Core stability of games on convex geometries

Yoshio Okamoto

The University of Tokyo

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Overview (1/2)

Traditional theory of cooperative games
(since von Neumann–Morgenstern, '44)

$\Downarrow\Downarrow$

Games with partial cooperation

- Games on graphs (Myerson, '77)
- .....
- Games on convex geometries (Bilbao, '98)

Overview (2/2)

The core of games on convex geometries

- Relationship between the extreme points and the greedy algorithm:

traditionally	Shapley ('71)
on convex geometries	Bilbao et al. ('97)

- Stability of the core:

traditionally	Shapley ('71)
on convex geometries	Okamoto ('01)

Organization of the talk

1. Introduction of games on convex geometries, including some examples
2. Description of the result
3. Explanation of the result
   - The core and stable sets
   - Convex games and quasi-convex games
4. Some remarks

Traditional cooperative games

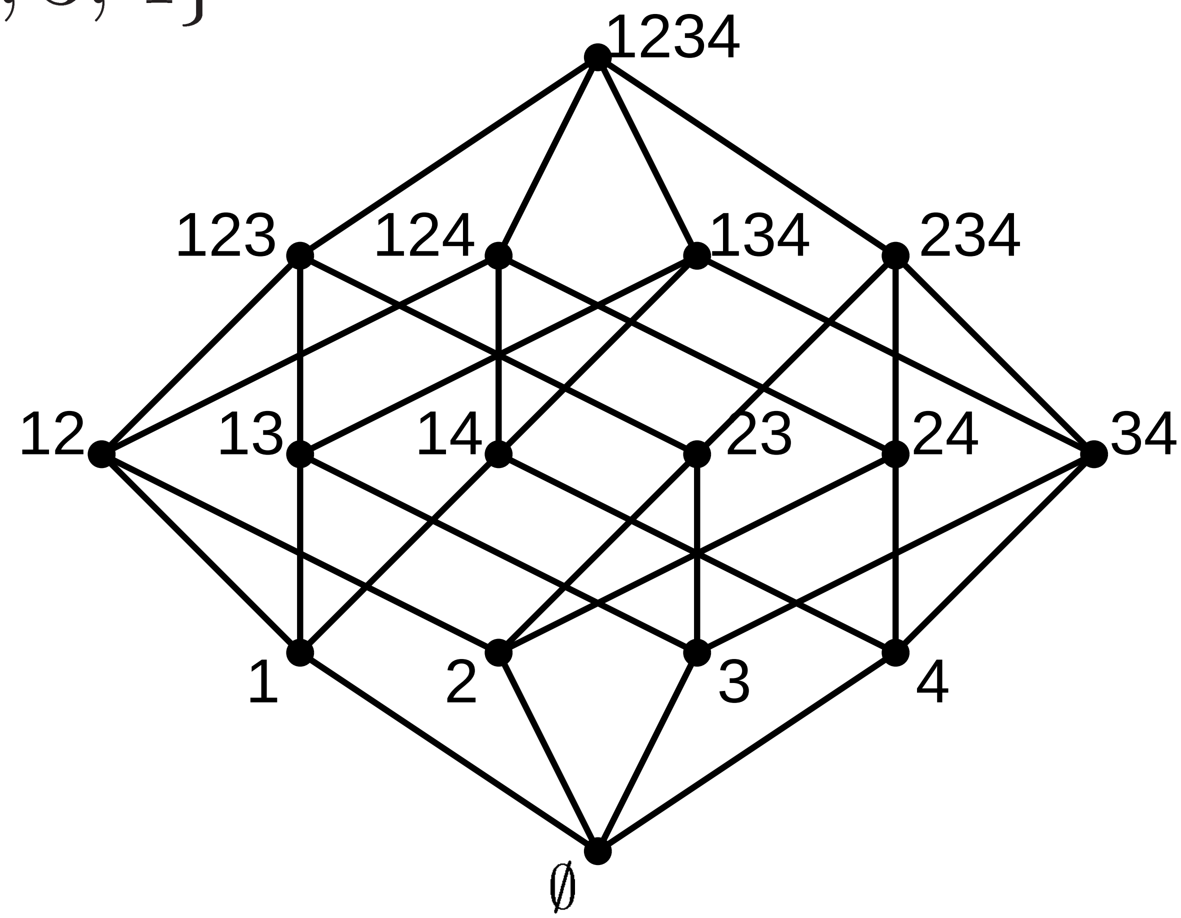
$N = \{1, \ldots, n\}$	players
$S \subseteq N$	coalition
$v : 2^N \to \mathbb{R}$	characteristic function
(representing the sum of the profit assigned to each)	
(N, v)	cooperative game (game)

Discuss:

how to allocate the total profit to each players

Coalitions of a traditional game

$N = \{1, 2, 3, 4\}$



This forms the Boolean lattice.

Games with partial cooperation

For traditional games

How realistic are all subsets of the players allowed to form coalitions?

- Political situations
- Cultural situations
- Geographical situations
- $\dots$

$\Longrightarrow$ Interest to games with partial cooperation

Games on convex geometries

(Bilbao, '98)

- The set of all players is allowed to form a coalition;
- the intersection of two coalitions is also a coalition;
- we extend a coalition to the set of all players by adding players repeatedly;

the characteristic function is defined only on the allowed coalitions.

Def. Convex geometries

(Edelman, '70; Jamison, '80)

$N = \{1, \ldots, n\}$ the players; $\mathcal{L} \subseteq 2^N$

Def. $\mathcal{L}$ is a convex geometry on N if

1. $\emptyset \in \mathcal{L}$, $N \in \mathcal{L}$,
2. $S, T \in \mathcal{L}$ imply $S \cap T \in \mathcal{L}$,
3. $S \in \mathcal{L} \setminus \{N\}$ implies $S \cup \{i\} \in \mathcal{L}$ for some $i \in N \setminus S$.

Rem. A convex geometry forms a lattice w.r.t. "$\subseteq$".

Def. Games on convex geometries

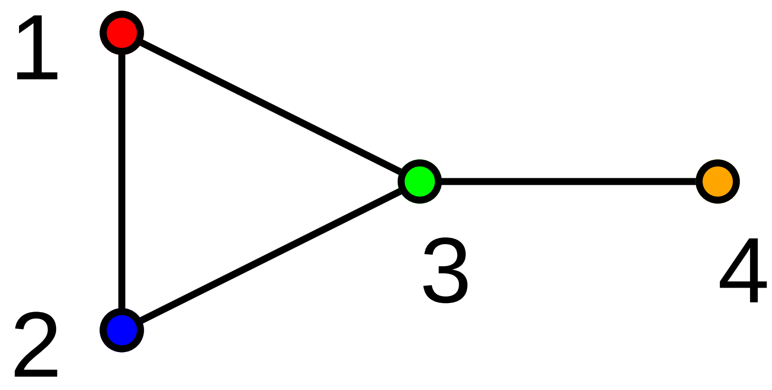
$N = \{1, \ldots, n\}$	players
$\mathcal{L} \subseteq 2^N$	convex geometry
$S \in \mathcal{L}$	coalition
$v : \mathcal{L} \to \mathbb{R}$	characteristic function
$(N, \mathcal{L}, v)$	game on $\mathcal{L}$

This includes several models of partial cooperation in the literature.

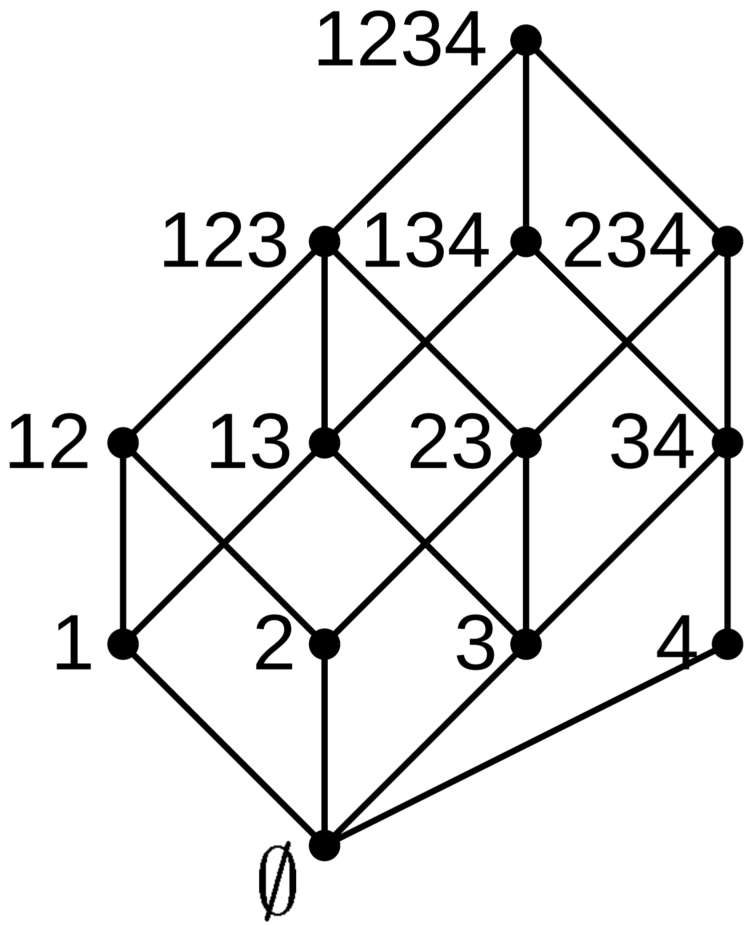
Ex.1. Communication graph

(Myerson, '77)

Two players have an edge if they can communicate.



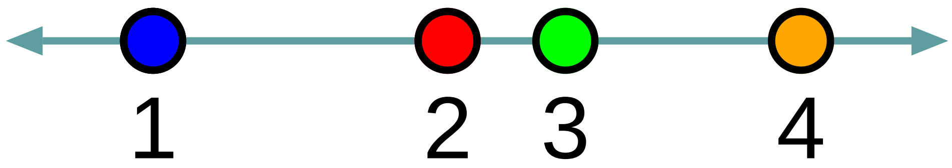
It is a convex geometry
if the graph satisfies
a certain condition.

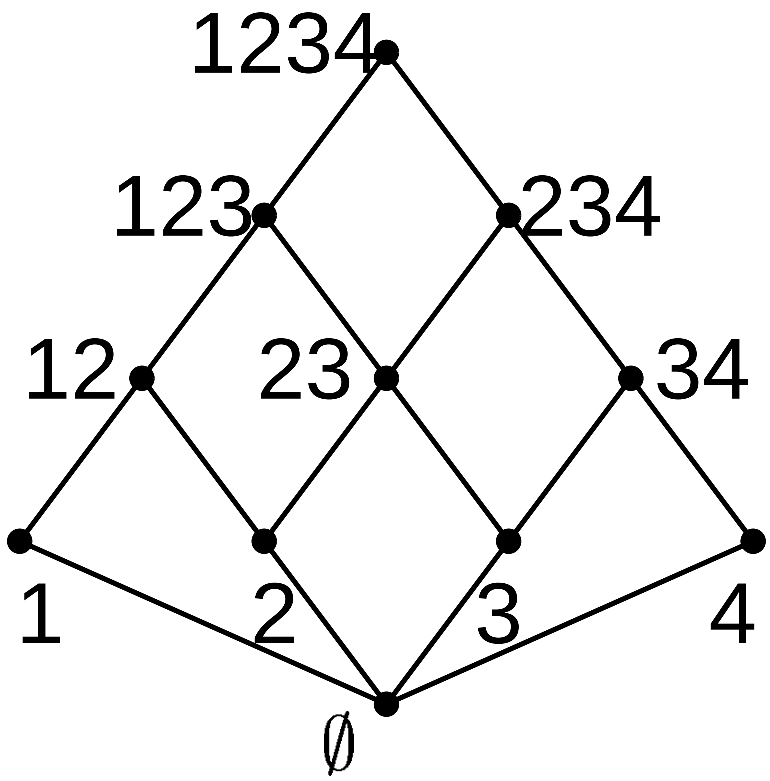


Ex.2. Policy order (Edelman, '97)

The players on the line according to their policies.

$S \in \mathcal{L} \iff$ There exists an interval exactly containing the players in S



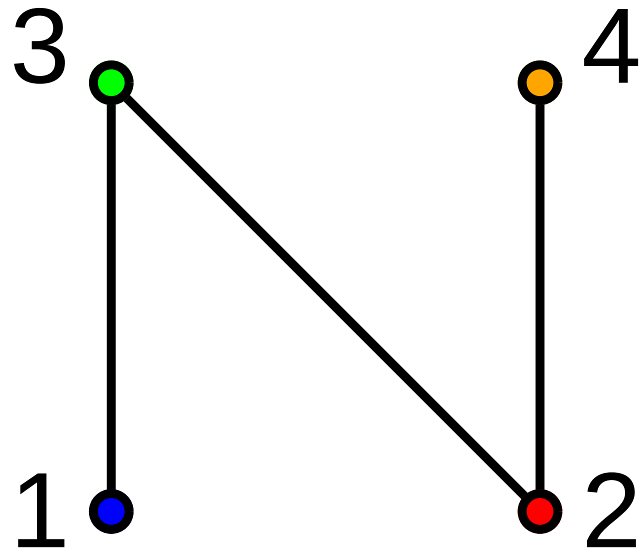


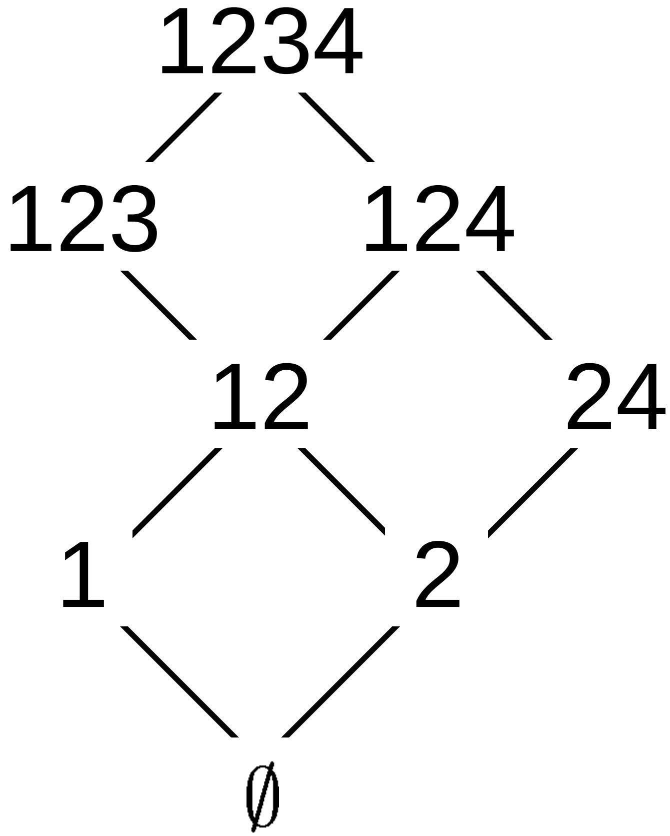
$\mathcal{L}$ is a convex geometry.

Ex.3. Hierarchy (Faigle–Kern, '92)

$P = (N, \preceq)$ a partially ordered set

$i \preceq j$ a coalition with j must have i.

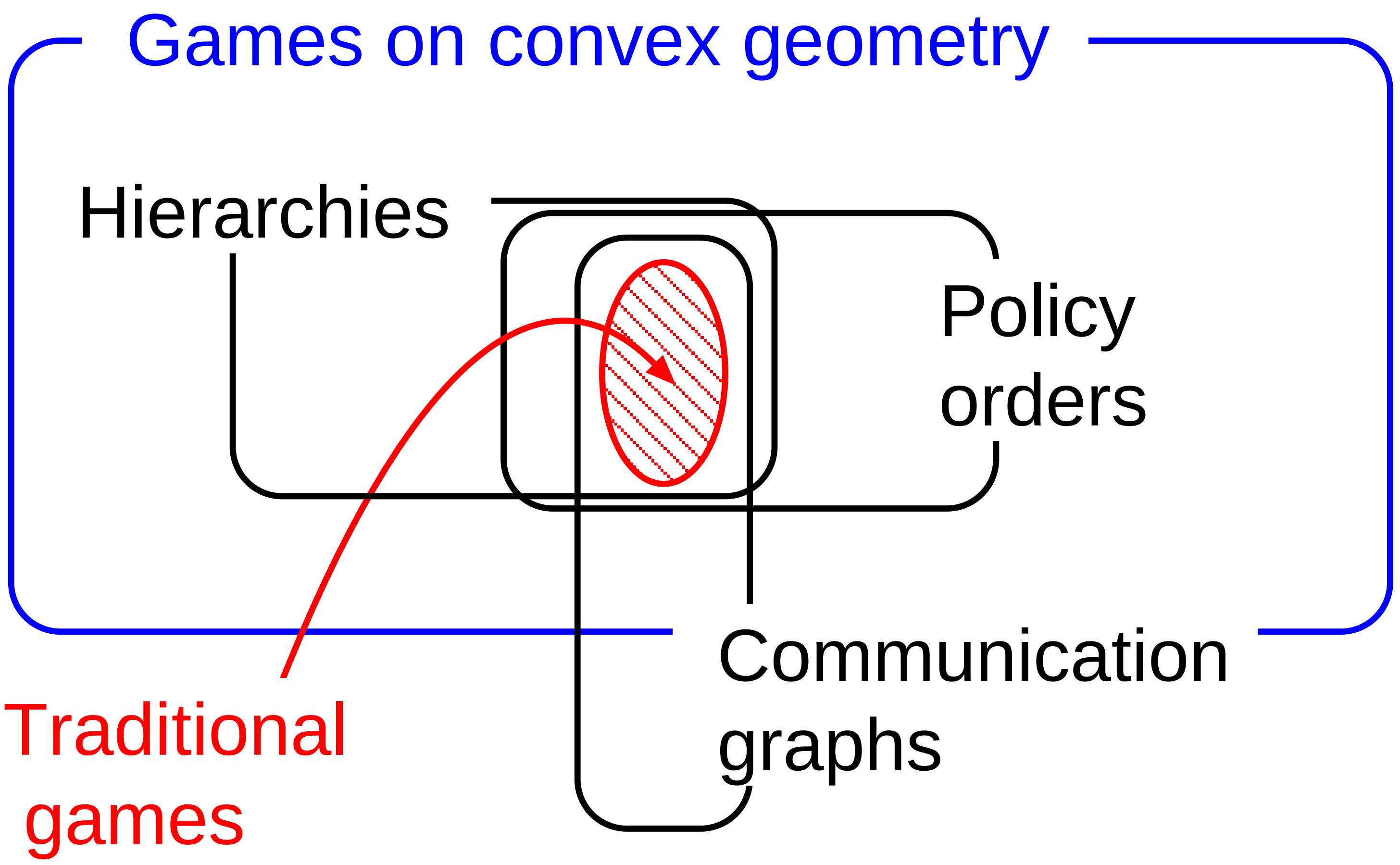




It is a convex geometry.

Motivation

How can we extend some results for traditional games to games on convex geometries?



The result

For the traditional situation (Shapley, '71)

The core of a convex game is
a unique stable set.

$\Downarrow\Downarrow\Downarrow$

. . . . $\boxed{\text{extension}}$ . . . .

$\Downarrow\Downarrow\Downarrow$

On convex geometries (Okamoto, '01)

The core of a quasi-convex game is
a unique stable set.

Explanation of the result

- Imputations, Domination
- The core, Stable sets
- Convex games
- Quasi-convex games

Imputations

$(N, \mathcal{L}, v)$ a game on a convex geometry $\mathcal{L}$

$x \in \mathbb{R}^N$ a payoff vector

Def. x is efficient if $\sum_{i=1}^{n} x_i = v(N)$.

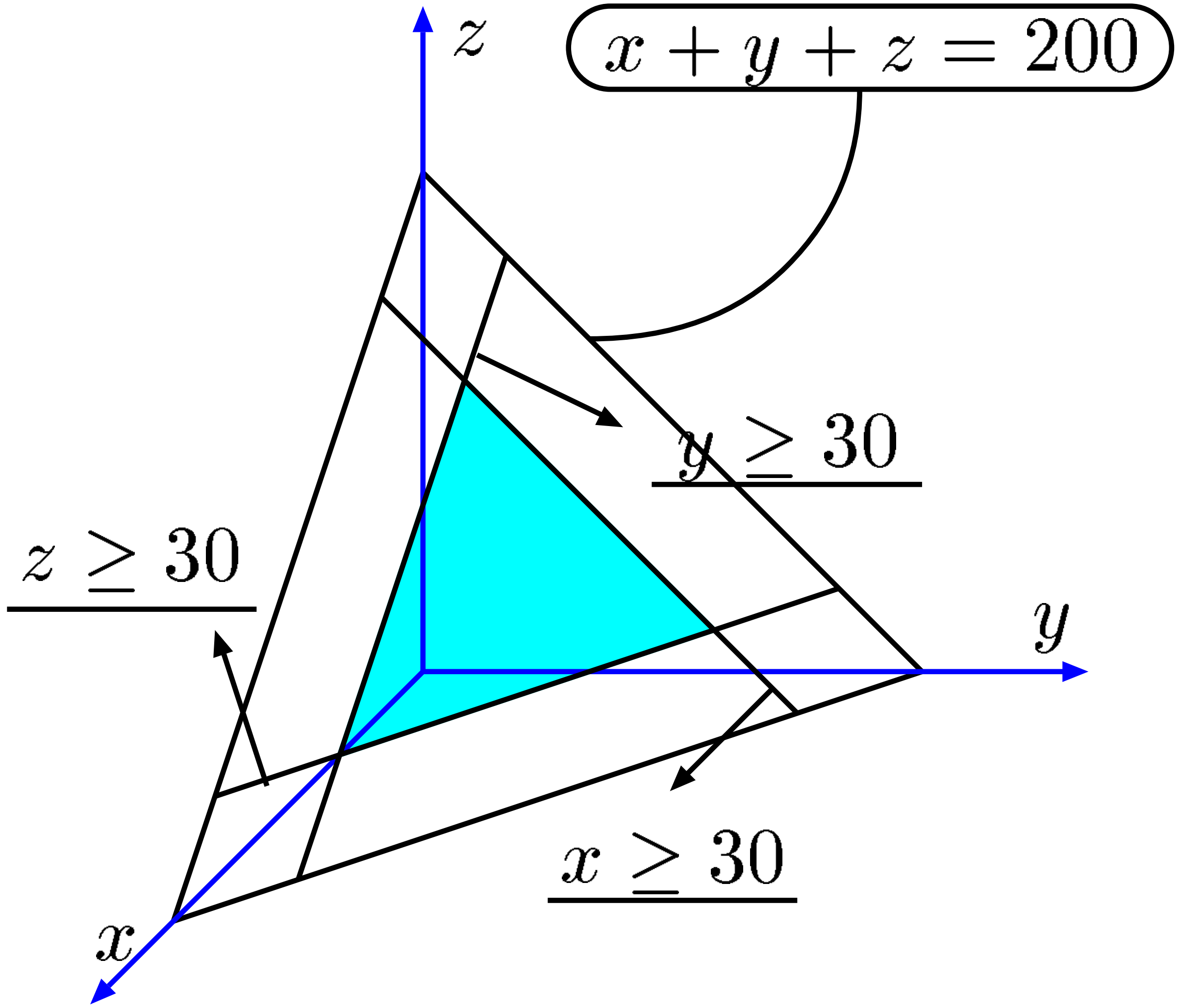
Def. x is individually rational if

$$x_i \geq v(\{i\}) \quad \text{for all } \{i\} \in \mathcal{L}.$$

Def. x is an imputation if
it is efficient and individually rational.

Example of imputations

$I(\mathcal{L}, v)$: the set of the imputations of $(N, \mathcal{L}, v)$.



$\boxed{\text{sky blue}}$

The set of the imputations $I(\mathcal{L}, v)$

(traditional game with 3 players)

Domination

$x, y \in I(\mathcal{L}, v)$ imputations

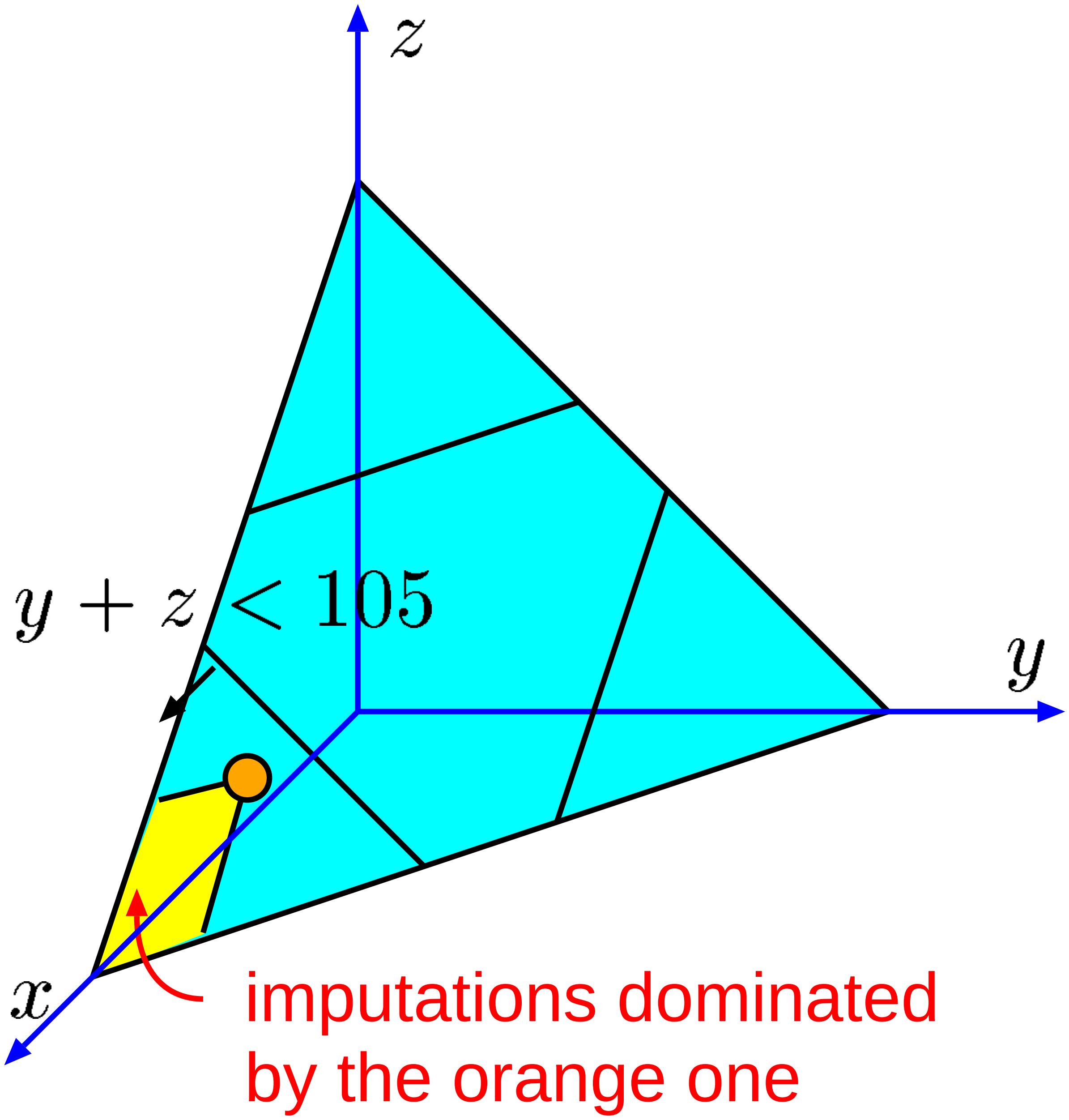
Def. y dominates x (notation: $x \prec y$) if

for some coalition $S \in \mathcal{L}$

1. $y_i > x_i$ for all $i \in S$,
   (the players in S prefer y to x.)

2. $\displaystyle\sum_{i \in S} y_i \leq v(S)$.
   (x_i and y_i for $i \in S$ are realized by $v(S)$.)

Example of domination



sky blue

The set of the imputations $I(\mathcal{L}, v)$

(traditional game with 3 players)

The core

Def. $x \in I(\mathcal{L}, v)$ is coalitionally rational if

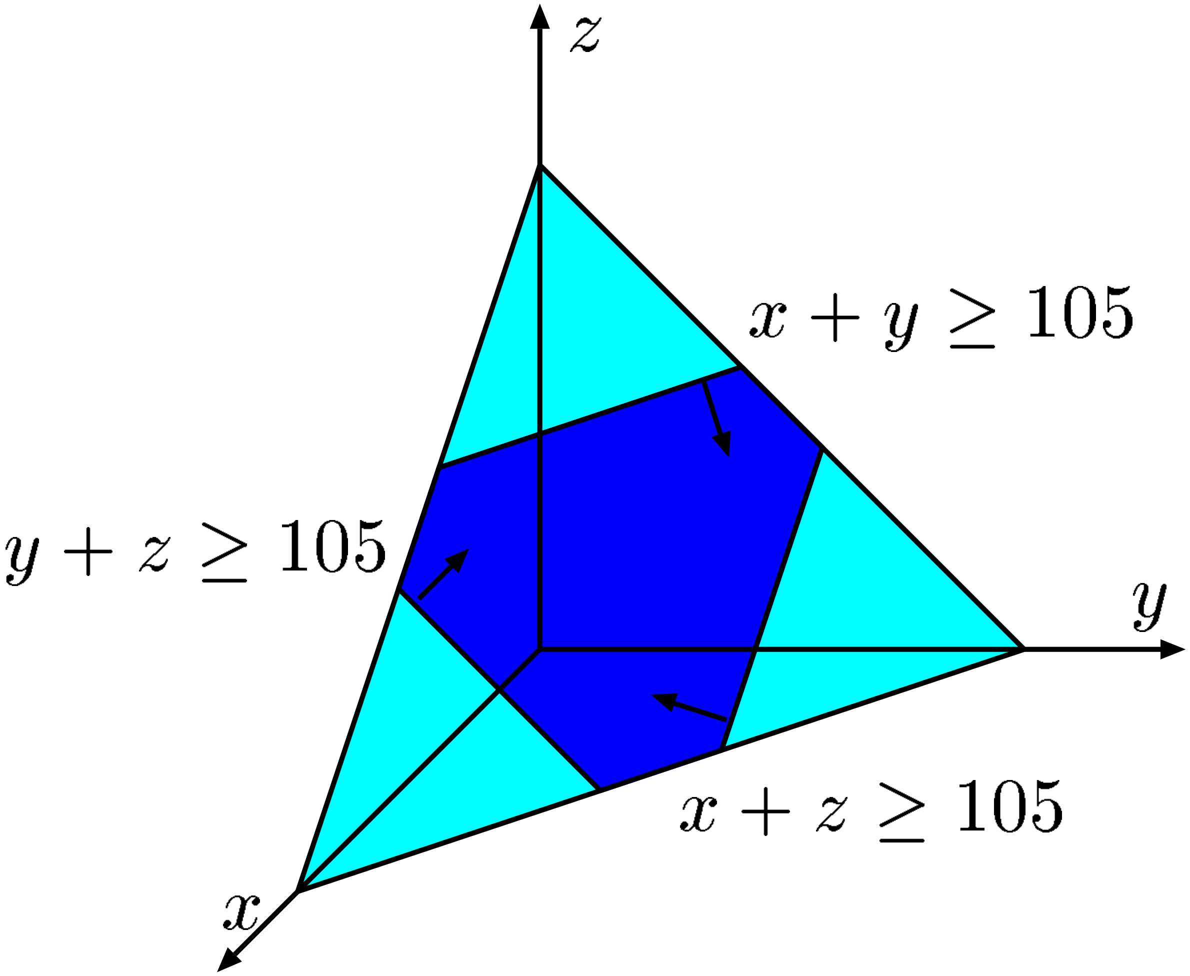
$$\sum_{i \in S} x_i \geq v(S) \quad \text{for all } S \in \mathcal{L}.$$

Def. The core is the set of the coalitionally rational imputations.

In other words

The core is the set of imputations which no other imputations dominates.

Example of the core



sky blue $I(\mathcal{L}, v)$

blue The core

(traditional game with 3 players)

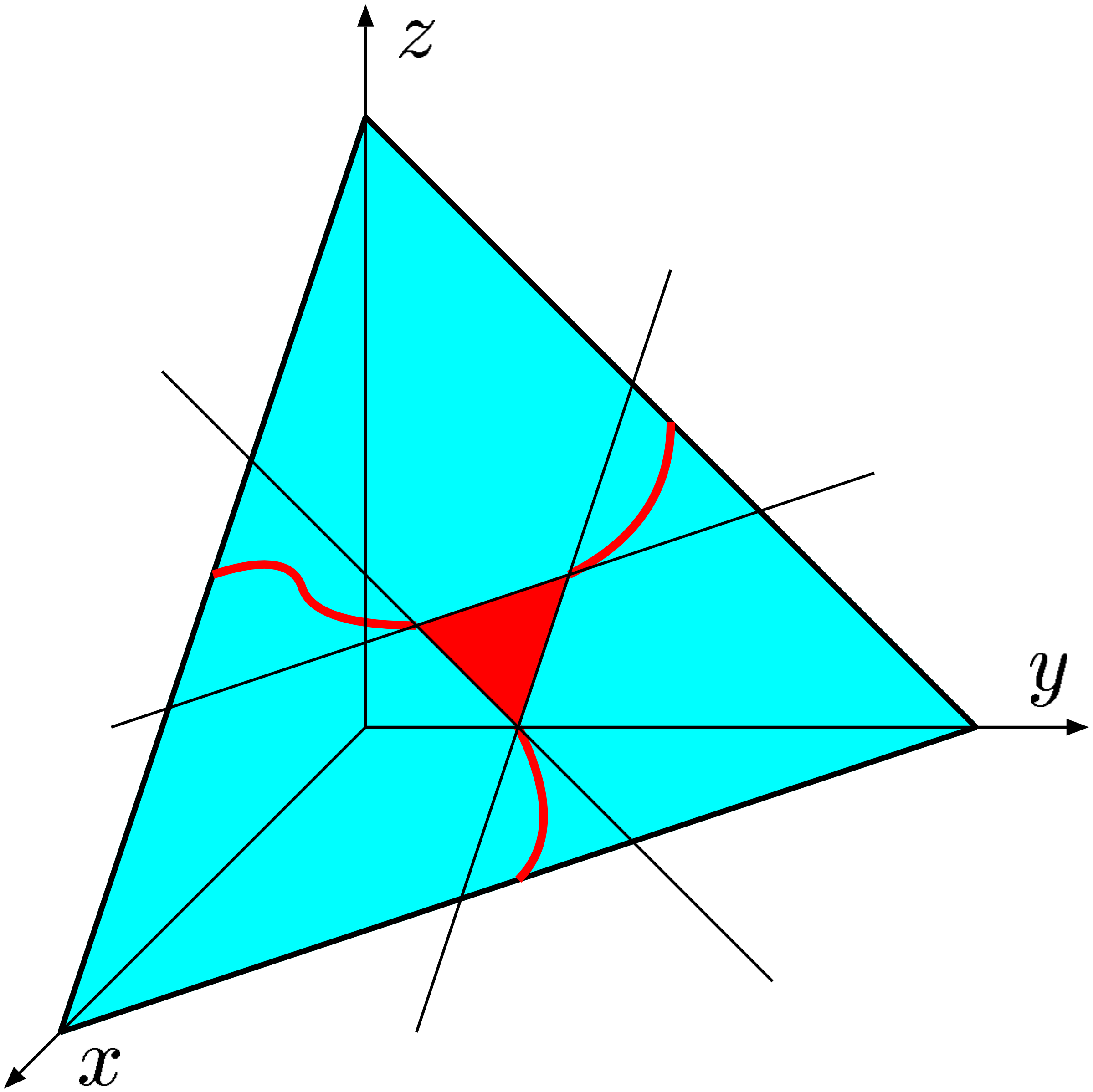
A stable set

Def. A set $K \subseteq I(v, \mathcal{L})$ is a stable set if

1. $\forall\ y \in I(v, \mathcal{L}) \setminus K$, $\exists\ x \in K$ s.t. $y \prec x$,
   (An imp. $\notin K$ is dominated by an imp. $\in K$.)
2. $x \not\prec y\ \forall\ x, y \in K$.
   (Imp.s $\in K$ never dominate each other.)

- A stable set represents
  - a minimal set of imp.s which dominates all the imp.s.
- Generally, a stable set is not unique.

Example of a stable set



sky blue $I(\mathcal{L}, v)$

red a stable set

(traditional game with 3 players)

Core stability of traditional games

For traditional games, Shapley ('71) showed:

The core of a convex game is a unique stable set

where, a game (N, v) is convex if

$$v(S) + v(T) \leq v(S \cap T) + v(S \cup T)$$

for all $S, T \subseteq N$.

. . . .

$\Rightarrow$ How about games on convex geometries??

Quasi-convex games

$\boxed{\text{Def.}}$ a game $(N, \mathcal{L}, v)$ is quasi-convex if

$$v(S) + v(T) \leq v(S \cap T) + v(S \cup T)$$

for all $S, T \in \mathcal{L}$ s.t. $S \cup T \in \mathcal{L}$.

(This definition is originated by Bilbao et al. ('97).)

. . . .

Quasi-convex games lead to core stability!!

Our result

(Okamoto, '01)

For games on convex geometries

> The core of a quasi-convex game is a unique stable set

This is an extension of Shapley's result.

Some remarks (1/2)

- Can other known results be extended to convex geometries?

  Consistency of the core of balanced games

Traditionally	Peleg ('86)
On convex geometries	Okamoto ('01)

Some remarks (2/2)

- How about other solution concepts?
  - Jiménez ('98):
    investigation for simple games
  - Bilbao–Edelman ('00):
    the Shapley value

  These are arranged in the book
  "Cooperative Games on Combinatorial Structures"
  by J. Bilbao, Kluwer, 2000.

Conclusion

We have seen that

> For games on convex geometries,
> the core of a quasi-convex game is a unique stable set

. . . .

You can get my slides from

> http://dali.c.u-tokyo.ac.jp/yoshio/.