Traveling salesman games with the Monge property

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Abstract

Several works indicate the relationship between well-solved combinatorial optimization problems and the core non-emptiness of cooperative games associated with them. In this note, we consider the core of traveling salesman games. We show that the core of traveling salesman games in which the distance matrix is a Monge matrix is non-empty. This is the first result for traveling salesman games related with a well-solved structure. Moreover, we show that the problem of testing the core non-emptiness of a given traveling salesman game is $\mathcal{NP}$-hard.

1 Introduction

Several works indicate the relationship between well-solved combinatorial optimization problems and the core non-emptiness of cooperative games associated with them. For examples, Deng–Ibaraki–Nagamochi [6] gives a necessary and sufficient condition for maximum packing games and minimum covering games to have the non-empty cores. It says that linear programming relaxation of a maximum packing problem (and a minimum covering problem, respectively) has an integral optimal solution if and only if the associated game has the non-empty core. A similar result is shown by Faigle–Kern [8] on partition games and by Goemans–Skutella [11] on facility location games. Moreover, Granot–Hamers–Tijs [12] investigates the core non-emptiness of delivery games in relation with structures of the underlying graphs. We refer two books [2, 5] for a lot of results and properties of cooperative games associated with combinatorial optimization problems.

In this note, we investigate traveling salesman games, introduced by Potters–Curiel–Tijs [17]. There are some works which discuss traveling salesman games. Tamir [18] shows that a symmetric traveling salesman game with at most four players always has a non-empty core, and that there exists a symmetric traveling salesman game with six players such that the core is empty. Kuipers [14] shows that a symmetric traveling salesman game with five players always has a non-empty core. Also, Potters–Curiel–Tijs [17] give an example of an asymmetric traveling game with four players which has the empty core, and show some conditions for an asymmetric traveling salesman game to have a non-empty core. In [5], we can see other conditions for a traveling salesman game to have a non-empty core.

Now, we show another sufficient condition for symmetric traveling salesman games to have a non-empty core. That is, a traveling salesman game in which the distance matrix is a Monge matrix has always a non-empty core. It is well-known that an (asymmetric) traveling salesman problem in which the distance matrix is a Monge matrix can be solved in linear time of the number of cities [16] while for a general distance matrix, traveling salesman problems are $\mathcal{NP}$-hard, even if it satisfies the triangle inequality. So this is the first result for traveling salesman games related with a well-solved structure. We know that Monge matrices are related with other well-solved cases for combinatorial optimization, which is surveyed by Burkard–Klinz–Rudolf [4]. Other well-solved cases for traveling salesman problems are surveyed in [3, 10], and some recent results appear in [1, 7, 13, 15], etc.

Moreover, some papers indicate the relationship between intractable combinatorial optimization problems and the intractability of testing the core non-emptiness of cooperative games associated with them. For example, Deng–Ibaraki–Nagamochi [6] shows that testing the core non-emptiness of a minimum coloring game is $\mathcal{NP}$-complete; Goemans–Skutella [11] shows that testing the core non-emptiness of a facility location game is $\mathcal{NP}$-complete. Notice that Deng–Ibaraki–Nagamochi [6] shows that for a minimum vertex cover game and a maximum independent set game, we can test the core non-emptiness in polynomial time, while a minimum vertex cover problem and a maximum independent set problem are known to be intractable. In this note, we show that the problem of testing the core non-emptiness of a given traveling salesman game is $\mathcal{NP}$-hard.

This note is organized as follows. The next section is devoted to some definitions and basic properties. In Section 3, we show that a traveling salesman game in which the distance matrix is a Monge matrix has always a non-empty core. In Section 4, we show that the problem of testing the core non-emptiness of a given traveling salesman game is $\mathcal{NP}$-hard. We mention the concluding remarks in the final section.

2 Traveling salesman games and Monge matrices

Let $N_0 = \{0, 1, \ldots, n\}$ and an $(n+1) \times (n+1)$-matrix D be given. We treat N_0 as the cities and D as the distance matrix. We assume that the diagonal components of D are all zero and the non-diagonal components of D are all positive. A *traveling salesman problem* (TSP, for short) is the problem to find a shortest tour; more formally

$$\text{minimize} \sum_{i=0}^{n} d[i, \tau(i)], \tag{1}$$

where $\tau(i)$ denotes the successor of i and $d[i,j]$ is the (i,j)-component of D. We denote the k-th successor and k-th predecessor of i by the $\tau^k(i)$ and $\tau^{-k}(i)$ respectively. A tour is denoted by $\tau = \langle 0, \tau(0), \tau^2(0), \ldots, \tau^n(0) \rangle$.

TSP is known as an $\mathcal{NP}$-hard problem, even if D satisfies the triangle inequality: $d[i,j] + d[i,k] \geq d[j,k]$ for all $i, j, k \in N_0$. Besides, some works say that if D satisfies a special condition then a shortest tour can be found in polynomial time of the number of cities. Among such conditions, the Monge property may be the most famous. A matrix D is a *Monge* matrix if D satisfies

$$d[l, m] + d[l', m'] \leq d[l, m'] + d[l', m] \tag{2}$$

for all $l < l'$ and $m < m'$. If a matrix D is a Monge matrix, then it is also said to have the *Monge property*. Note that a Monge matrix does not need to satisfy the triangle inequality.

It is well-known that Monge matrices relate with some well-solved combinatorial optimization problems. The next proposition is the basic result on Monge matrices and TSP. A tour $\langle 0, i_1, i_2, \ldots, i_r, n, j_1, j_2, \ldots, j_{n-r-1} \rangle$ is called a *pyramidal* tour if $i_1 < i_2 < \cdots < i_r$ and $j_1 > j_2 > \cdots > j_{n-r-1}$ (see Figure 1).

Proposition 2.1 ([10]). *If D is a Monge matrix, then there exists a shortest tour which is pyramidal.*

Generally, a shortest pyramidal tour can be found in $\mathrm{O}(n^2)$ by the dynamic programming technique [10]. Moreover, Park [16] shows that a shortest pyramidal tour for TSP with a Monge distance matrix can be found in $\mathrm{O}(n)$ using the structure of Monge matrices.

Let $N = N_0 \setminus \{0\} = \{1, 2, \ldots, n\}$. For $S \subseteq N$, we define $c(S)$ as the distance of a shortest tour of $S \cup \{0\}$. We call the pair (N, c) a *traveling salesman game*. In terminology of cooperative

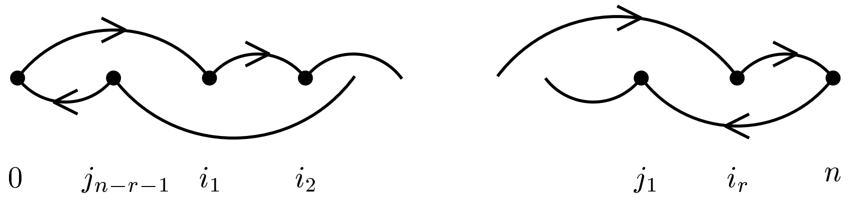

Figure 1: A pyramidal tour.

game theory, N is called the *players* and c is the *characteristic function*. If we consider traveling salesman games, we assume that the distance matrix D is symmetric, i.e., $d[i,j] = d[j,i]$ for all $i.j \in N_0$. Traveling salesman games are arisen from the following cost allocation problem, originated by Fishburn–Pollak [9]. A professor has visited starting from his home institute 0, the universities $1, 2, \ldots, n$ which invited him and has returned to his home. The total cost of the trip should be paid by the inviting universities. The problem is to find a "fair" rule for the allocation of the total cost among these universities.

In cooperative game theory, the core is frequently used for finding a fair allocation rule. For a traveling salesman game (N, c), the *core* is defined as

$$\mathrm{Core}(N, c) = \left\{ x \in \mathbb{R}^N : x(N) = c(N) \text{ and } x(S) \leq c(S) \text{ for all } S \subseteq N \right\}, \tag{3}$$

where $x(S)$ means $\sum_{i \in S} x_i$.

Here, the next theorem is the main result of this note.

Theorem 2.2. *Consider a traveling salesman problem on the cities N_0 with a symmetric distance matrix D. If D is a Monge matrix, then the core of the traveling salesman game (N, c) is non-empty.*

For a traveling salesman game (N, c) and $T \subseteq N$, we define the *subgame* (T, c_T) as $c_T(S) = c(S)$ for all $S \subseteq T$. Since every submatrix of a Monge matrix is also a Monge matrix, every subgame of a traveling salesman game also has the non-empty core by Theorem 2.2. In cooperative game theory, a game with the property that every subgame has the non-empty core is called *totally balanced*. Hence, Theorem 2.2 immediately leads to the following corollary.

Corollary 2.3. *Consider a traveling salesman problem on the cities N_0 with a symmetric distance matrix D. If D is a Monge matrix, then the traveling salesman game (N, c) is totally balanced.*

On the other hand, for a general traveling salesman game, it may be hard to test that the core is non-empty, as is indicated by the next result.

Theorem 2.4. *Consider a traveling salesman problem on the cities N_0 with a symmetric distance matrix D. The problem of testing the core non-emptiness of the traveling salesman game (N, c) is $\mathcal{NP}$-hard.*

3 Proof of Theorem 2.2

We need a lemma for proving Theorem 2.2.

Lemma 3.1. *Consider a traveling salesman problem on the cities N_0 with a symmetric distance matrix D. Let (N, c) be a traveling salesman game. If D is a Monge matrix, then for $S \subseteq N$ with $n \in S$,*

$$c(S) - c(S \setminus \{n\}) \geq c(N) - c(N \setminus \{n\}).$$

Proof. Let $S = \{p_1, p_2, \ldots, p_{s-1}, n\}$, where $s = |S|$. We denote the distance of a tour τ by $\Delta(\tau)$. Moreover, we denote a shortest pyramidal tour of $S \cup \{0\}$ and $(S \setminus \{n\}) \cup \{0\}$ by

$$\begin{aligned}
\pi^{(S)} &= \langle 0, i_1^{(S)}, i_2^{(S)}, \ldots, i_r^{(S)}, n, j_1^{(S)}, j_2^{(S)}, \ldots, j_{s-r-1}^{(S)} \rangle, \text{ and} \\
\pi^{(S \setminus \{n\})} &= \langle 0, i_1^{(S-1)}, i_2^{(S-1)}, \ldots, i_{r'}^{(S-1)}, p_{s-1}, j_1^{(S-1)}, j_2^{(S-1)}, \ldots, j_{s-r'-1}^{(S-1)} \rangle,
\end{aligned}$$

respectively. By Proposition 2.1, we have $c(S) = \Delta(\pi^{(S)})$ and $c(S \setminus \{n\}) = \Delta(\pi^{(S \setminus \{n\})})$. Since the tour $\pi^{(S)}$ is pyramidal, $p_{s-1} = i_r^{(S)}$ or $j_1^{(S)}$. By the symmetry of D, we can assume that $p_{s-1} = i_r^{(S)}$ without loss of generality. (If $p_{s-1} = j_1^{(S)}$, then we only reverse the order of $\pi^{(S)}$.) Then we define a tour $\sigma^{(S \setminus \{n\})}$ as

$$\sigma^{(S \setminus \{n\})} = \langle 0, i_1^{(S)}, i_2^{(S)}, \ldots, i_{r-1}^{(S)}, p_{s-1}, j_1^{(S)}, j_2^{(S)}, \ldots, j_{s-r-1}^{(S)} \rangle.$$

We have $\Delta(\pi^{(S \setminus \{n\})}) \leq \Delta(\sigma^{(S \setminus \{n\})})$, since $\pi^{(S \setminus \{n\})}$ is a shortest tour of $(S \setminus \{n\}) \cup \{0\}$. So it holds that

$$c(S) - c(S \setminus \{n\}) = \Delta(\pi^{(S)}) - \Delta(\pi^{(S \setminus \{n\})}) \geq \Delta(\pi^{(S)}) - \Delta(\sigma^{(S \setminus \{n\})}). \tag{4}$$

We now consider extending the tours $\pi^{(S)}$ of $S \cup \{0\}$ and $\sigma^{(S \setminus \{n\})}$ of $(S \setminus \{n\} \cup \{0\})$ to tours $\tau^{(N)}$ of $N \cup \{0\}$ and $\tau^{(N \setminus \{n\})}$ of $(N \setminus \{n\}) \cup \{0\}$, respectively, by inserting the cities not in S step by step in the following way.

Step 1. Set $\tau^{(N)} = \pi^{(S)}$, $\tau^{(N \setminus \{n\})} = \sigma^{(S \setminus \{n\})}$ and $T = N \setminus S$.

Step 2. Choose the leftmost city $i \in T$, i.e., $i \in T$ such that $i \leq j$ for any $j \in T$. Since $\tau^{(N)}$ and $\tau^{(N \setminus \{n\})}$ are pyramidal, there exists a unique city j such that $j < i < \tau^{(N)}(j)$. Set $j' = \tau^{(N)}(j)$.

Step 2.1. If $j' \neq n$, then reset as $\tau^{(N)}(j) = (\tau^{(N)})^{-1}(j') = \tau^{(N \setminus \{n\})}(j) = (\tau^{(N \setminus \{n\})})^{-1}(j') = i$. Remark that this insertion keeps the value $\Delta(\tau^{(N)}) - \Delta(\tau^{(N \setminus \{n\})})$ equal (see Figure 2(i)).

Step 2.2. If $j' = n$, then first set $n' = \tau^{(N)}(n)$. By the construction, we have $\tau^{(N \setminus \{n\})}(j) = n'$. Now reset as $\tau^{(N)}(j) = (\tau^{(N)})^{-1}(n) = \tau^{(N \setminus \{n\})}(j) = (\tau^{(N \setminus \{n\})})^{-1}(n') = i$. Remark that

$$d[j, n] - d[j, n'] \geq d[i, n] - d[i, n'], \tag{5}$$

by the Monge property (see Figure 2(ii)). Therefore, this insertion never makes the value $\Delta(\tau^{(N)}) - \Delta(\tau^{(N \setminus \{n\})})$ increase.

Step 3. We remove i from T. If $T = \emptyset$, then we proceed to the next step. Otherwise, we return to Step 2. Remark that $\tau^{(N)}$ and $\tau^{(N \setminus \{n\})}$ are kept pyramidal.

Step 4. Now $(\tau^{(N)})^{-1}(n) = n - 1$ and $\tau^{(N)}(n) = \tau^{(N \setminus \{n\})}(n - 1)$.

Step 4.1. If $\tau^{(N)} = n - 2$, then this construction is terminated.

Step 4.2. Otherwise, i.e., $\tau^{(N)} < n - 2$, then set $j = \tau^{(N)}(n)$ and $k = (\tau^{(N)})^{-1}(n - 2)$. Note that $j = \tau^{(N \setminus \{n\})}(n - 1)$ and $k = (\tau^{(N \setminus \{n\})})^{-1}(n - 2)$. Now reset as $\tau^{(N)}(n) = \tau^{(N \setminus \{n\})}(n-1) = n-2$, $\tau^{(N)}(n-2) = \tau^{(N \setminus \{n\})}(n-2) = j$ and $\tau^{(N)}(k) = \tau^{(N \setminus \{n\})}(k) = n - 1$, which terminates this construction. Notice that this step never makes the value $\Delta(\tau^{(N)}) - \Delta(\tau^{(N \setminus \{n\})})$ increase, because it holds that

$$d[n, j] - d[n - 1, j] \geq d[n, n - 2] - d[n - 1, n - 2]$$

by the Monge property (see Figure 2(iii)).

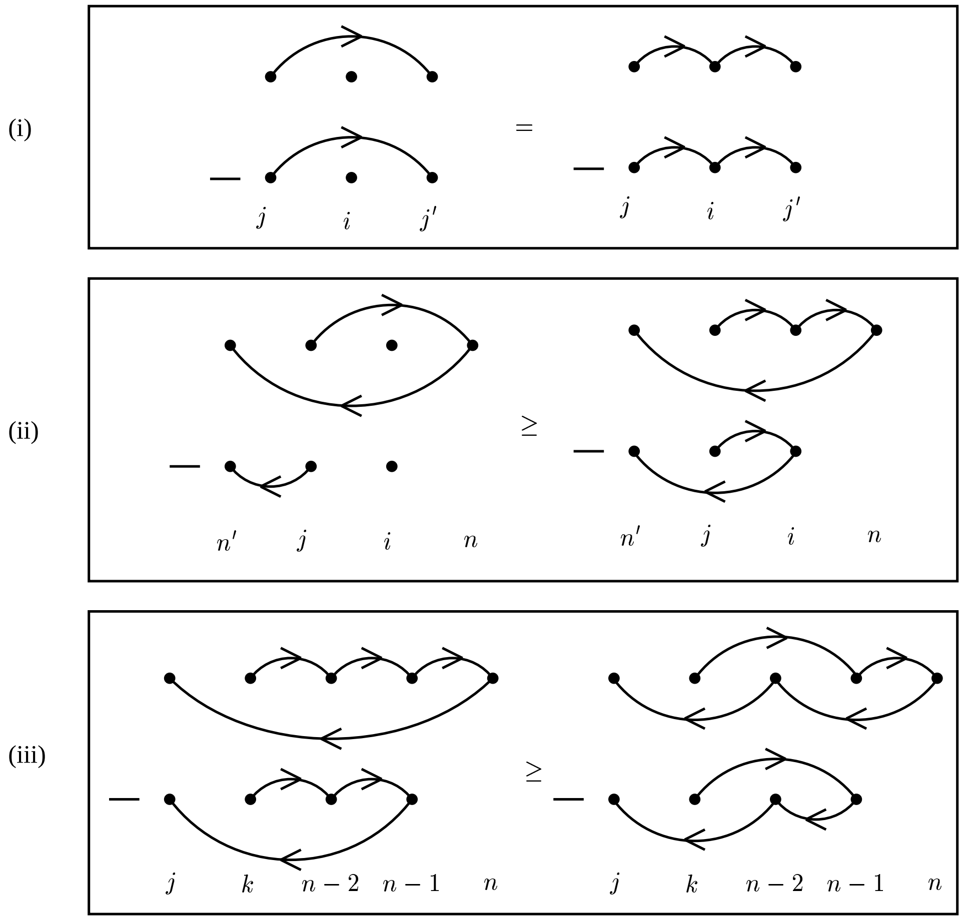

Figure 2: How to construct $\tau^{(N)}$ and $\tau^{(N\setminus\{n\})}$. Boxes (i), (ii) and (iii) correspond to Step 2.1, Step 2.2 and Step 4.2, respectively.

Thus we finish constructing $\tau^{(N)}$ and $\tau^{(N\setminus\{n\})}$. The construction above yields that

$$\Delta(\pi^{(S)}) - \Delta(\sigma^{S\setminus\{n\}}) \geq \Delta(\tau^{(N)}) - \Delta(\tau^{(N\setminus\{n\})}). \tag{6}$$

Remark that by the above construction we have $\tau^{(N)}(n) = \tau^{(N\setminus\{n\})}(n-1) = n-2$, $(\tau^{(N)})^{-1}(n-1) = (\tau^{(N\setminus\{n\})})^{-1}(n-1)$ and $(\tau^{(N)})^{-1}(n) = n-1$. Moreover,

$$\Delta(\tau^{(N)}) - \Delta(\tau^{(N\setminus\{n\})}) = d[n-1,n] + d[n,n-2] - d[n-1,n-2].$$

Let $\pi^{(N\setminus\{n\})}$ be a shortest pyramidal tour of $(N \setminus \{n\}) \cup \{0\}$. Since $\pi^{(N\setminus\{n\})}$ is pyramidal, we have $n-2 = \pi^{(N\setminus\{n\})}(n-1)$ or $n-2 = (\pi^{(N\setminus\{n\})})^{-1}(n-1)$. By the symmetry of D, we can assume that $n-2 = \pi^{(N\setminus\{n\})}(n-1)$ without loss of generality. (If not so, then we only reverse the order of $\pi^{(N\setminus\{n\})}$.)

Now we construct a tour $\sigma^{(N)}$ of $N \cup \{0\}$ using $\pi^{(N\setminus\{n\})}$: first set $\sigma^{(N)} = \pi^{(N\setminus\{n\})}$ and consider inserting n into $\sigma^{(N)}$ as $\sigma^{(N)}(n-1) = n$ and $\sigma^{(N)}(n) = n-2$. Here,

$$\begin{aligned}
\Delta(\tau^{(N)}) - \Delta(\tau^{(N\setminus\{n\})}) &= d[n-1,n] + d[n,n-2] - d[n-1,n-2] \\
&= (\Delta(\pi^{(N\setminus\{n\})}) + d[n-1,n] + d[n,n-2]) \\
&\quad -(\Delta(\pi^{(N\setminus\{n\})}) + d[n-1,n-2]) \\
&= (\Delta(\sigma^{(N)}) + d[n-1,n-2]) \\
&\quad -(\Delta(\pi^{(N\setminus\{n\})}) + d[n-1,n-2]) \\
&= \Delta(\sigma^{(N)}) - \Delta(\pi^{(N\setminus\{n\})}).
\end{aligned} \tag{7}$$

If we denote a shortest pyramidal tour of $N \cup \{0\}$ by $\pi^{(N)}$, then $\Delta(\sigma^{(N)}) \geq \Delta(\pi^{(N)})$. Together with inequalities (4), (6) and (7), it holds that

$$c(S) - c(S \setminus \{n\}) \geq \Delta(\pi^{(N)}) - \Delta(\pi^{(N\setminus\{n\})}) = c(N) - c(N \setminus \{n\}),$$

where the last equality is led from Proposition 2.1. Thus we complete the proof. $\square$

Now we prove Theorem 2.2.

Proof of Theorem 2.2. We prove it by induction in terms of the number of the cities. Let $N_h = \{1, 2, \ldots, h\}$ for $1 \leq h \leq n$. Suppose that the theorem holds for N_{h-1}. So we can pick up $\bar{x} \in \mathrm{Core}(N_{h-1}, c) \subseteq \mathbb{R}^{h-1}$. Remark that $\bar{x}$ satisfies that $\bar{x}(N_{h-1}) = c(N_{h-1})$ and $\bar{x}(S) \leq c(S)$ for all $S \subseteq N_{h-1}$.

Define $\bar{x}' \in \mathbb{R}^h$ as

$$\bar{x}'_i = \begin{cases} \bar{x}_i & (i \in N_{h-1}), \\ c(N_h) - c(N_{h-1}) & (i = h). \end{cases} \tag{8}$$

So we have

$$\begin{aligned} \bar{x}'(N_h) &= \bar{x}'(N_{h-1}) + \bar{x}'_h \\ &= \bar{x}(N_{h-1}) + c(N_h) - c(N_{h-1}) \\ &= c(N_{h-1}) + c(N_h) - c(N_{h-1}) \\ &= c(N_h). \end{aligned}$$

Now we show $\bar{x}'(S) \leq c(S)$ for all $S \subseteq N_h$. For any $S \subseteq N_h$ with $h \notin S$, we have $\bar{x}'(S) = \bar{x}(S) \leq c(S)$. Moreover for any $S \subseteq N_h$ with $h \in S$,

$$\begin{aligned} \bar{x}'(S) &= \bar{x}'(S \setminus \{h\}) + \bar{x}'_h \\ &= \bar{x}(S \setminus \{h\}) + c(N_h) - c(N_{h-1}) \\ &\leq c(S \setminus \{h\}) + c(N_h) - c(N_{h-1}) \\ &\leq c(S), \end{aligned}$$

where the last inequality uses Lemma 3.1. This completes the proof. $\square$

4 Proof of Theorem 2.4

We state the problem more formally:

Core Non-emptiness of Traveling Salesman Games	
Instance:	cities N_0 and an $(n+1) \times (n+1)$ symmetric distance matrix D,
Question:	is the core of the traveling salesman game (N, c) non-empty?

To prove $\mathcal{NP}$-hardness of Core Non-emptiness of Traveling Salesman Game, we use Hamiltonian Path, a famous $\mathcal{NP}$-complete problem:

Hamiltonian Path	
Instance:	a graph $G = (N, E)$,
Question:	does G have a Hamiltonian path, i.e., a simple spanning path?

Proof of Theorem 2.4. We reduce Hamiltonian Path to Core Non-emptiness of Traveling Salesman Games. For a given graph $G = (N, E)$, let $N = \{1, \ldots, n\}$ and $N_0 = N \cup \{0\}$. We define an $(n+1) \times (n+1)$ symmetric distance matrix D as

$$d[i,j] = \begin{cases} \dfrac{n}{1+n} & (\{i,j\} \in E,\ i = 0,\ \text{or } j = 0), \\ 3 & (\text{otherwise}). \end{cases}$$

Let (N, c) be a traveling salesman game derived from the cities N_0 and the distance matrix D. We now show that G has a Hamiltonian path if and only if the game (N, c) has a non-empty core, which completes the reduction. Suppose that G has a Hamiltonian path. Then $c(N) = n$ and $c(S) = \dfrac{n(|S|+1)}{n+1}$ for all $S \subseteq N$. Let $x = (x_1, \ldots, x_n)$ be $x_i = 1$ for all $i \in N$. We can check that x belongs to the core, since $x(N) = n = c(N)$ and $x(S) = |S| \leq c(S)$ for all $S \subseteq N$.

Suppose that G does not have a Hamiltonian path. Let $\tau = \langle 0, \tau(0), \tau^2(0), \ldots, \tau^n(0)\rangle$ be a shortest tour of N_0. We define a subgraph $\bar{G} = (N, \bar{E})$ of G as $\bar{E} = E \cap \{\{i, \tau(i)\} : i \in V\}$. Let P be the set of vertices which consist of a longest path in $\bar{G}$. Since G has no Hamiltonian path, $\bar{G}$ also has no Hamiltonian path. This implies $P \neq N$. Therefore, at least one of the endpoints of P is incident to the edge whose cost is 3 in the tour τ. Hence,

$$c(N \setminus P) \leq c(N) - c(P) + \frac{3n}{1+n} - 3 < c(N) - c(P). \tag{9}$$

Here, we assume that the core is non-empty, i.e., there exists $x \in \mathbb{R}^N$ such that $x(N) = c(N)$ and $x(S) \leq c(S)$ for all $S \subseteq N$. Now we have

$$x(N) = x(P) + x(N \setminus P) \leq c(P) + c(N \setminus P) < c(N),$$

using the inequality (9). This contradicts to the fact that $x(N) = c(N)$. So, the core is empty. Thus, we complete the reduction. □

5 Concluding remarks

We mention some open problems on the core non-emptiness of traveling salesman games.

1. We have shown that a traveling salesman game is totally balanced if the distance matrix is a Monge matrix. However, we do not know whether the Monge property is also necessary. If it is not necessary, what is the necessary and sufficient condition for the traveling salesman game to be totally balanced?

   Note that the Monge property is not sufficient for the core non-emptiness of traveling salesman games. For example, let $N_0 = \{0, 1, 2, 3, 4\}$ and

$$D = \begin{bmatrix} 0 & 3 & 1 & 3 & 3 \\ 3 & 0 & 3 & 4 & 6 \\ 1 & 3 & 0 & 2 & 3 \\ 3 & 4 & 2 & 0 & 4 \\ 3 & 6 & 3 & 4 & 0 \end{bmatrix}.$$

   Then, the associated traveling salesman game has a non-empty core. This example has appeared in [17].

2. A game (N, c) is *submodular* if $c(S) + c(T) \geq c(S \cup T) + c(S \cap T)$ for all $S, T \subseteq N$. It is known that submodularity implies the totally balancedness of the game. So the Monge property may imply the submodularity of the traveling salesman game.

3. Proposition 2.1 says that there exists a shortest tour which is pyramidal for TSP with the Monge property. In addition, there are other properties of distance matrices which derive the same result. These appear in [1, 3, 7, 10, 15]. So do these other properties guarantee the totally balancedness of the associated traveling salesman games?

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