Matroid representation of clique complexes

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Combinatorics
Geometry
Computation

My interests

Combinatorial models of optimization

- matroids, polymatroids, independence systems
- oriented matroids
- polytopes, polyhedra, simplicial complexes
- greedoids, antimatroids, closure systems
- LP-type problems
- unique sink orientation

Today's topic $\rightarrow$ matroids, independence systems

Background

- An independence system can be represented as the intersection of finitely many matroids.
- The clique complex cliq(G) of a graph G is an independence system.

Question

- How many matroids do we need for clique complexes?

Result

- We characterize how many we need.

Def: independence system

V a nonempty finite set

Def.: an independence system on V is a nonempty family $\mathcal{I}$ of subsets of V satisfying

$$X \in \mathcal{I} \quad \Longrightarrow \quad Y \in \mathcal{I} \quad \text{for all } Y \subseteq X.$$

Example: $V = \{1,2,3,4\}$;

- an example of independence systems
  $\{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{2,3\}, \{3,4\}, \{1,2,3\}\}$.
- NOT an example of independence systems
  $\{\emptyset, \{1\}, \{2\}, \{3\}, \qquad \{1,2\}, \{1,3\}, \{2,3\}, \{3,4\}, \{1,2,3\}\}$.

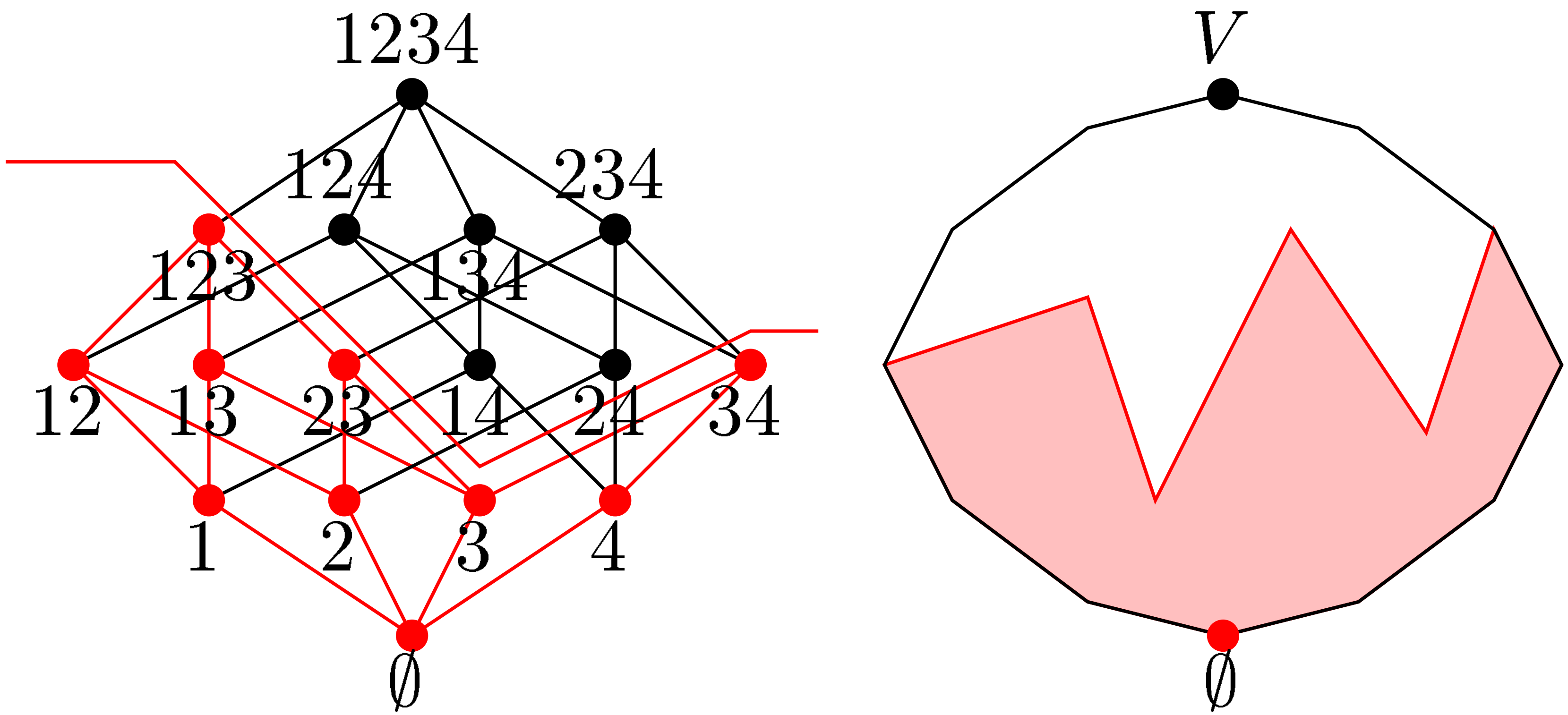
Remark: also called an abstract simplicial complex.

Typical independence systems from ...

- From graphs
  - the family of the cliques
  - the family of the forests
  - the family of the matchings
  - the family of vertices which share a common vertex.
- From partially ordered sets
  - the family of all chains
  - the family of all antichains
- From polytopes
  - the family of all proper faces of a simplicial polytope
- From optimization
  - the set of all feasible solutions of a knapsack problem
- From game theory
  - the family of losing coalitions of a weighted majority game

As order ideals of the Boolean lattices

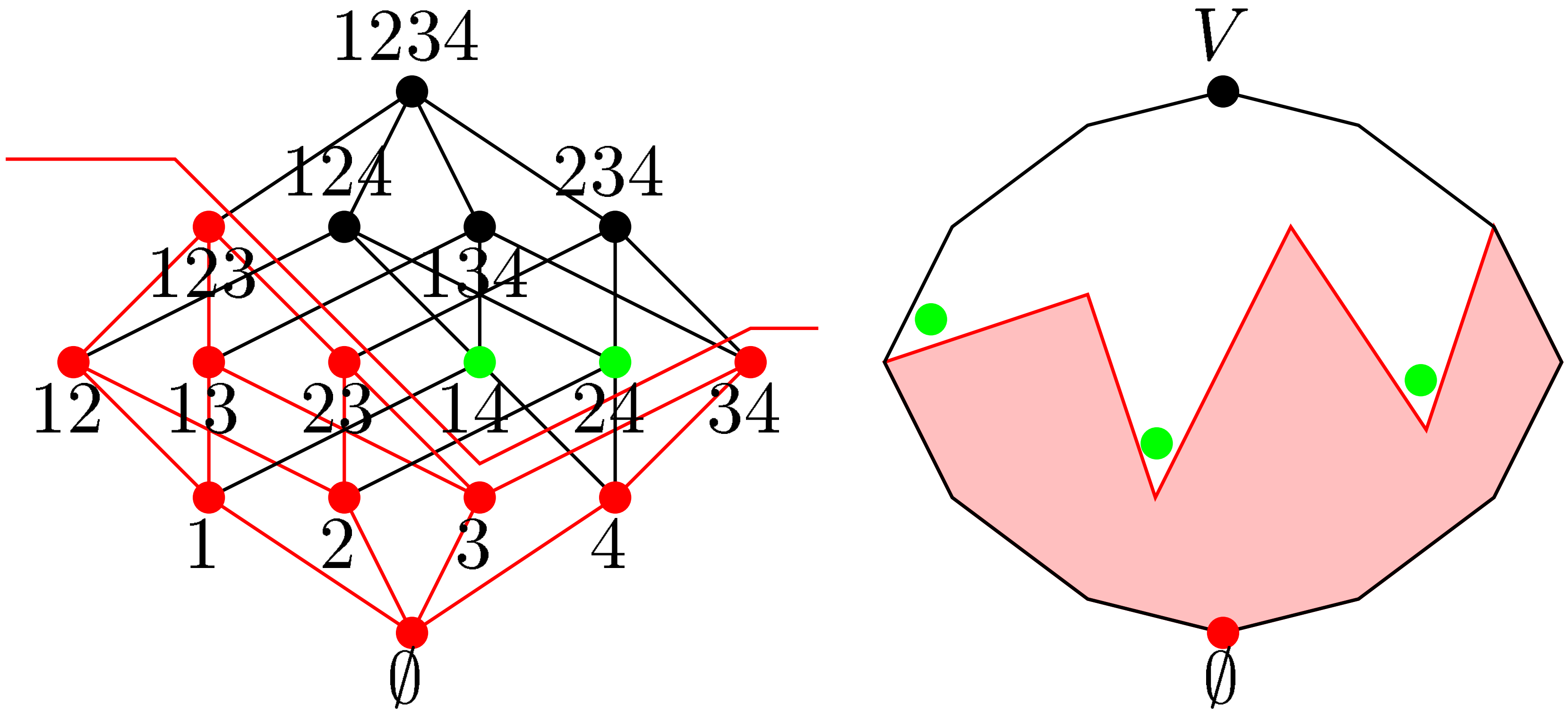
$\mathcal{I} = \{\emptyset, 1, 2, 3, 4, 12, 13, 23, 34, 123\}.$



(known as the face poset of $\mathcal{I}$,
when we see $\mathcal{I}$ as an abstract simplicial complex)

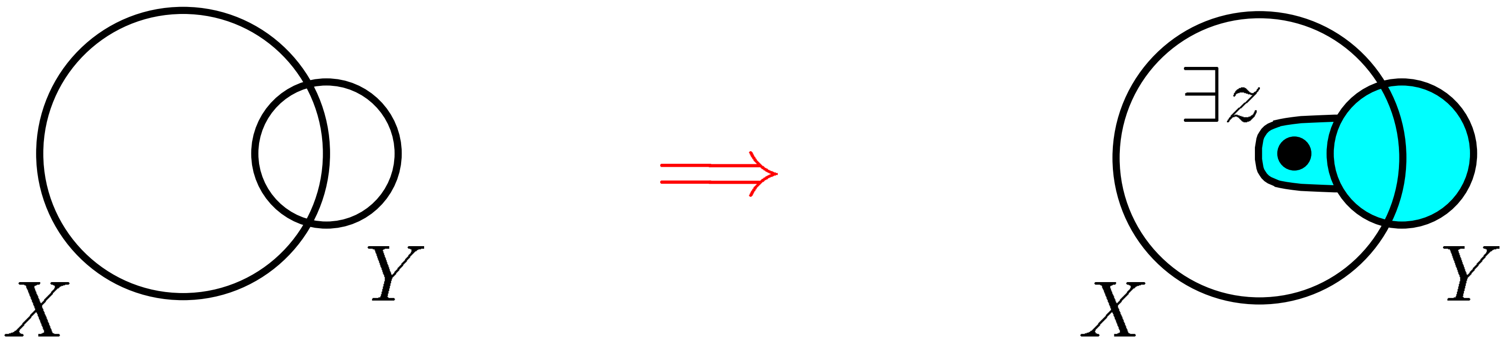
$\mathcal{I}$ an independence system on V

Def.: $C \subseteq V$ is called a circuit if $C \notin \mathcal{I}$ and C is minimal w.r.t. this property.



Def.: An independence system $\mathcal{I}$ is called a matroid if $\mathcal{I}$ satisfies the following augmentation axiom:

$$X, Y \in \mathcal{I}, |X| > |Y| \Longrightarrow \exists z \in X \setminus Y \text{ s.t. } \underline{Y \cup \{z\}} \in \mathcal{I}.$$



The intersection of two matroids $\mathcal{I}_1$, $\mathcal{I}_2$ on V is

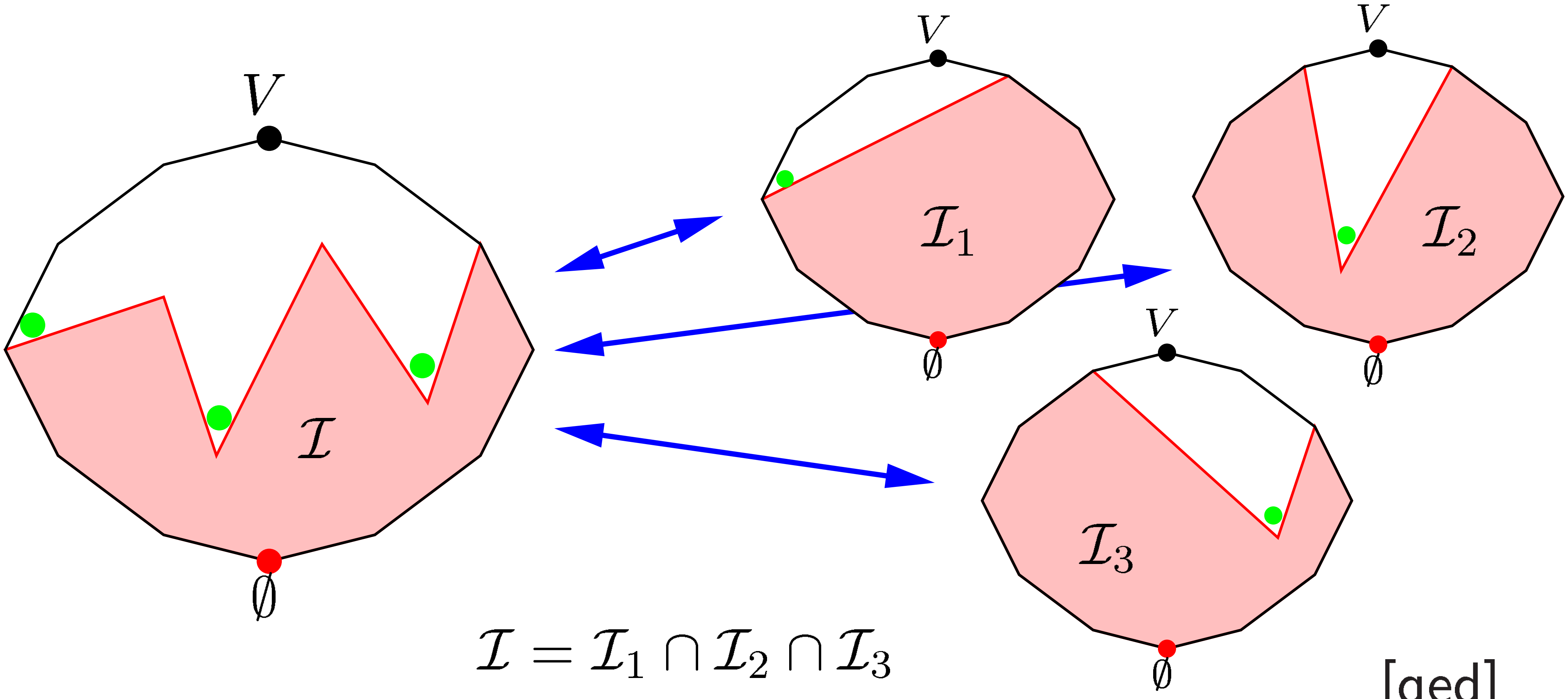
$$\mathcal{I}_1 \cap \mathcal{I}_2 = \{X \subseteq V : X \in \mathcal{I}_1 \text{ and } X \in \mathcal{I}_2\}.$$

The intersection of more matroids is defined analogously.

★ The intersection of matroids is not always a matroid but it is still an independence system. Moreover...

Fact: Every independence system can be represented as the intersection of finitely many matroids.

Proof. An indep. sys. with only one circuit is a matroid.



$$\mathcal{I} = \mathcal{I}_1 \cap \mathcal{I}_2 \cap \mathcal{I}_3$$

[qed]

Question:

How many matroids do we need to
represent a given independence system as their intersection??

The proof in the prev. slide says that:

this number $\leq$ the number of circuits of $\mathcal{I}$.

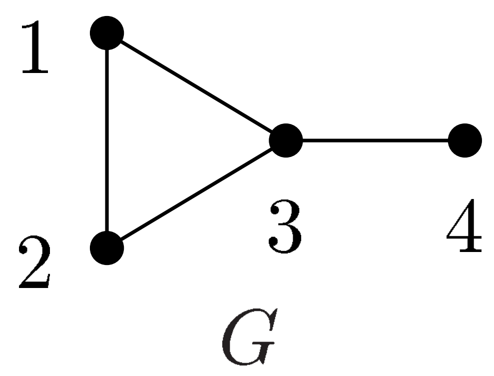
But, we can do better.

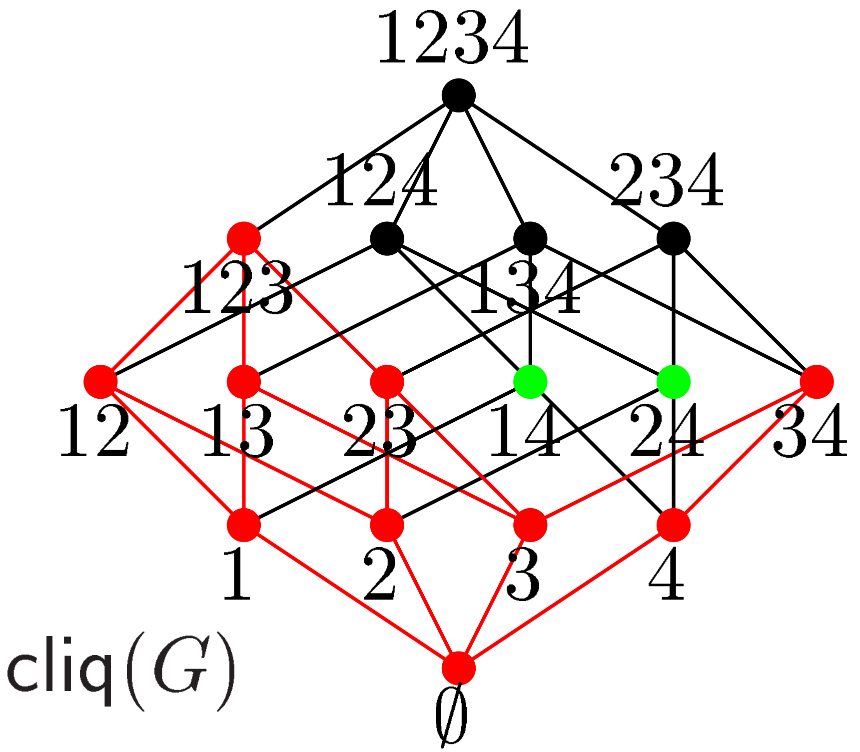
We investigate this number for clique complexes.

$G = (V, E)$ a graph

Def.: The clique complex of G is the family of all cliques of G.

Example: $V = \{1, 2, 3, 4\}$





Notation: $\mathrm{cliq}(G) =$ the clique complex of G.

Question (again)

Question:

How many matroids do we need to
represent a given clique complex as their intersection??

Result:

We investigate and characterize these numbers.

. . . .

However, you may ask...

(1) Why do we consider such representation??

(2) Why do we consider clique complexes??

Motivation (1)

$\boxed{\text{Q.}}$ Why do we consider such representation?

$\boxed{\text{Ans.}}$ in the following proposition.

$\boxed{\text{Prop.:}}$ (Jenkyns, '76; Korte–Hausmann, '78)

(Translation of their prop. to clique complexes)

cliq(G) is the intersection of k matroids

$\Longrightarrow$ Greedy Algorithm is a k-approximation alg. for $\textsc{MaxWeightedCliqueProblem}$ in G.

Q. Why do we consider clique complexes?

Ans. The class of clique complexes includes some important classes like the classes of

- the family of the matchings of a graph G
  (the matching complex of G)
  $=$ cliq(the complement of the line graph of G)
- the family of the chains of a partially ordered set P
  (the order complex of P)
  $=$ cliq(the comparability graph of P)
- ...

In the rest of my talk

(1) clique complex which is a matroid

- Key concept: partition matroid

(2) clique complex which is the intersection of k matroids

- Key concept: stable-set partition

(3) two other results

Partition matroid

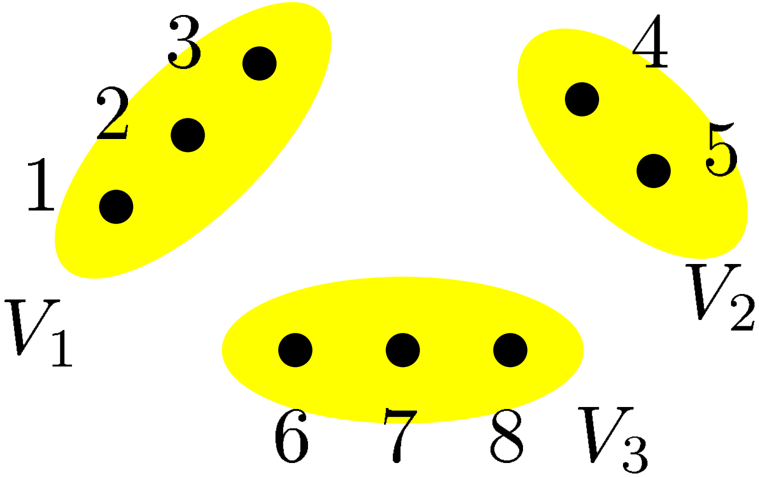
V a nonempty finite set

$\mathcal{P} = \{V_1, V_2, \ldots, V_r\}$ a partition of V

Def.: A partition matroid $\mathcal{I}(\mathcal{P})$ of $\mathcal{P}$ is defined as

$$\mathcal{I}(\mathcal{P}) = \{X \subseteq V : |X \cap V_i| \leq 1 \text{ for all } i = 1, \ldots, r\}.$$

Example: $\mathcal{P} = \{V_1, V_2, V_3\}$

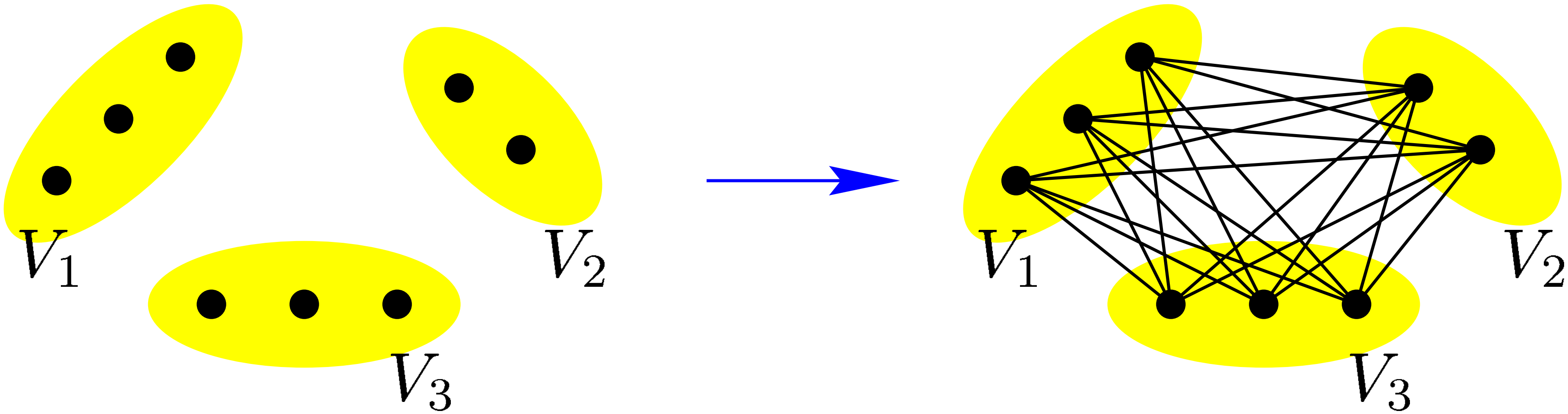


$\mathcal{I}(\mathcal{P}) =$
$\{146, 147, 148, 156, 157, 158,$
$246, 247, 248, 256, 257, 258,$
$346, 347, 348, 356, 357, 358,$
and their subsets$\}$

Notice: $\mathcal{I}(\mathcal{P})$ is really a matroid.

Lem.: Partition matroids are clique complexes.

Proof. Given a partition $\mathcal{P} = \{V_1, \ldots, V_r\}$ of V.



Construct a complete r-partite graph $G_{\mathcal{P}}$ s.t.
each part of $G_{\mathcal{P}}$ corresponds to V_i.

Then, $\mathrm{cliq}(G_{\mathcal{P}}) = \mathcal{I}(\mathcal{P})$. [qed]

Remark: G is complete r-partite
$\Longrightarrow$ $\mathrm{cliq}(G)$ is a (partition) matroid.

Thm. 1: G a graph

G is complete r-partite $\Longleftrightarrow$ $\mathrm{cliq}(G)$ is a matroid.

Proof.

[$\Longrightarrow$] ... is the previous lemma.

[$\Longleftarrow$] ... needs two more lemmas.

Lem. 1a:

every circuit of cliq(G) is a nonedge of G.

Lem. 1b: $\mathcal{I}$ a matroid on V s.t. $\{v\} \in \mathcal{I}$ for every $v \in V$

$x, y, z \in V$

$$\begin{array}{l} \{x, y\} \text{ is a circuit of } \mathcal{I} \\ \{y, z\} \text{ is a circuit of } \mathcal{I} \end{array} \implies \{x, z\} \text{ is also a circuit of } \mathcal{I}.$$

In particular: Two-element circuits give a partition of V.

Proofs of Lemmas. Omitted.

Want to show:

cliq(G) is a matroid $\Longrightarrow$ G is complete r-partite.

Proof (cont'd). Assume: cliq(G) is a matroid.

From Lem. 1a:

Every circuit of cliq(G) is a nonedge of G.

From Lem. 1b and our assumption:

These circuits give a partition $\mathcal{P} = \{V_1, \dots, V_r\}$ of V s.t. V_i is a stable set of G for all $i = 1, \dots, r$.

Therefore:

G is a complete r-partite graph with the partition $\mathcal{P}$.

[qed]

In the rest of my talk

(1) clique complex which is a matroid

- Key concept: partition matroid

(2) clique complex which is the intersection of k matroids

- Key concept: stable-set partition

(3) two other results

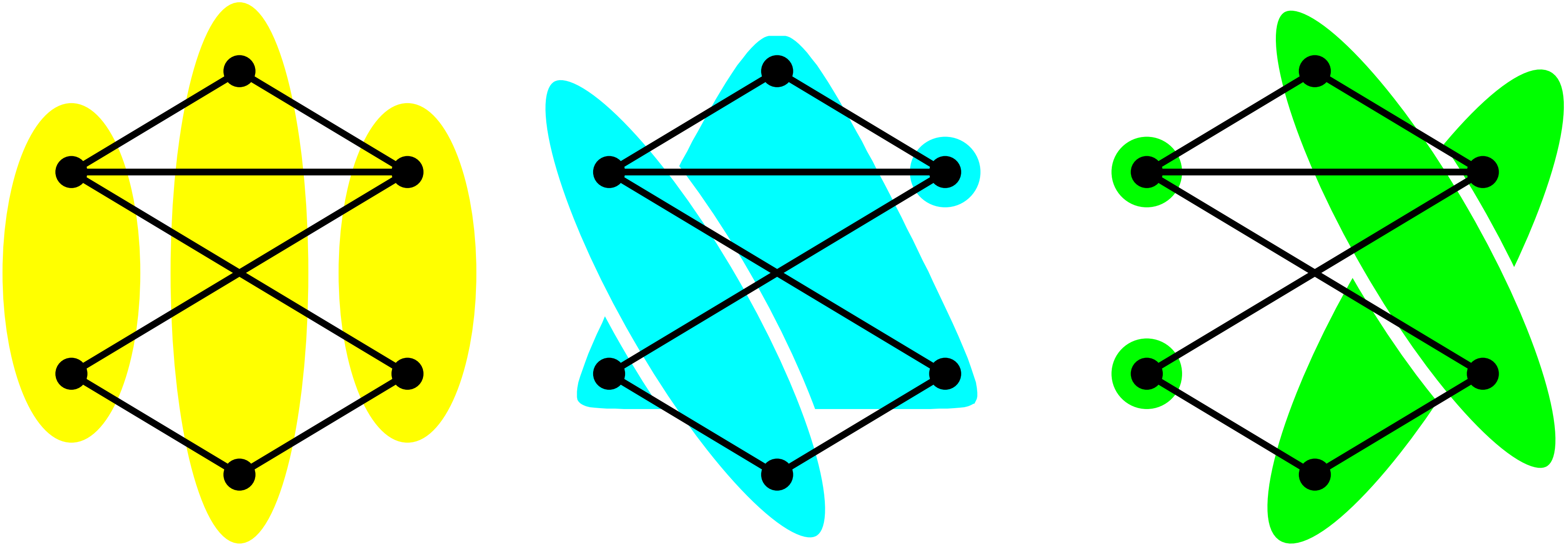
$G = (V, E)$ a graph

Def.: A stable-set partition of G

is a partition $\mathcal{P} = \{V_1, \ldots, V_r\}$ of V

s.t. V_i is a stable set of G for all $i = 1, \ldots, r$.

Example:



Construct a graph from a stable-set partition

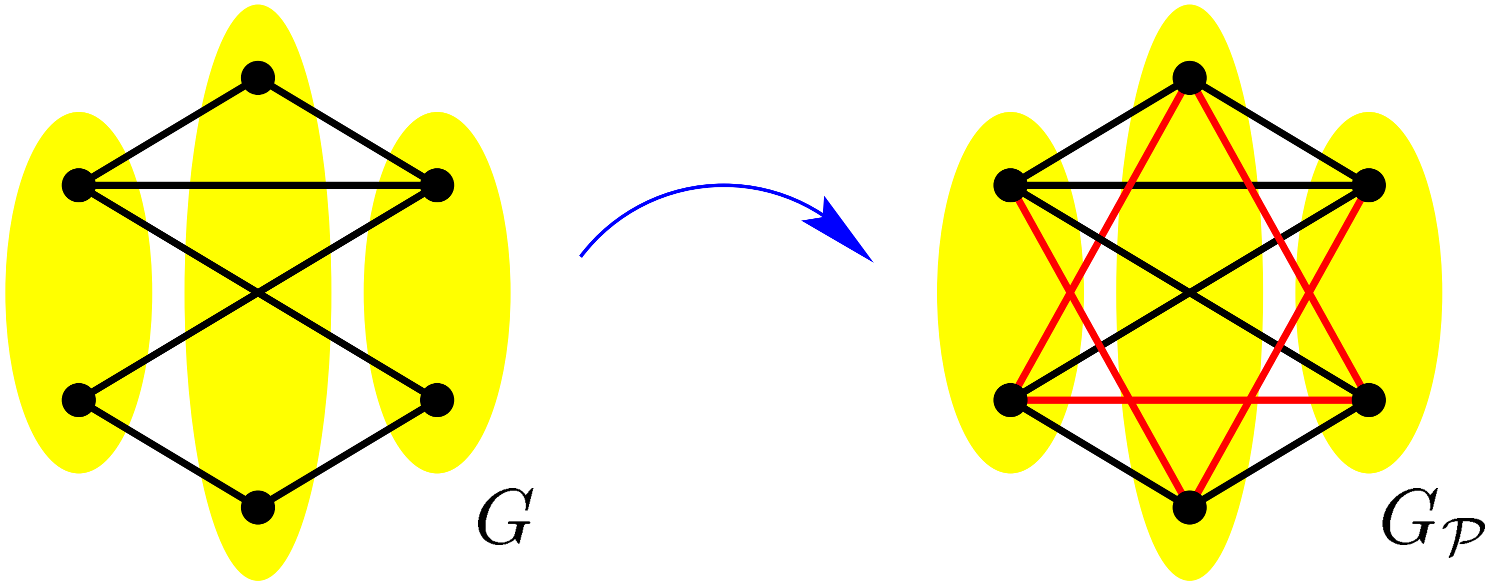
G a graph
$\mathcal{P}$ a stable-set partition of G

Construct a complete r-partite graph $G_{\mathcal{P}}$ from $\mathcal{P}$.

Observation: An edge of G is also an edge of $G_{\mathcal{P}}$.

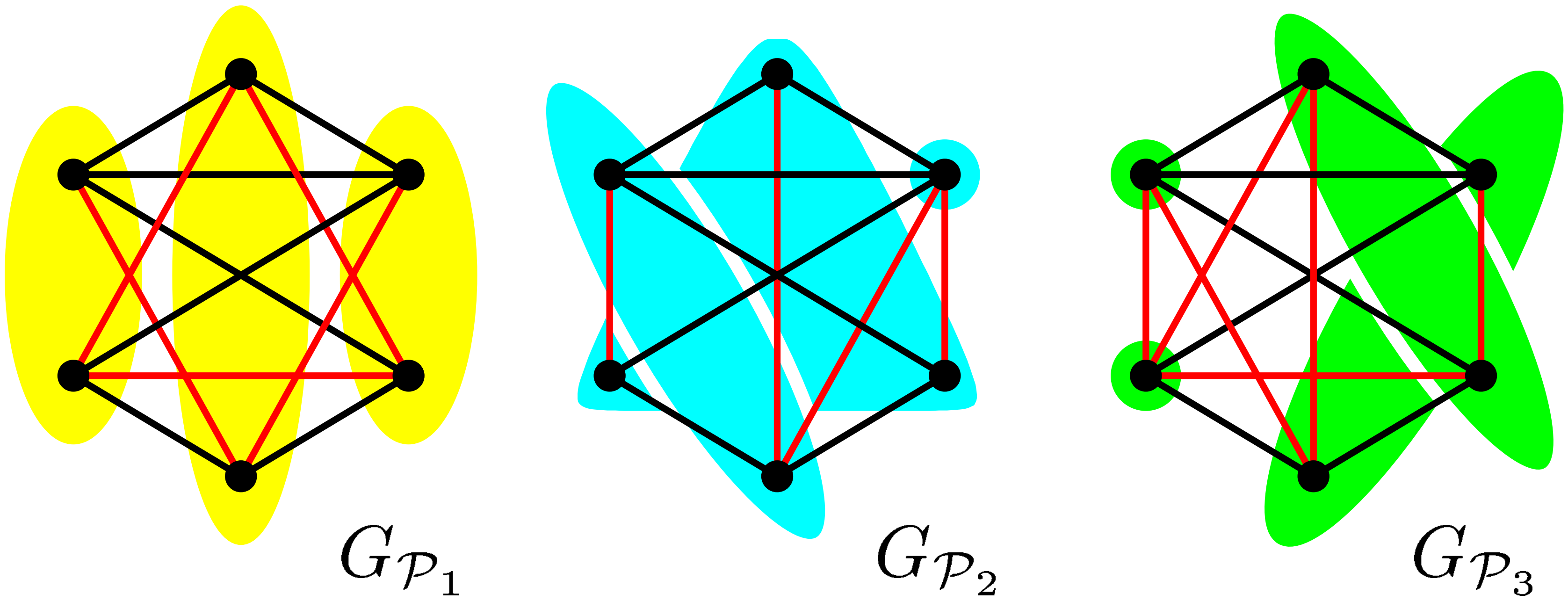
Therefore: A clique of G is also a clique of $G_{\mathcal{P}}$.

This means: $\text{cliq}(G) \subseteq \text{cliq}(G_{\mathcal{P}}) = \mathcal{I}(\mathcal{P})$.



Generally, $\mathrm{cliq}(G) \subseteq \mathrm{cliq}(G_{\mathcal{P}_1}) \cap \mathrm{cliq}(G_{\mathcal{P}_2}) \cap \mathrm{cliq}(G_{\mathcal{P}_3})$.

However, in this example,



$$\begin{aligned} \mathrm{cliq}(G) &= \mathrm{cliq}(G_{\mathcal{P}_1}) \cap \mathrm{cliq}(G_{\mathcal{P}_2}) \cap \mathrm{cliq}(G_{\mathcal{P}_3}) \\ &= \quad \mathcal{I}(\mathcal{P}_1) \quad \cap \quad \mathcal{I}(\mathcal{P}_2) \quad \cap \quad \mathcal{I}(\mathcal{P}_3). \end{aligned}$$

That is because

There is no red edge contained in all of these graphs.

$G = (V, E)$ a graph

$\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k$ stable-set partitions of G

$$\text{cliq}(G) = \bigcap\{\mathcal{I}(\mathcal{P}_i) : i = 1, \ldots, k\}$$

$$\Updownarrow$$

There is no red edge contained in all $G_{\mathcal{P}_i}$'s.

$$\left(\begin{array}{l} \text{For every nonedge } \{u, v\} \notin E \\ \exists\ S \in \bigcup\{\mathcal{P}_i : i = 1, \ldots, k\} \text{ s.t. } \{u, v\} \subseteq S. \end{array} \right)$$

Moreover...

Thm. 2: $G = (V, E)$ a graph

cliq(G) is represented as the intersection of k matroids

$$\Updownarrow$$

$\exists\ k$ stable-set partitions $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k$ of G s.t.
cliq$(G) = \bigcap\{\mathcal{I}(\mathcal{P}_i) : i = 1, \ldots, k\}$

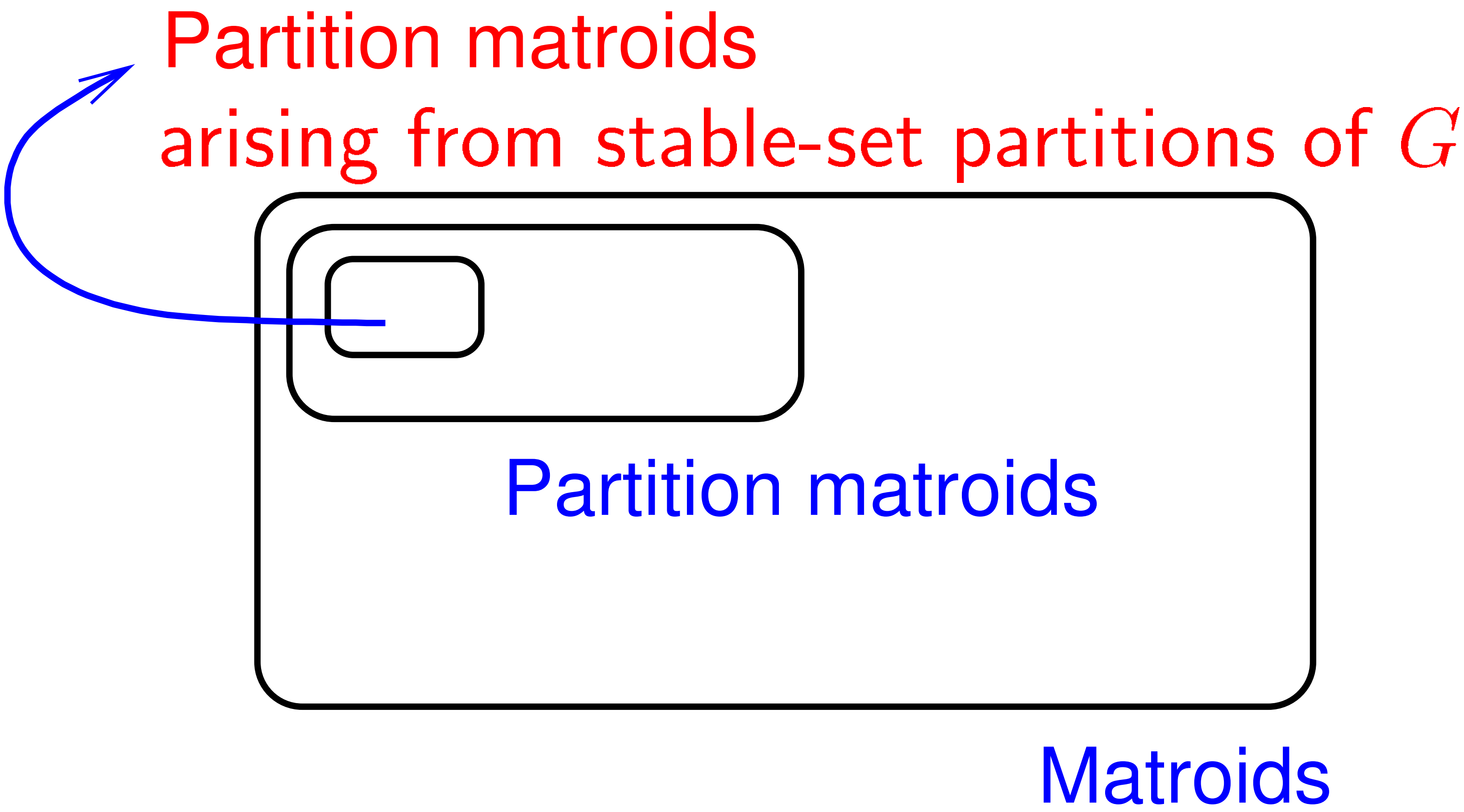
$$\Updownarrow$$

$\exists\ k$ stable-set partitions $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k$ of G s.t.
there is no red edge contained in all $G_{\mathcal{P}_i}$'s.

$$\left(\begin{array}{l} \text{For every nonedge } \{u, v\} \not\in E \\ \exists\ S \in \bigcup\{\mathcal{P}_i : i = 1, \ldots, k\} \text{ s.t. } \{u, v\} \subseteq S \end{array} \right)$$

Proof. Omitted.

Given a graph G.



When we want a representation of $\mathrm{cliq}(G)$ by k matroids,
just find k partition matroids from stable-set partitions of G.

What does this thm mean? (2)

Consider the following decision problem.

Instance: a graph G and a positive integer k

Question: ? $\exists$ k matroids $\mathcal{I}_1, \ldots, \mathcal{I}_k$
s.t. $\mathrm{cliq}(G) = \mathcal{I}_1 \cap \cdots \cap \mathcal{I}_k$?

Thm 2 implies: This problem $\in \mathcal{NP}$.

The talk plan

In the rest of my talk

(1) clique complex which is a matroid

- Key concept: partition matroid

(2) clique complex which is the intersection of k matroids

- Key concept: stable-set partition

(3) two other results

$\boxed{\text{Thm. 3:}}$ G a graph

cliq(G) is the intersection of two matroids

$$\Updownarrow$$

The stable-set graph of G is bipartite.

$\boxed{\text{Proof.}}$ Omitted.

$\boxed{\text{Note:}}$ Combining $\begin{cases} \text{Thm. 3 and} \\ \text{a result by Protti–Szwarcfiter ('02),} \end{cases}$

we can tell whether

the clique complex of a given graph
is the intersection of two matroids or not

in polynomial time.

A graph can be seen as an independence system.

Thm. 4: G a graph

$$G \text{ is the intersection of } k \text{ matroids}$$
$$\Updownarrow$$
$$\mathrm{cliq}(G) \text{ is the intersection of } k \text{ matroids.}$$

Proof. Omitted.

Summary

Question

- How many matroids do we need to represent a given clique complex as their intersection??

Result

- $\mathrm{cliq}(G)$ is the intersection of k matroids

$$\Updownarrow$$

$\mathrm{cliq}(G)$ is the intersection of k partition matroids arising from stable-set partitions.

[End of the talk]