcal{I}_1 \text{ and } X \in \mathcal{I}_2\}.$$

- The intersection of more indep. systems is defined analogously.

Intersection of independence systems

The intersection of indep. systems $\mathcal{I}_1, \mathcal{I}_2$ on V is

$$\mathcal{I}_1 \cap \mathcal{I}_2 = \{X \subseteq V : X \in \mathcal{I}_1 \text{ and } X \in \mathcal{I}_2\}.$$

- The intersection of more indep. systems is defined analogously.
- The intersection of indep. systems is again an indep. system.

Intersection of independence systems

The intersection of indep. systems $\mathcal{I}_1, \mathcal{I}_2$ on V is

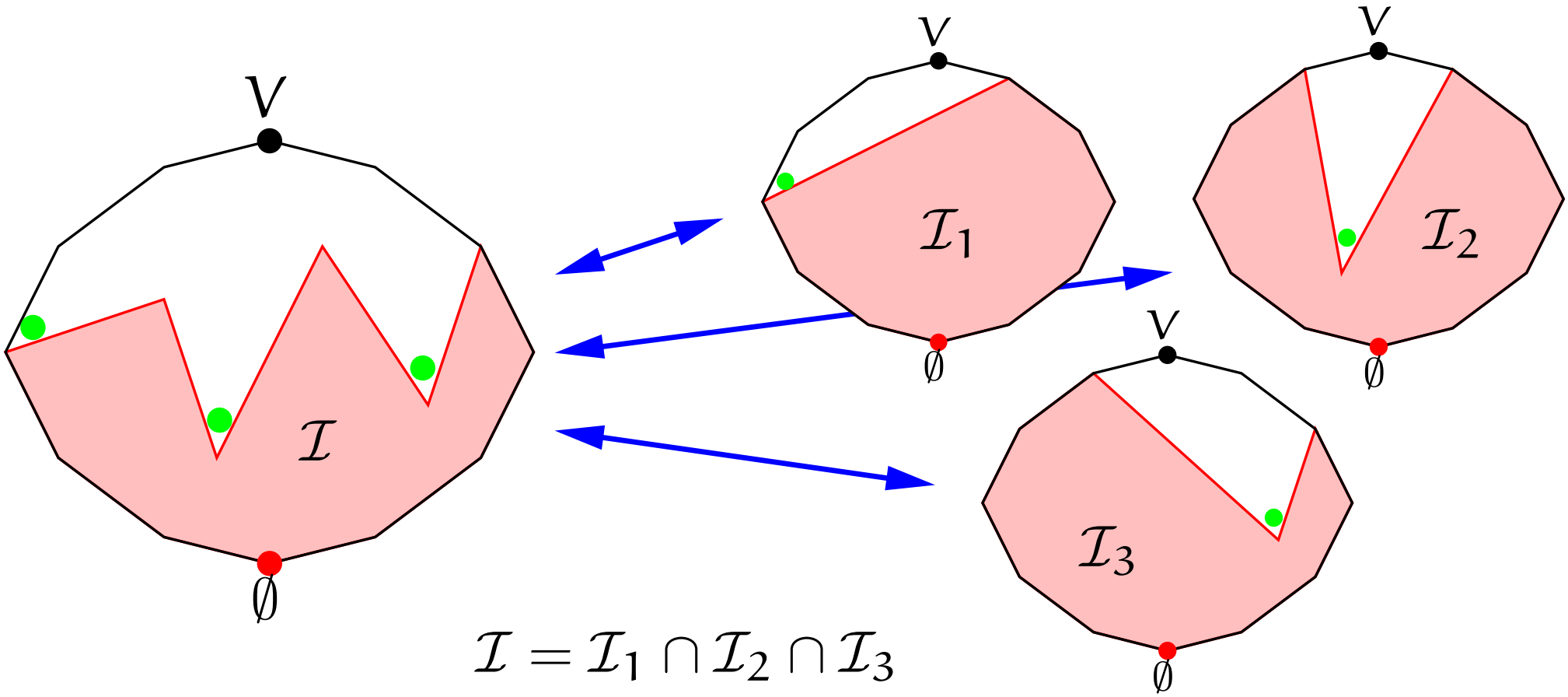
$$\mathcal{I}_1 \cap \mathcal{I}_2 = \{X \subseteq V : X \in \mathcal{I}_1 \text{ and } X \in \mathcal{I}_2\}.$$

- The intersection of more indep. systems is defined analogously.
- The intersection of indep. systems is again an indep. system.
- The intersection of matroids is an indep. system but not necessarily a matroid. However ...

An important fact

Fact: Every indep. system is the intersection of finitely many matroids.

Proof. Min. sets not in $\mathcal{I}$ are shown by ●.



$$\mathcal{I} = \mathcal{I}_1 \cap \mathcal{I}_2 \cap \mathcal{I}_3$$

Question:

How many matroids do we need to represent a given indep. system as their intersection??

. . . . . .

The proof in the prev. page says that:

this number $\leq$ the number of $\bullet$ for $\mathcal{I}$.

But, we might do better.

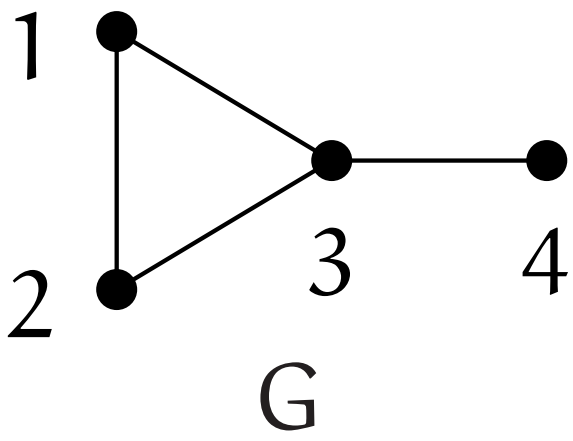
★ We study this number for clique complexes.

Def.: Clique complex

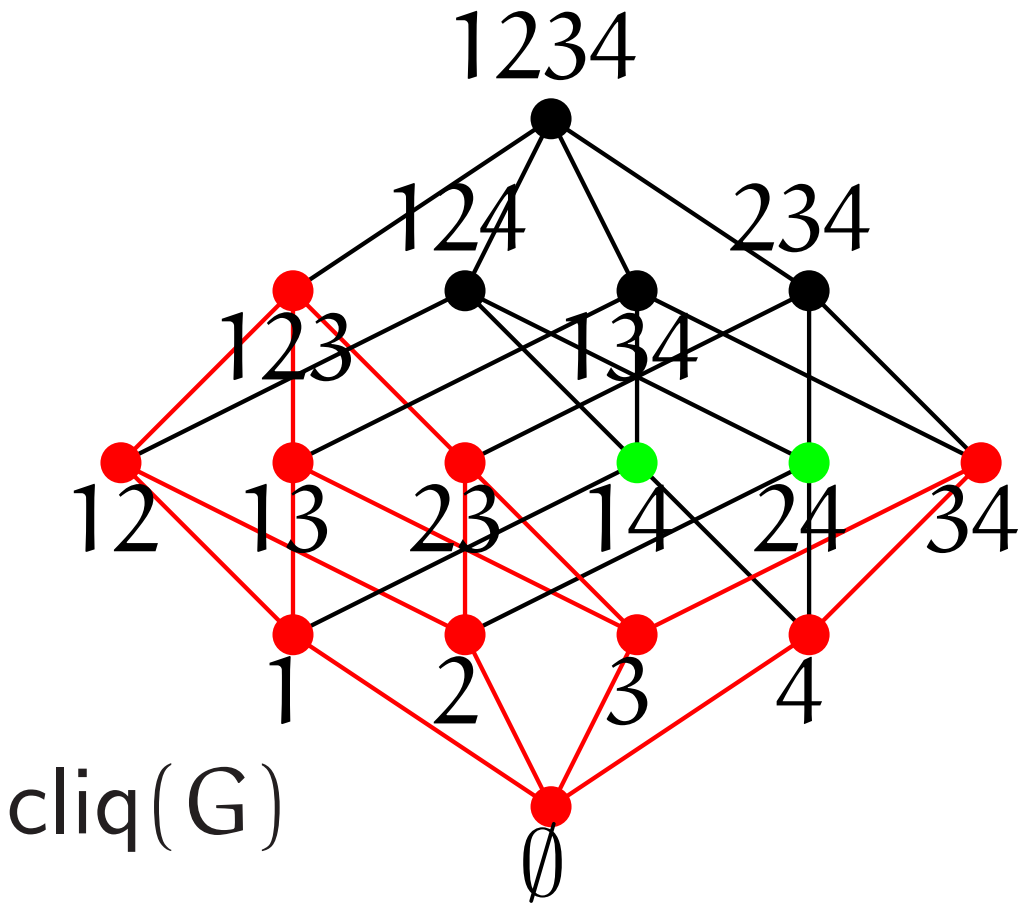
Def.: The clique complex of a graph $G = (V, E)$ is the family of all cliques of G.

Example:

$V = \{1, 2, 3, 4\}$



G



cliq(G)

Question (again)

Question:

How many matroids do we need to represent a clique complex as their intersection??

Result:

We investigate and characterize these numbers.

Question (again)

Question:

How many matroids do we need to represent a clique complex as their intersection??

Result:

We investigate and characterize these numbers.

. . . . . .

However, you may ask...

(1) Why do we consider such a representation??

(2) Why do we consider clique complexes??

Motivation (1)

Q. Why do we consider such a representation?

Ans. in the following proposition.

Motivation (1)

Q. Why do we consider such a representation?

Ans. in the following proposition.

Prop.: (Jenkyns, '76; Korte–Hausmann, '78)

(Translation of their prop. to clique complexes)

$\mathrm{cliq}(G)$ is the intersection of k matroids
$\Longrightarrow$ Greedy Alg. is a k-approx. alg. for
Max weighted clique problem in G.

★ $k \approx$ how complex G is w.r.t. clique problem.

Motivation (2)

Q. Why do we consider clique complexes?

Motivation (2)

Q. Why do we consider clique complexes?

Ans. The class of clique complexes includes some important classes like the classes of

- the family of the matchings of a graph G
  $=$ cliq(the complement of the line graph of G)
  studied by Fekete–Firla–Spille '03
  in the same framework as ours.
- the family of the chains of a poset P
  $=$ cliq(the comparability graph of P)
- ...

The talk plan

In the rest of my talk

(1) clique complex
which is a matroid
- Key concept: partition matroid

(2) clique complex
which is the intersection of k matroids
- Key concept: stable-set partition

(3) other results

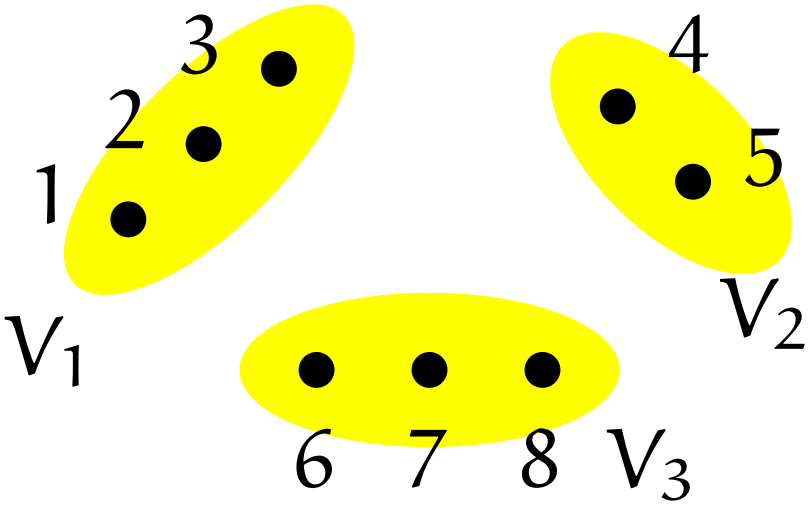
Def.: Partition matroid

V a nonempty finite set
$\mathcal{P} = \{V_1, V_2, \ldots, V_r\}$ a partition of V

Def.: A partition matroid $\mathcal{I}(\mathcal{P})$ of $\mathcal{P}$ is

$$\mathcal{I}(\mathcal{P}) = \{X \subseteq V : |X \cap V_i| \leq 1 \text{ for all } i = 1, \ldots, r\}.$$

Example: $\mathcal{P} = \{V_1, V_2, V_3\}$



$\mathcal{I}(\mathcal{P}) =$
$\{146, 147, 148, 156, 157, 158,$
$246, 247, 248, 256, 257, 258,$
$346, 347, 348, 356, 357, 358,$
and their subsets$\}$

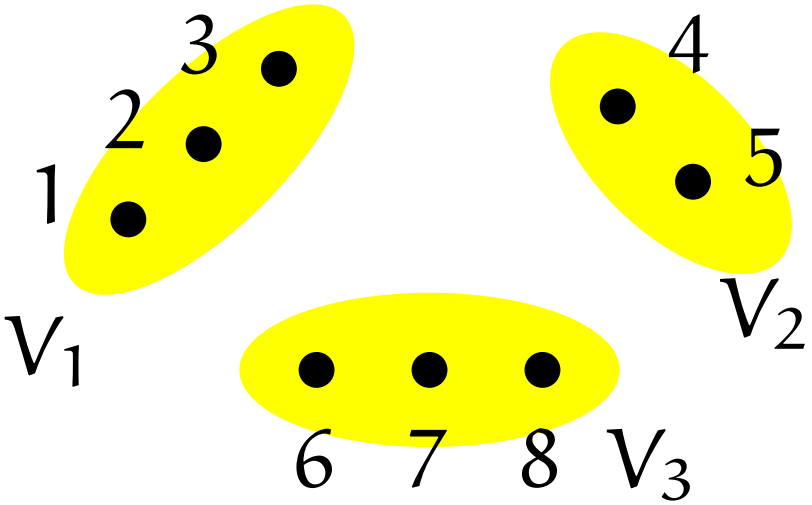
Def.: Partition matroid

V a nonempty finite set
$\mathcal{P} = \{V_1, V_2, \ldots, V_r\}$ a partition of V

Def.: A partition matroid $\mathcal{I}(\mathcal{P})$ of $\mathcal{P}$ is

$$\mathcal{I}(\mathcal{P}) = \{X \subseteq V : |X \cap V_i| \leq 1 \text{ for all } i = 1, \ldots, r\}.$$

Example: $\mathcal{P} = \{V_1, V_2, V_3\}$

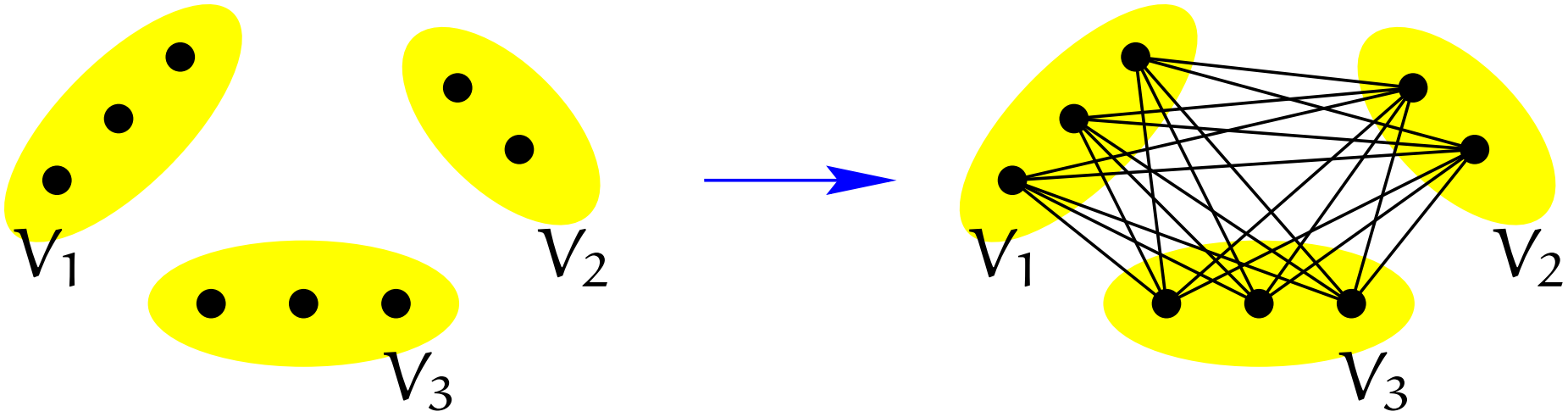


Notice: In fact, $\mathcal{I}(\mathcal{P})$ is a matroid.

Partition matroids are clique complexes

Lem.: Partition matroids are clique complexes.

Proof. Given a partition $\mathcal{P} = \{V_1, \ldots, V_r\}$ of V.



Construct a complete r-partite graph $G_{\mathcal{P}}$ s.t. each part of $G_{\mathcal{P}}$ corresponds to V_i.

Then, $\mathrm{cliq}(G_{\mathcal{P}}) = \mathcal{I}(\mathcal{P})$. [qed]

Consequence

Remark: G a graph

G is complete multipartite

$\Downarrow$

cliq(G) is a matroid.

Matroidal clique complexes

Thm 1: G a graph

G is complete multipartite

$\Updownarrow$

cliq(G) is a matroid.

Proof. Omitted.

The talk plan

In the rest of my talk

(1) clique complex
which is a matroid
- Key concept: partition matroid

(2) clique complex
which is the intersection of k matroids
- Key concept: stable-set partition

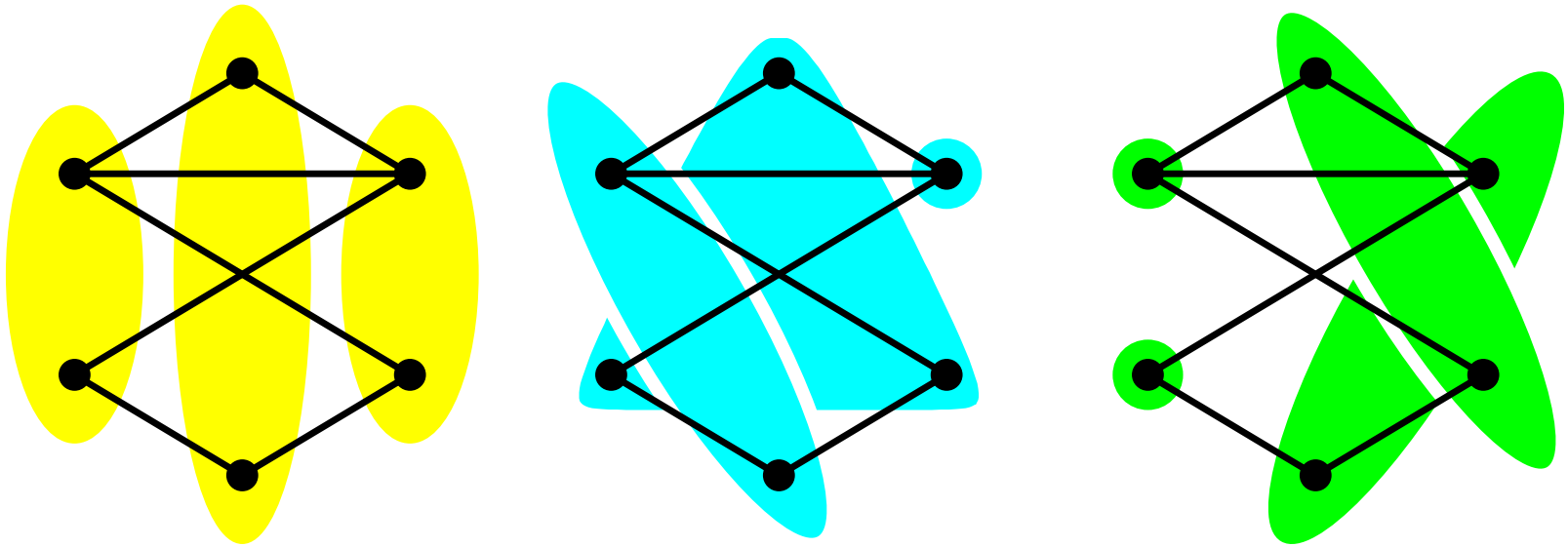
(3) other results

Def.: Stable-set partitions

$G = (V, E)$ a graph

Def.: A stable-set partition of G is a partition $\mathcal{P} = \{V_1, \ldots, V_r\}$ of V s.t. V_i is a stable set of G $(i = 1, \ldots, r)$.

Example:



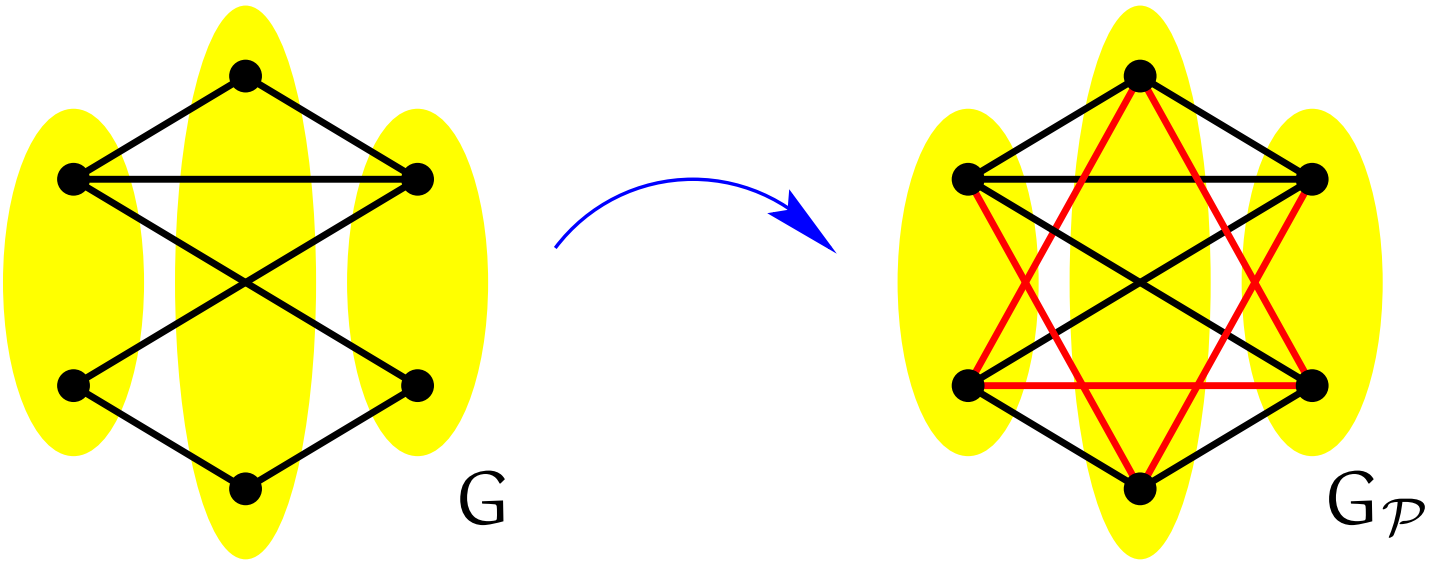
Construct a graph from a stable-set partition

G a graph
$\mathcal{P}$ a stable-set partition of G

Construct a complete r-partite graph $G_{\mathcal{P}}$ from $\mathcal{P}$.

Observation: An edge of G is an edge of $G_{\mathcal{P}}$.

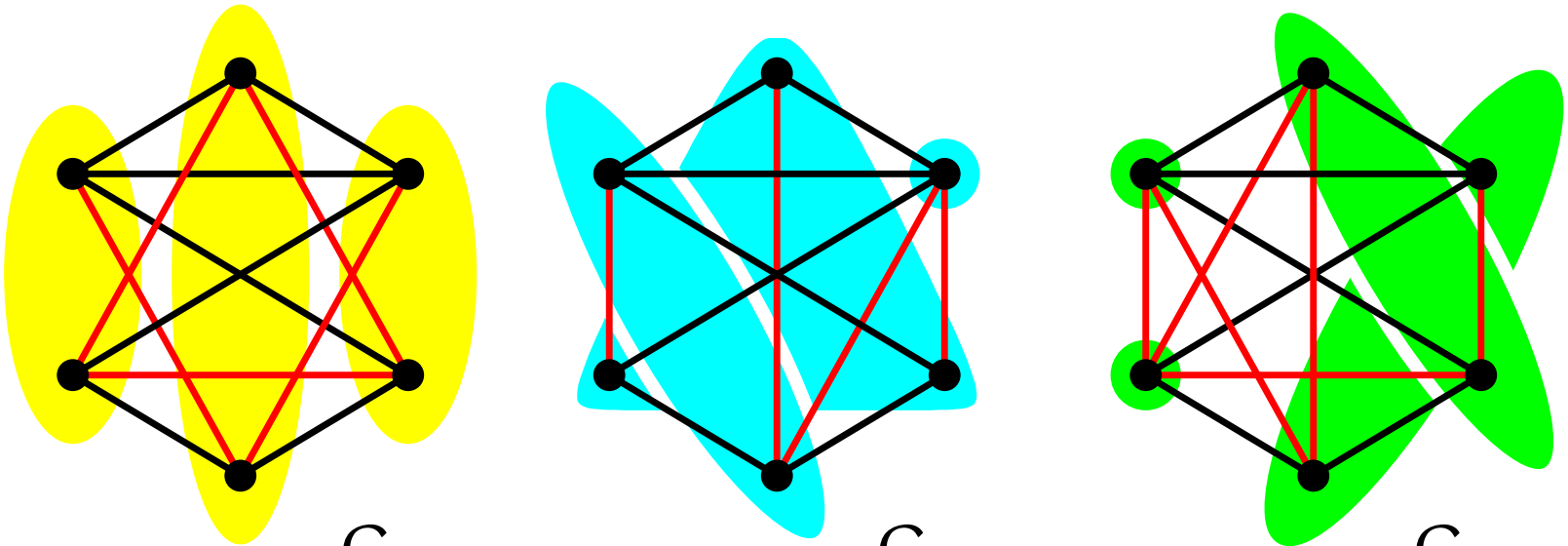
This means: $\mathsf{cliq}(G) \subseteq \mathsf{cliq}(G_{\mathcal{P}}) = \mathcal{I}(\mathcal{P})$.



Three stable-set partitions

Generally,
$\mathrm{cliq}(G) \subseteq \mathrm{cliq}(G_{\mathcal{P}_1}) \cap \mathrm{cliq}(G_{\mathcal{P}_2}) \cap \mathrm{cliq}(G_{\mathcal{P}_3})$.

However, in this example,



$G_{\mathcal{P}_1}$ $G_{\mathcal{P}_2}$ $G_{\mathcal{P}_3}$

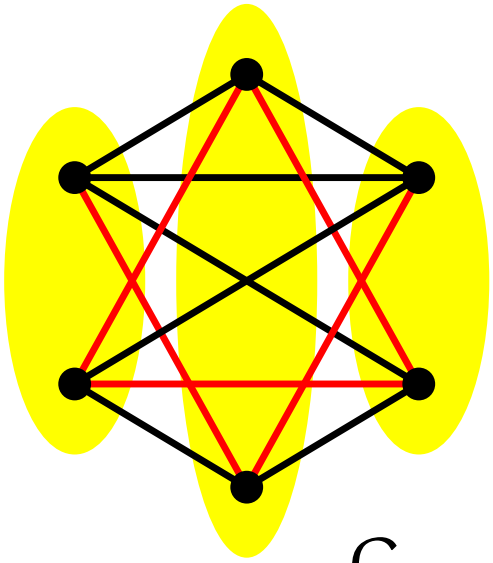
$$\mathrm{cliq}(G) = \mathrm{cliq}(G_{\mathcal{P}_1}) \cap \mathrm{cliq}(G_{\mathcal{P}_2}) \cap \mathrm{cliq}(G_{\mathcal{P}_3})$$

That is because

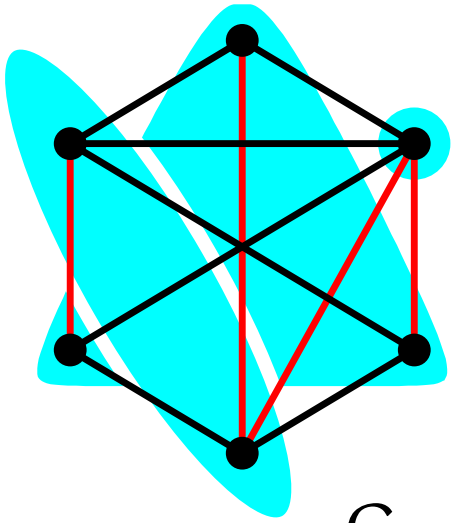
Three stable-set partitions

Generally,
$\mathrm{cliq}(G) \subseteq \mathrm{cliq}(G_{\mathcal{P}_1}) \cap \mathrm{cliq}(G_{\mathcal{P}_2}) \cap \mathrm{cliq}(G_{\mathcal{P}_3})$.

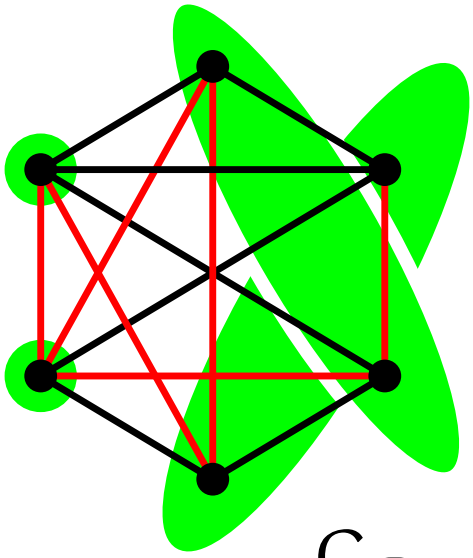
However, in this example,



$G_{\mathcal{P}_1}$



$G_{\mathcal{P}_2}$



$G_{\mathcal{P}_3}$

$$\mathrm{cliq}(G) = \mathrm{cliq}(G_{\mathcal{P}_1}) \cap \mathrm{cliq}(G_{\mathcal{P}_2}) \cap \mathrm{cliq}(G_{\mathcal{P}_3})$$

No red edge is contained in all of these graphs.

To summarize

$G = (V, E)$ a graph

$\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k$ stable-set partitions of G

$$\text{cliq}(G) = \bigcap \{\mathcal{I}(\mathcal{P}_i) : i = 1, \ldots, k\}$$

$$\Updownarrow$$

No red edge is contained in all $G_{\mathcal{P}_i}$'s.

In other words...

Remark: $G = (V, E)$ a graph

$\exists$ k stable-set partitions $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k$ of G s.t. $\text{cliq}(G) = \bigcap\{\mathcal{I}(\mathcal{P}_i) : i = 1, \ldots, k\}$

$\Updownarrow$

$\exists$ k stable-set partitions $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k$ of G s.t. no red edge is contained in all $G_{\mathcal{P}_i}$'s.

Thm 2: $G = (V, E)$ a graph

$\mathrm{cliq}(G)$ is the intersection of k matroids

$\Updownarrow$

$\exists$ k stable-set partitions $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k$ of G s.t. $\mathrm{cliq}(G) = \bigcap\{\mathcal{I}(\mathcal{P}_i) : i = 1, \ldots, k\}$

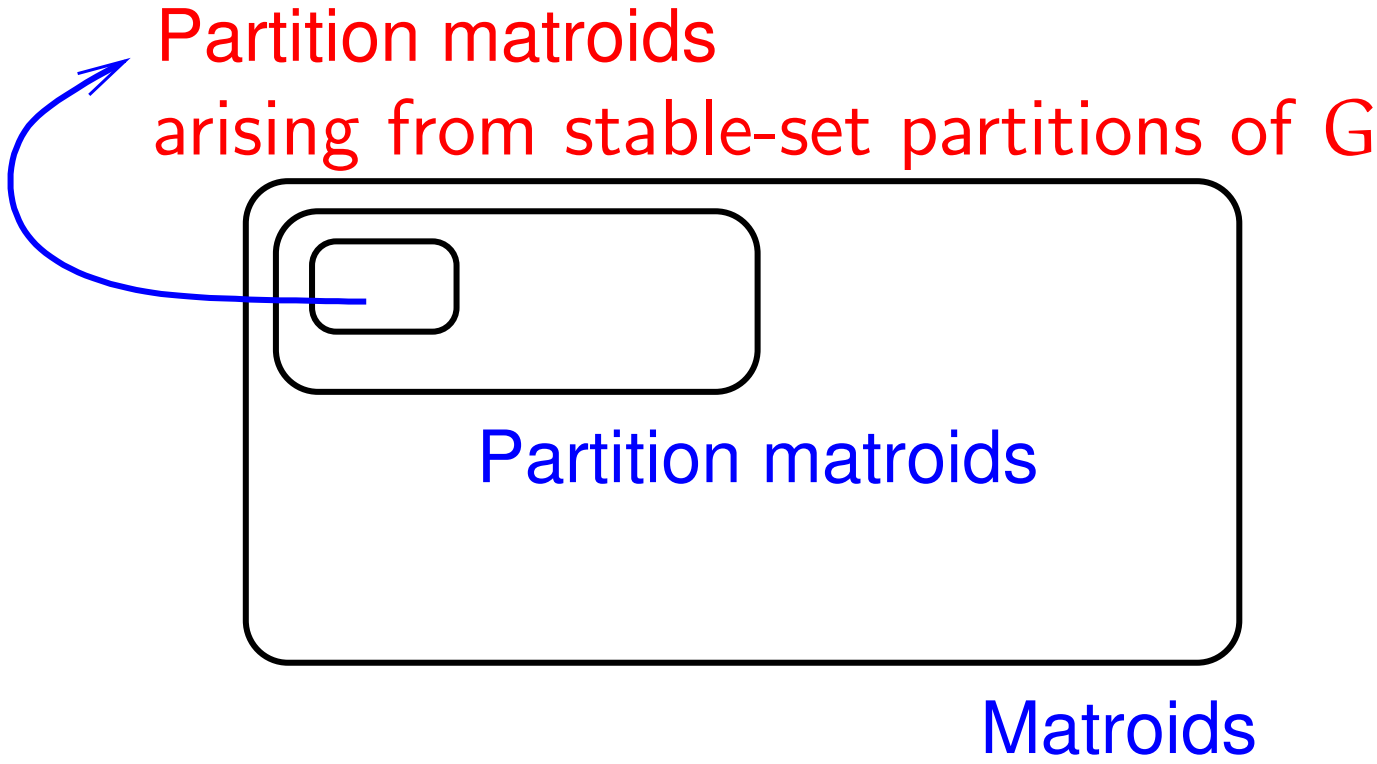
$\Updownarrow$

$\exists$ k stable-set partitions $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k$ of G s.t. no red edge is contained in all $G_{\mathcal{P}_i}$'s.

Proof. Omitted.

What does this theorem mean? (1)

Given a graph G.



When we want a representation of cliq(G) by k matroids, just search k partition matroids from stable-set partitions of G.

What does this theorem mean? (2)

Consider the following decision problem.

Instance: a graph G and a positive integer k

Question: ?? $\exists$ k matroids $\mathcal{I}_1, \ldots, \mathcal{I}_k$
s.t. $\text{cliq}(G) = \mathcal{I}_1 \cap \cdots \cap \mathcal{I}_k$??

Thm 2 implies: This problem $\in \mathcal{NP}$.

The talk plan

In the rest of my talk

(1) clique complex
which is a matroid
- Key concept: partition matroid

(2) clique complex
which is the intersection of k matroids
- Key concept: stable-set partition

(3) other results
(five more pages!!!)

Clique complexes represented by two matroids

Thm 3: G a graph

cliq(G) is the intersection of two matroids

$\Updownarrow$

The stable-set graph of G is bipartite.

Clique complexes represented by two matroids

Thm 3: G a graph

cliq(G) is the intersection of two matroids

$\Updownarrow$

The stable-set graph of G is bipartite.

Note: Combining $\begin{cases} \text{Theorem 3 and} \\ \text{Protti–Szwarcfiter ('02),} \end{cases}$

we can tell, in polynomial time, whether

the clique complex of a given graph
is the intersection of two matroids or not.

Complexity issue

Consider the following decision problem.

Fix: a positive integer k

Instance: a graph G

Question: ? $\exists$ k matroids $\mathcal{I}_1, \ldots, \mathcal{I}_k$
s.t. $\mathrm{cliq}(G) = \mathcal{I}_1 \cap \cdots \cap \mathcal{I}_k$?

What is the complexity of this problem for each k?

k	1	2	3	4	$\cdots$
	$\mathcal{P}$	$\mathcal{P}$	?	?	$\cdots$

Graphs as independence systems

A graph can be seen as an independence system:

$$X \in G \quad \Leftrightarrow \quad X = \begin{cases} \emptyset & \text{or} \\ \{v\} & \text{or} \\ \{u, v\} & \text{if it's an edge.} \end{cases}$$

Thm 4: G a graph

G is the intersection of k matroids

$\Updownarrow$

cliq(G) is the intersection of k matroids.

Number of matroids we need

Let $k(G) = \min\{k : \text{cliq}(G) \text{ is the intersection of } k \text{ matroids } \}$.

$k(n) = \max\{k(G) : G \text{ has } n \text{ vertices } \}$.

Thm 5: $k(n) = n - 1$.

Summary

Question

- How many matroids do we need to represent a given clique complex as their intersection??

Result

- cliq(G) is the intersection of k matroids

cliq(G) is the intersection of k partition matroids from stable-set partitions of G

[End of the talk]