Submodularity of some classes of the combinatorial optimization games

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August 20–28, 2002

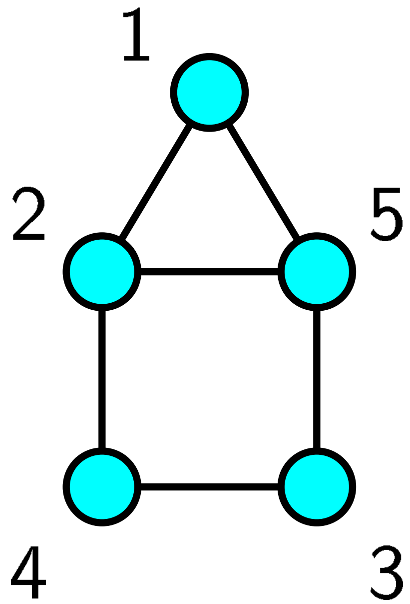
International Congress of Mathematicians 2002
Beijing, China

Combinatorics
Geometry
Computation

★★ Definition: minimum coloring game ★★

$G = (V, E)$ a graph

- A k-coloring of G
  - a surjective map $c : V \to \{1, \ldots, k\}$ s.t. if $\{u, v\} \in E$, then $c(u) \neq c(v)$.
- The chromatic number $\chi(G)$ of G
  - the minimum k s.t. a k-coloring of G exists.
- The minimum coloring game on G
  - a function $\chi_G : 2^V \to \mathbb{N}$ def'd as $\chi_G(S) = \chi(G[S])$, where $G[S]$ is the subgraph induced by $S \subseteq V$.



S	χ_G	S	χ_G	S	χ_G	S	χ_G
$\emptyset$	0	14	1	123	2	245	2
1	1	15	2	124	2	345	2
2	1	23	2	125	3	1234	2
3	1	24	1	134	2	1235	3
4	1	25	2	135	2	1245	3
5	1	34	2	145	2	1345	2
12	2	35	1	234	2	2345	2
13	1	45	2	235	2	12345	3

Characterization of submodular minimum coloring games

$\boxed{\text{Aim}}$ Fill in "......" in the following statement:

χ_G is submodular $\iff$ G satisfies "......".

Def.: A function $\gamma : 2^X \to \mathbb{R}$ is submodular if

$$\gamma(S) + \gamma(T) \geq \gamma(S \cup T) + \gamma(S \cap T)$$

for any $S, T \subseteq X$.

★★ Why is it called a "game"? ★★

Def.: A cooperative game is a function

$$\gamma : 2^X \to \mathbb{R} \text{ satisfying } \gamma(\emptyset) = 0.$$

- X is a finite set, and an element of X is called a player.
- For $S \subseteq X$, $\gamma(S)$ is the total cost owed by S when the players in S cooperate.

In our case: $X := V$, and $\gamma := \chi_G$.

Goal of cooperative game theory

- Allocate the total cost $\gamma(X)$ to each player "fairly".
- Or, what is a "fair allocation"?

★★ Why do we consider submodularity? (1) ★★

Submodularity has some prominent properties.

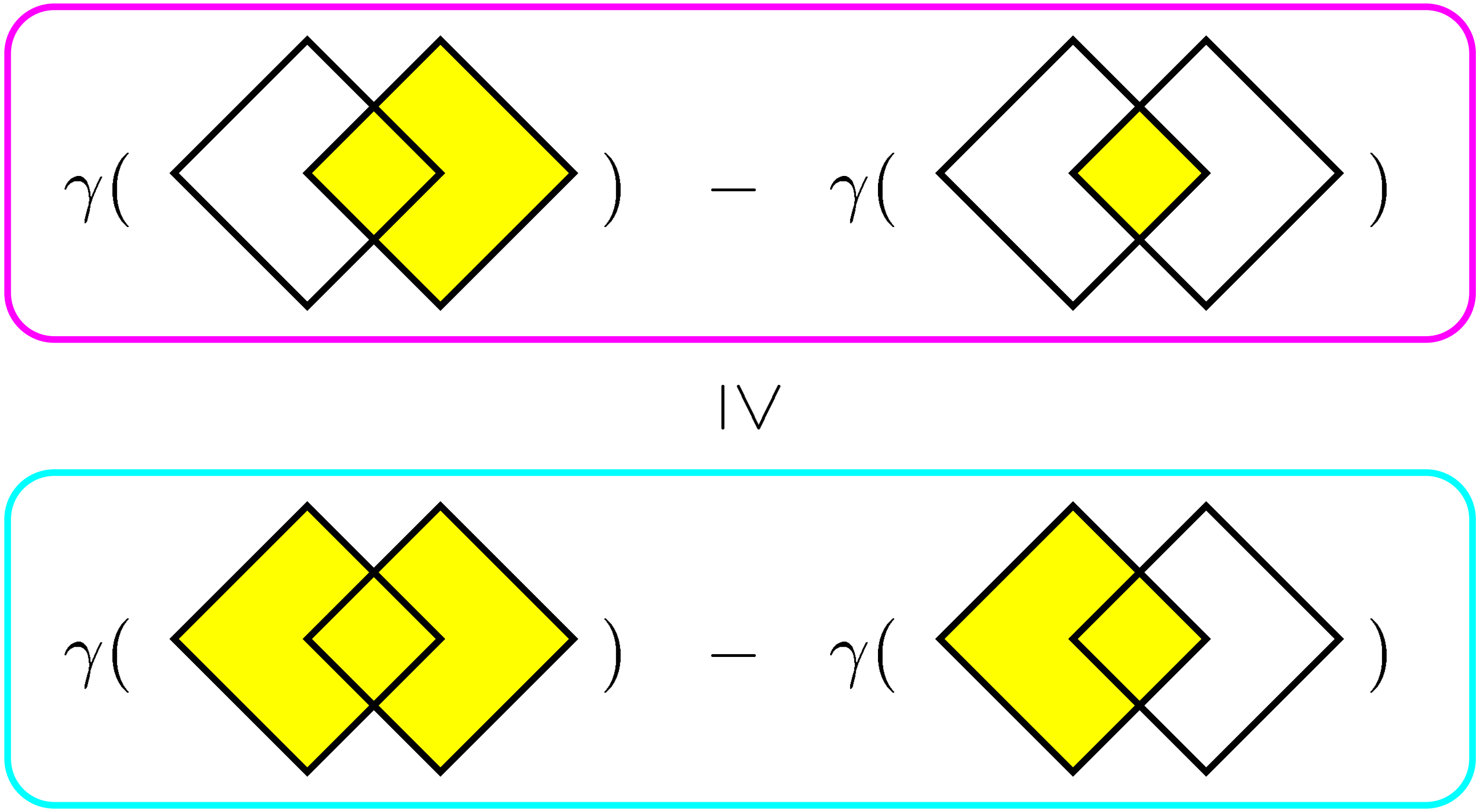
- The core is nonempty. (Shapley, '71)
- The core is a unique von Neumann–Morgenstern solution. (Shapley, '71)
- The Shapley value is the barycenter of the core. (Shapley, '71)
- The bargaining set coincides with the core. (Maschler–Peleg–Shapley, '72)
- The kernel coincides with the nucleolus. (Maschler–Peleg–Shapley, '72)
- The τ-value can be computed in poly time. (Tijs, '81)

★★ Why do we consider submodularity? (2) ★★

Submodularity has a quite natural interpretation.

$$\gamma(S) + \gamma(T) \geq \gamma(S \cup T) + \gamma(S \cap T)$$

$$\Longleftrightarrow \boxed{\gamma(T) - \gamma(S \cap T)} \geq \boxed{\gamma(S \cup T) - \gamma(S)}$$

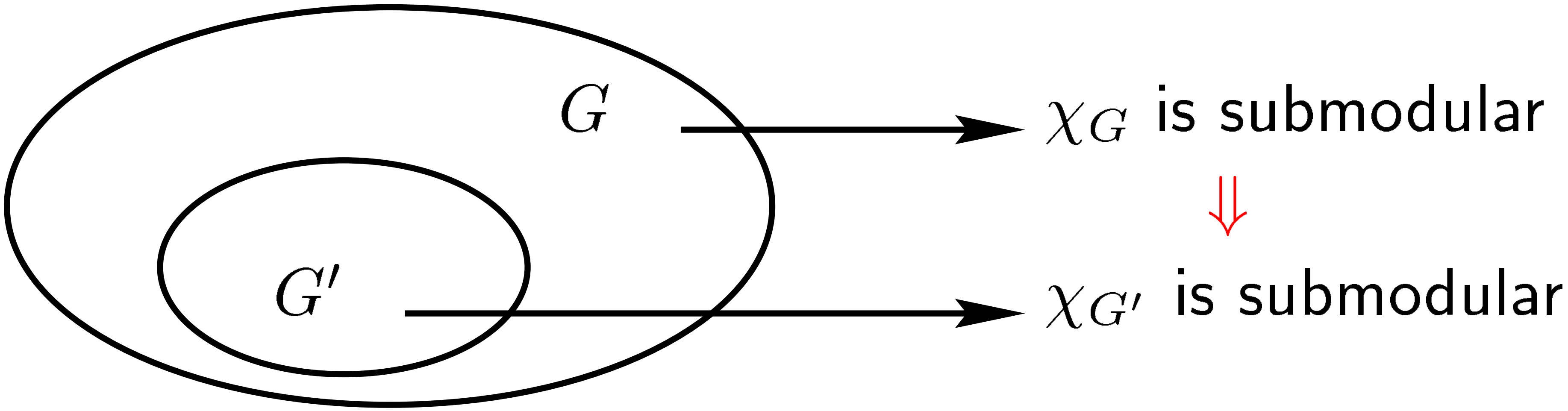


First observation

Observation:

Let G' be an induced subgraph of G

χ_G is submodular $\Longrightarrow$ $\chi_{G'}$ is also submodular.



This implies:

In order to characterize submodular minimum coloring games,

We only have to list up "forbidden induced subgraphs."

Now our aim is more clarified.

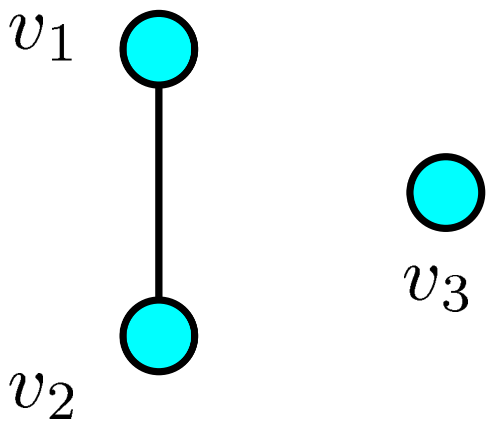
$\boxed{\text{Aim:}}$ Determine $G'_1, G'_2, \ldots$ in the following statement:

$$\chi_G \text{ is submodular}$$
$$\Updownarrow$$
$$G \text{ doesn't contain } G'_1, G'_2, \ldots \text{ as its induced subgraph.}$$

$G'_1, G'_2, \ldots$ are called the forbidden induced subgraphs of the property that "χ_G is submodular."

Second observation

For the following graph $G = K^1 \cup K^2$, χ_G is not submodular.



Set $S = \{v_1, v_3\}, T = \{v_2, v_3\}$

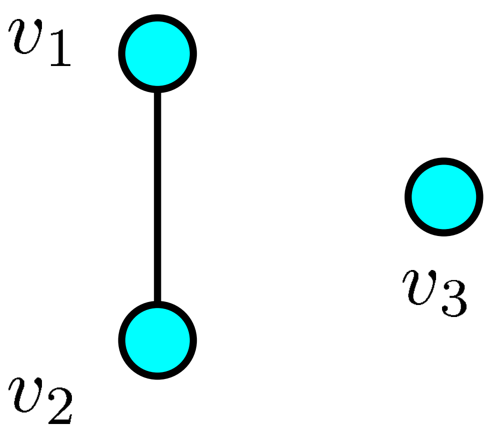
$$\underbrace{\chi_G(S)}_{1} + \underbrace{\chi_G(T)}_{1} \quad < \quad \underbrace{\chi_G(S \cup T)}_{2} + \underbrace{\chi_G(S \cap T)}_{1}.$$

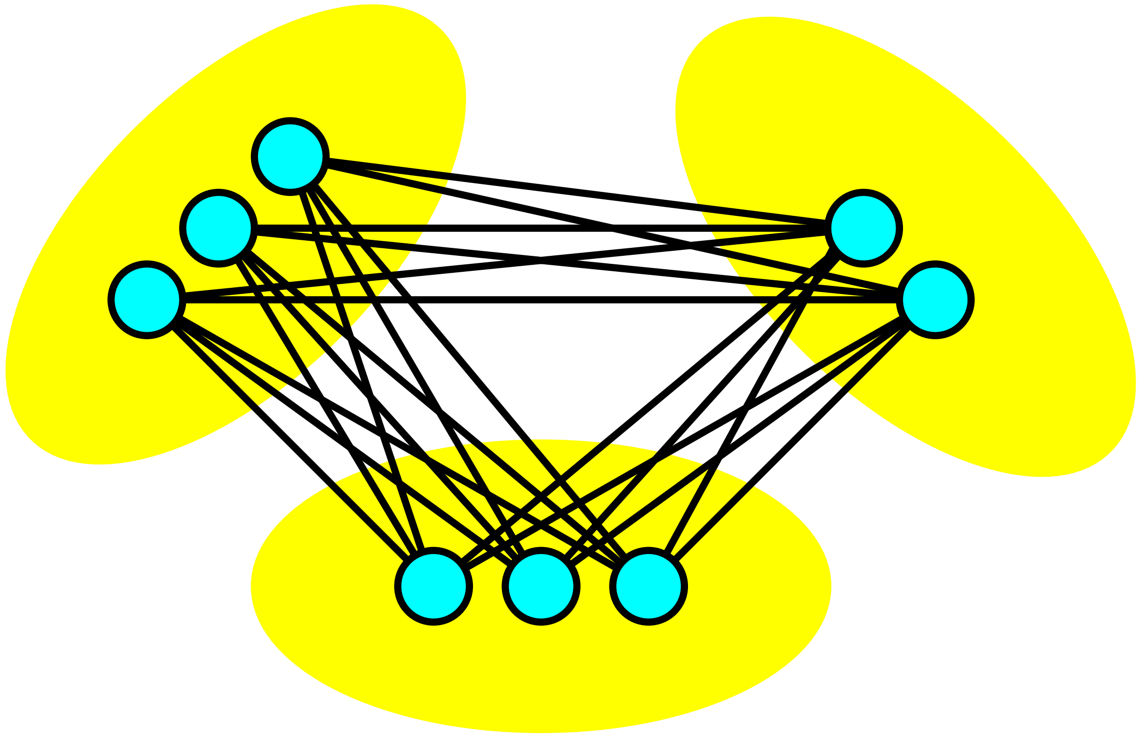
Thm. The following statements are equivalent.

(1) χ_G is submodular

(2) G doesn't contain $K^1 \cup K^2$ as its induced subgraph

(3) G is a complete r-partite graph.



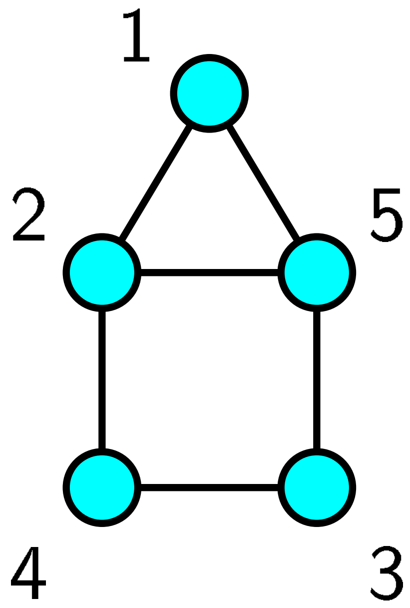


We will see that

our result also lead to the characterization of matroidal clique complexes.

$G = (V, E)$ a graph

The clique complex of G is the family of all cliques of G



The clique complex is
$\{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\},$
$\{1,2\}, \{1,5\}, \{2,4\}, \{2,5\}, \{3,4\}, \{3,5\},$
$\{1,2,5\}\}$

Matroids and the rank functions

A matroid is a pair (X, ρ) of

- a nonempty finite set X (called the ground set) and
- a function $\rho : 2^X \to \mathbb{N}$ (called the rank function) satisfying

(R1) $0 \leq \rho(S) \leq |S| \qquad (S \subseteq X)$;
(R2) $\rho(S) \leq \rho(T) \qquad (S \subseteq T \subseteq X)$;
(R3) $\rho(S) + \rho(T) \geq \rho(S \cup T) + \rho(S \cap T) \qquad (S, T \subseteq X)$.

Example

- $X =$ a set of vectors, $\rho(S) = \dim \operatorname{span}(S)$
- $X =$ the edge set of a graph, $\rho(S) =$ the max. size of forests contained in S

For any graph G,

- the minimum coloring game χ_G satisfies **(R1)** and **(R2)**.
- Our theorem implies
  χ_G satisfies **(R3)** $\Longleftrightarrow$ G is complete r-partite.

  (R1) $0 \leq \rho(S) \leq |S| \quad (S \subseteq X)$;
  (R2) $\rho(S) \leq \rho(T) \quad (S \subseteq T \subseteq X)$;
  (R3) $\rho(S) + \rho(T) \geq \rho(S \cup T) + \rho(S \cap T) \quad (S, T \subseteq X)$.

Therefore:

(V, χ_G) is a matroid $\Longleftrightarrow$ G is complete r-partite.

★★ The char. of matroidal clique complexes ★★

(X, ρ) a matroid

The family of independent sets of (X, ρ) is

$$\mathcal{I}(X, \rho) = \{S \subseteq X : \rho(S) = |S|\},$$

When G is complete r-partite, (V, χ_G) is a matroid and

$$\mathcal{I}(V, \chi_G) = \text{the clique complex of } G.$$

So, the following are equivalent.

- G is complete r-partite.
- The clique complex of G is the family of independent sets of a matroid determined by χ_G.

★★ Concluding summary ★★

For minimum coloring games and clique complexes,
we showed the following are equivalent.

(1) χ_G is submodular

(2) G doesn't contain $K^1 \cup K^2$ as its induced subgraph

(3) G is a complete r-partite graph.

(4) (V, χ_G) is a matroid.

(5) The clique complex of G is the family of independent set of a matroid.

Note: We have similar arguments for
min vertex cover games and matching complexes.