Core Stability of Games on Convex Geometries

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Abstract

The concepts of core and stability consist of a part of the main topics of cooperative game theory. We consider a game on a convex geometry, which is introduced by Bilbao. We show that if a game is quasi-convex, then the core is stable. This result can be seen as an extension of a result by Shapley for traditional cooperative games.

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1 Introduction

The concepts of core and stability consist of a part of the main topics of cooperative game theory [7]. We consider a game on a convex geometry, introduced by Bilbao [1]. We show that if a game on a convex geometry is quasi-convex, then the core is a unique stable set of the game. This result can be seen as an extension of a result by Shapley [8] for traditional cooperative games.

Recall the definition of a convex geometry. Let $N = \{1, 2, \ldots, n\}$ be a finite set. A *convex geometry* $\mathcal{L}$ on N is a family of subsets of N with the properties

$$\emptyset \in \mathcal{L}, N \in \mathcal{L}, \tag{1}$$

$$S, T \in \mathcal{L} \text{ imply } S \cap T \in \mathcal{L}, \tag{2}$$

$$S \in \mathcal{L} \setminus \{N\} \text{ implies } S \cup \{i\} \in \mathcal{L} \text{ for some } i \in N \setminus S. \tag{3}$$

For a convex geometry $\mathcal{L}$ on N, we call an element i of $S \in \mathcal{L}$ an *extreme point* of S if $S \setminus \{i\} \in \mathcal{L}$. The set of the extreme points of S is denoted by $\mathrm{ex}(S)$, called the *extreme set* of S. See [3, 5] for a further theory of convex geometries.

A *cooperative game* on a convex geometry $\mathcal{L}$ on N is a function $v : \mathcal{L} \to \mathbb{R}$ with $v(\emptyset) = 0$. The *players* are the elements of N and the *coalitions* are the members of $\mathcal{L}$.

A cooperative game on a convex geometry is introduced by Bilbao [1] as a model of cooperation structures in which only certain coalitions are allowed to form. This includes

some situations introduced former. Indeed, if $\mathcal{L} = 2^N$, then it is a game in which all coalitions are allowed, which we call here a *traditional* game. If $\mathcal{L}$ is defined via a partially ordered set $P = (N, \leq)$ as

$$\mathcal{L} = \{S \subseteq N : S \text{ is an order ideal of } P\}, \tag{4}$$

then $\mathcal{L}$ forms a convex geometry, which is a model introduced by Faigle–Kern [4]. Besides, let G be a connected graph with the vertex set N. Myerson [6] introduced a communication game. Define $\mathcal{L}$ as

$$\mathcal{L} = \{S \subseteq N : G[S] \text{ is connected } \}, \tag{5}$$

where $G[S]$ means the subgraph induced by S. If G is a block-graph, that is, every 2-connected component forms a complete graph, then $\mathcal{L}$ is a convex geometry.

2 The cores and the stable sets

In this section, we introduce some solution concepts. Let $\mathcal{L}$ be a convex geometry on the players N, and $v : \mathcal{L} \to \mathbb{R}$ be a game on $\mathcal{L}$. The set of *pre-imputations* of the game v on $\mathcal{L}$ is

$$I^*(\mathcal{L}, v) = \left\{ x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N) \right\}, \tag{6}$$

and the set of *imputations* is

$$I(\mathcal{L}, v) = \{x \in I^*(\mathcal{L}, v) : x_i \geq v(\{i\}) \text{ for all } \{i\} \in \mathcal{L}\}. \tag{7}$$

We say that a pre-imputation is a *Pareto-optimal* or *efficient* pay-off vector, and that an imputation is an *individually rational* pre-imputation. Note that the set of imputations is possibly empty and not necessarily bounded. Indeed, we have the following result. A convex geometry is *atomic* if $\{i\} \in \mathcal{L}$ for all $i \in N$.

Proposition 2.1 ([1]). *Let $\mathcal{L}$ be a convex geometry on N, and v be a game on $\mathcal{L}$.*

1. *If $\mathcal{L}$ is not atomic, then $I(\mathcal{L}, v) \neq \emptyset$.*
2. *If $\mathcal{L}$ is atomic, then*

$$I(\mathcal{L}, v) \neq \emptyset \text{ if and only if } v(N) \geq \sum_{i \in N} v(\{i\}). \tag{8}$$

3. *$I(\mathcal{L}, v)$ is bounded if and only if $\mathcal{L}$ is atomic.*

The *core* of the game v on $\mathcal{L}$ is defined as

$$\mathrm{Core}(\mathcal{L}, v) = \left\{ x \in I^*(\mathcal{L}, v) : \sum_{i \in S} x_i \geq v(S) \text{ for all } S \in \mathcal{L} \right\}. \tag{9}$$

We say that an imputation in the core is *coalitionally rational*. For the core, the next proposition is known.

Proposition 2.2 ([2]). *Let $\mathcal{L}$ be a convex geometry on N, and v be a game on $\mathcal{L}$.*

1. *The core* $\mathrm{Core}(\mathcal{L}, v)$ *is either empty or a pointed polyhedron.*

2. *Moreover, if* $\mathcal{L}$ *is atomic, then* $\mathrm{Core}(\mathcal{L}, v)$ *is bounded.*

Next, we introduce the stable set. Let $x, y \in I(\mathcal{L}, v)$ for a game v on a convex geometry $\mathcal{L}$, and $S \in \mathcal{L}$. We say that y *dominates* x *via* S if

$$\sum_{i \in S} y_i \leq v(S) \quad \text{and} \tag{10}$$

$$y_i > x_i \quad \text{for all } i \in S, \tag{11}$$

and denote it by $x \prec_S y$. A *stable set*, or the *von Neumann–Morgenstern solution* [9], of a game v on a convex geometry $\mathcal{L}$ is a set $K \subseteq I(\mathcal{L}, v)$ of imputations satisfying the following two conditions:

1. for all $y \in I(\mathcal{L}, v) \setminus K$, there exist $x \in K$ and $S \in \mathcal{L}$ such that $y \prec_S x$,

2. for all $x, y \in K$ and $S \in \mathcal{L}$, $x \not\prec_S y$ and $y \not\prec_S x$.

The first condition is called the *external stability*, while the second one is called the *internal stability*.

Remark 2.3. 1. *A stable set is not necessarily unique and possibly empty.*

2. *If two collections of imputations are distinct stable sets, then they do not include each other.*

3. *If the core is nonempty and so is any stable set of the game, then the core is always included by every stable set.*

A well-known fact on stable sets is that, if $\mathcal{L} = 2^N$ and a game v is convex, then the core is a unique stable set [8], where a game $v : 2^N \to \mathbb{R}$ is *convex* or *supermodular* if we have

$$v(S) + v(T) \leq v(S \cap T) + v(S \cup T) \tag{12}$$

for all $S, T \in 2^N$. In the next section, we show that if a game on a convex geometry is quasi-convex, then the stable set is unique and coincides with the core.

3 Quasi-convex game and core stability

A game $v : \mathcal{L} \to \mathbb{R}$ is *quasi-convex* or *quasi-supermodular* if for all $S, T \in \mathcal{L}$ with $S \cup T \in \mathcal{L}$, we have

$$v(S) + v(T) \leq v(S \cap T) + v(S \cup T). \tag{13}$$

The quasi-convexity may be first defined in [2]. Obviously, if $\mathcal{L} = 2^N$, then a quasi-convex game is a convex game.

Lemma 3.1 ([2, Proposition 4]). *A game* v *on a convex geometry* $\mathcal{L}$ *is quasi-convex if and only if for all* $S, T \in \mathcal{L}$ *such that* $T \subseteq S$ *and for all* $i \in \mathrm{ex}(S) \cap T$*, we have*

$$v(S) - v(S \setminus \{i\}) \geq v(T) - v(T \setminus \{i\}).$$

Let $\mathcal{L}$ be a convex geometry, v be a game on $\mathcal{L}$ and

$$\mathcal{C} : \emptyset = S_0 \subsetneq S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_{n-1} \subsetneq S_n = N \tag{14}$$

be a maximal chain of $\mathcal{L}$. Then, the *marginal worth vector* $a^{\mathcal{C}} \in \mathbb{R}^N$ with respect to the maximal chain $\mathcal{C}$ in the game v is defined by

$$a_i^{\mathcal{C}} = v(S_{\mathcal{C}}(i)) - v(S_{\mathcal{C}}(i) \setminus \{i\}) \quad \text{for all } i \in N, \tag{15}$$

where $S_{\mathcal{C}}(i) = \bigcap\{S \in \mathcal{C} : i \in S\}$, that is, $S_{\mathcal{C}}(i)$ is the minimal set of $\mathcal{C}$ containing i. Note that $i \in \mathrm{ex}(S_{\mathcal{C}}(i))$ from the minimality of $S_{\mathcal{C}}(i)$.

Lemma 3.2 ([2, Theorem 5]). *A game v on a convex geometry $\mathcal{L}$ is quasi-convex if and only if $a^{\mathcal{C}}$ is in* $\mathrm{Core}(\mathcal{L}, v)$ *for every maximal chain $\mathcal{C}$ of $\mathcal{L}$.*

Now, we show the main result.

Theorem 3.3. *Let $\mathcal{L}$ be a convex geometry on the players N, and $v : \mathcal{L} \to \mathbb{R}$ be a quasi-convex game on $\mathcal{L}$. Then a stable set of the game $(\mathcal{L}, v)$ coincides with the core.*

Proof. First, we show that the core satisfies the internal stability. Suppose that $x, y \in \mathrm{Core}(\mathcal{L}, v)$ satisfy $x \prec_S y$ for some $S \in \mathcal{L}$. Namely, $\sum\{y_i : i \in S\} \leq v(S)$ and $y_i > x_i$ for all $i \in S$. Then we have $\sum\{x_i : i \in S\} < v(S)$, which contradicts the fact that x is in the core.

Next, we consider the external stability. Take an arbitrary $y \in I(\mathcal{L}, v) \setminus \mathrm{Core}(\mathcal{L}, v)$. Define $g : \mathcal{L} \setminus \{\emptyset\} \to \mathbb{R}$ as

$$g(S) = \frac{v(S) - \sum_{i \in S} y_i}{|S|}. \tag{16}$$

Let S^* be a maximizer of g. Since $y \notin \mathrm{Core}(\mathcal{L}, v)$ and S^* is a maximizer of g, we have $g(S^*) > 0$. We pick up a maximal chain $\mathcal{C}$ of $\mathcal{L}$ such that $S^* \in \mathcal{C}$. Now, define x as

$$x_i = \begin{cases} y_i + g(S^*) & (i \in S^*), \\ a_i^{\mathcal{C}} & (i \in N \setminus S^*). \end{cases} \tag{17}$$

We have $x_i > y_i$ for all $i \in S^*$. Moreover,

$$\sum_{i \in S^*} x_i = \sum_{i \in S^*} y_i + |S^*| g(S^*) = v(S^*), \tag{18}$$

$$\sum_{i \in N} x_i = v(S^*) + \sum_{i \in N \setminus S^*} a_i^{\mathcal{C}} = \sum_{i \in S^*} a_i^{\mathcal{C}} + \sum_{i \in N \setminus S^*} a_i^{\mathcal{C}} = \sum_{i \in N} a_i^{\mathcal{C}} = v(N). \tag{19}$$

Therefore, we have $y \prec_{S^*} x$. It remains to show that $x \in \mathrm{Core}(\mathcal{L}, v)$.

Take an arbitrary $T \in \mathcal{L}$. Then,

$$\begin{aligned} \sum_{i \in T} x_i &= \sum_{i \in S^* \cap T} x_i + \sum_{i \in T \setminus S^*} x_i = \sum_{i \in S^* \cap T} y_i + |S^* \cap T| g(S^*) + \sum_{i \in T \setminus S^*} a_i^{\mathcal{C}} \\ &\geq \sum_{i \in S^* \cap T} y_i + |S^* \cap T| g(S^* \cap T) + \sum_{i \in T \setminus S^*} a_i^{\mathcal{C}} = v(S^* \cap T) + \sum_{i \in T \setminus S^*} a_i^{\mathcal{C}}. \end{aligned} \tag{20}$$

Let us recall that $S_{\mathcal{C}}(i)$ is the minimal set of $\mathcal{C}$ containing i. From (3), there must be a numbering $T \setminus S^* = \{i_1, i_2, \ldots, i_m\}$ such that $S_{\mathcal{C}}(i_1) \subsetneq S_{\mathcal{C}}(i_2) \subsetneq \cdots \subsetneq S_{\mathcal{C}}(i_m)$. Notice that

$S^* \subsetneq S_{\mathcal{C}}(i_1)$. Set $S_j = S_{\mathcal{C}}(i_j) \cap T \in \mathcal{L}$ for all $j \in \{1, \ldots, m\}$. Then, we have $S_j = \{i_1, \ldots, i_j\}$ and $i_j \in \mathrm{ex}(S_j)$. By Lemma 3.1,

$$a^{\mathcal{C}}_{i_j} = v(S_{\mathcal{C}}(i_j)) - v(S_{\mathcal{C}}(i_j) \setminus \{i_j\}) \geq v(S_j) - v(S_{j-1}), \tag{21}$$

and we have

$$\sum_{i \in T \setminus S^*} a^{\mathcal{C}}_i \geq \sum_{j=1}^{m} (v(S_j) - v(S_{j-1})) = v(T) - v(S^* \cap T). \tag{22}$$

Hence,

$$v(S^* \cap T) + \sum_{i \in T \setminus S^*} a^{\mathcal{C}}_i \geq v(S^* \cap T) + v(T) - v(S^* \cap T) \geq v(T). \tag{23}$$

This concludes that $x \in \mathrm{Core}(\mathcal{L}, v)$. □

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